Chapter 1

Introduction

Exercises

1. A mothball initially has radius 0.5 cm and slowly evaporates.

(a) If V denotes the volume of the mothball and r the radius, use the chain rule for differentiation to show that

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

(b) Suppose that the rate of change of the volume is proportional to the surface area of the mothball. Express this condition as a differential equation for r as a function of t.

(c) Find a formula for the radius as a function of time, assuming that after 30 days the radius is 0.25 cm. How long before the mothball disappears altogether?

Solution.

(a) Since $V = 4\pi r^3/3$ the formula for dV/dt follows immediately.

(b) This says that

$$\frac{dV}{dt} = -4k\pi r^2$$

for some constant $k > 0$. We can now eliminate dV/dt from the two equations to leave

$$4\pi r^2 \frac{dr}{dt} = -4k\pi r^2$$

or $dr/dt = -k$. This shows that r is decreasing at a constant rate $-k$.

(c) It is easy to see the solution to this differential equation. In fact all solutions are of the form $r(t) = -kt + \text{constant}$. The initial condition $r(0) = 0.5$ shows that the constant is 0.5. At $t = 30$ we have $r(30) = -30k + 0.5 = 0.25$, so $k = 0.25/30$ and the dependence of r on t is given by

$$r(t) = 0.5 - 0.25t/30.$$

Setting $r = 0$ gives $t = 30 \times 0.5/0.25 = 60$ days for the time at which the mothball disappears.

2. A car is travelling at 100 km/h on a level road when it runs out of fuel. Its speed v starts to decrease according to the formula

$$\frac{dv}{dt} = -kv,$$

where k is a constant. One kilometre after running out of fuel its speed has fallen to 50 km/h. Use the chain rule substitution

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

to solve the differential equation. How far will the car travel from the point where it runs out of fuel?

How long after running out of fuel will the car come to a stop? Is the model reasonable?

Answer. $x(t) = 2(1 - e^{-50t})$.

As $t \to \infty$, $x \to 2$. The car travel 2km, but this takes infinite time.

Chapter 2

The Definite Integral: Definition

Exercises

1. Partition the interval $[1, 2]$ into N subintervals of equal length,

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{N-1},\ x_N]\,.$$

(a) Show that $x_i = (N + i)/N$.

(b) Show that the maximum value of $f(x) = 1/x$ on $[x_{i-1}, x_i]$ is $N/(N + i - 1)$ and the minimum value is $N/(N + i)$.

(c) Show that the upper and lower Riemann sums for $f(x) = 1/x$ on $[1, 2]$ with N subintervals are

$$\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N}$$

and

$$\frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{2N-1}\,.$$

(d) Find a value of N such that the difference between the upper and lower sums is less than 10^{-6}.

Solution.

(a) Since the interval $[1, 2]$ has length 1, splitting it into N equal subintervals gives subintervals of length $1/N$. Thus $x_0 = 1$, $x_1 = 1 + \frac{1}{N}$, $x_2 = 1 + \frac{2}{N}$, and so on. That is, $x_i = 1 + \frac{i}{N} = \frac{N+i}{N}$.

(b) Since $f(x) = 1/x$ is a decreasing function, its minimum value on the interval $[x_{i-1}, x_i]$ occurs at the right hand end of the interval—that is, at $x = x_i$. Similarly, the maximum value occurs at the left hand end of the interval, where $x = x_{i-1}$. So the maximum value is $f(x_{i-1}) = \frac{1}{x_{i-1}} = \frac{N}{N+i-1}$ and the minimum value is $f(x_i) = \frac{1}{x_i} = \frac{N}{N+i}$.

(c) Since there are N subintervals $\Delta x = 1/N$. We have just seen that the minimum and maximum values of $1/x$ on the ith subinterval are $m_i = \frac{N}{N+i}$ and $M_i = \frac{N}{N+i-1}$ respectively. Summing up over the range $1 \le i \le N$ and multiplying by Δx gives the required formulas.

(d) Inspection of the formulas for the upper and lower sums shows that they differ by

$$\frac{1}{N} - \frac{1}{2N} = \frac{1}{2N} .$$

We can make this less than 10^{-6} by choosing $2N > \frac{1}{10^{-6}} = 10^6$. Hence $N = 500001$ will do.

Chapter 3

The Definite Integral: Properties

Exercises

1. Given that

$$\int_{-3}^{1} f(x)\,dx = -2\,, \quad \int_{1}^{2} f(x)\,dx = 5\,, \quad \int_{-3}^{2} g(x)\,dx = 8\,,$$

evaluate, where possible:

(a) $\displaystyle\int_{-3}^{2} (f(x) + g(x))\,dx$, (b) $\displaystyle\int_{2}^{-3} \frac{g(x)}{2}\,dx$, (c) $\displaystyle\int_{-3}^{2} f(x)g(x)\,dx$.

Solution.

(a)

$$\int_{-3}^{2} (f(x) + g(x))\,dx = \int_{-3}^{2} f(x)dx + \int_{-3}^{2} g(x)\,dx$$
$$= \left(\int_{-3}^{1} f(x)dx + \int_{1}^{2} f(x)dx\right) + \int_{-3}^{2} g(x)dx$$

(b) $\int_{2}^{-3} \frac{g(x)}{2}dx = -\int_{-3}^{2} \frac{g(x)}{2}dx = -\frac{1}{2}\int_{-3}^{2} g(x)dx = -\frac{1}{2} \times 8 = -4.$

(c) The information given is not sufficient, as the integral of a product is not expressible in terms of the integrals of the individual factors. In particular the integral of a product is not in general equal to the product of th integrals. For example, if $f(x) = g(x) = 1$ then

$$\int_{0}^{2} f(x)\,dx = \int_{0}^{2} g(x)\,dx = \int_{0}^{2} f(x)g(x)\,dx = 2.$$

So

$$\int_{0}^{2} f(x)g(x)\,dx \neq \int_{0}^{2} f(x)\,dx \times \int_{0}^{2} g(x)\,dx\,.$$

Chapter 4

The Definite Integral: Applications

Exercises

1. Consider the part of the hyperbola $x^2 - y^2 = a^2$ in the first quadrant and between the lines $x = a$ and $x = b$, where $0 < a < b$. Find the volume obtained by rotating this curve

(a) about the x-axis, (b) about the y-axis.

Solution.

(a) Since $y = \sqrt{x^2 - a^2}$,

$$V = \pi \int_a^b (x^2 - a^2)\, dx = \pi \left(\frac{x^3}{3} - a^2 x \right) \Big|_a^b = \frac{\pi (b-a)^2 (b+2a)}{3} = \frac{\pi (b^3 - 3a^2 b + 2a^3)}{3}.$$

(b) Now

$$V = 2\pi \int_a^b x\sqrt{x^2 - a^2}\, dx = 2\pi \int_a^b \frac{d}{dx} \left(\frac{1}{3} (x^2 - a^2)^{\frac{3}{2}} \right) dx = \frac{2\pi}{3} (b^2 - a^2)^{\frac{3}{2}}.$$

2. Let R be the region in the first quadrant bounded by the coordinate axes and the parabola $y = 4 - x^2$. Use the disc method to calculate the volume of the solid formed by revolving R about the y-axis. Use the shell method to check your answer.

Solution.

Consider a horizontal strip of thickness Δy from whose lower edge is the line segment joining the point $(0, y)$ to the point (x, y) in the first quadrant which lies on the parabola $y = 4 - x^2$. Then $x = \sqrt{4 - y}$. Revolving the line segment about the y-axis produces circle of area $\pi x^2 = \pi(4 - y)$; hence revolving the strip produces a disc of volume $\pi(4 - y)\Delta y$. By covering R with a collection of such strips (non-overlapping) and summing the volumes of the corresponding discs, we obtain an estimate of the required volume; furthermore, this estimate approaches the exact value as Δy decreases to 0. But the sum we obtain as an estimate is a Riemann sum for the integral $\int_0^4 \pi(4 - y)\, dy$. So the required volume is $\int_0^4 \pi(4 - y)\, dy = \pi(4y - \frac{y^2}{2})\big]_0^4 = 8\pi$.

Chapter 5

Integrals as Functions

Exercises

1. When applying the Fundamental Theorem it is important to check that the conditions for the theorem are satisfied. In particular, discontinuities in the function or its derivative can invalidate the formula. Consider the function $f(x) = 1/x^2$. This is not defined at $x = 0$ but everywhere else it has antiderivative $-1/x$. Since $f(x) > 0$ the integral over any interval should certainly be positive. Show that an attempt to apply the Fundamental Theorem over the interval $[-1, 1]$ (ignoring the difficulties at $x = 0$) leads to the contradictory result

$$\int_{-1}^{+1} \frac{dx}{x^2} = -2\,.$$

Answer.

$$\int_{-1}^{+1} \frac{dx}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{+1} = -1 - (+1) = -2.$$

2. Differentiate with respect to x

(a) $F(x) = \displaystyle\int_0^x \sin^3 t\,dt$ (b) $F(x) = \displaystyle\int_0^{x^3} \sin^3 t\,dt.$

Hint: Think of the integral as a function of a function and use the chain rule.

Solution. The first part of the Fundamental Theorem says that if $F(x) = \int_a^x f(t)\,dt$, then

$$\frac{dF}{dx} = f(x).$$

(a) $\dfrac{dF(x)}{dx} = \sin^3 x$, by the Fundamental Theorem.

(b) Write $g(x) = x^3$. Then we have to differentiate $F\big(g(x)\big)$.

$$\begin{aligned}\frac{d}{dx}\Big(F\big(g(x)\big)\Big) &= F'\big(g(x)\big)\cdot g'(x) \quad \text{(chain rule)}\\ &= F'(x^3)\cdot 3x^2 = \sin^3(x^3)\cdot 3x^2.\end{aligned}$$

3. Find the derivative of $\int_0^x u^2 e^{u^2}\,du$. Hence find the derivative of $\int_{\cos x}^{x^2} u^2 e^{u^2}\,du$.

Solution. By the Fundamental Theorem of Calculus, $\int_0^x u^2 e^{u^2}\,du = F(x) - F(0)$, where $F(x)$ is an arbitrary antiderivative of $x^2 e^{x^2}$. So

$$\frac{d}{dx}\int_0^x u^2 e^{u^2}\,du = \frac{d}{dx}\left(F(x) - F(0)\right) = F'(x) = x^2 e^{x^2}.$$

Furthermore,

$$\begin{aligned} \frac{d}{dx}\int_{\cos x}^{x^2} u^2 e^{u^2}\,du \qquad &= \tfrac{d}{dx}(F(x^2) - F(\cos x)) \\ &= F'(x^2)2x - F'(\cos x)(-\sin x) \\ &= 2x(x^2)^2 e^{(x^2)^2} + \sin x(\cos x)^2 e^{(\cos x)^2} \\ &= 2x^5 e^{x^4} + (\sin x)(\cos^2 x)(e^{\cos^2 x})\,. \end{aligned}$$

4. Use the definition of the general exponential function to verify the following formulas (for any $a > 0$).

(a) $a^x a^y = a^{x+y}$ (b) $a^0 = 1$ (c) $\ln(a^x) = x\ln(a)$.

Show also that

$$\frac{d}{dx}x^a = ax^{a-1}$$

(for $x > 0$).

Solution. (Last assertion only)

$$\begin{aligned} \frac{dx^a}{dx} &= \frac{de^{a\ln x}}{dx} \\ &= e^{a\ln x} \times \frac{a\ln x}{dx} \\ &= x^a \times \frac{a}{x} \\ &= ax^{a-1} \end{aligned}$$

Chapter 6

Integration Techniques: I

Exercises

1. Use substitutions to evaluate the following integrals

(a) $\displaystyle\int x^{-1}\log x\,dx$ (b) $\displaystyle\int e^{\sqrt{x}}\,dx$

(c) $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$ (d) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^4}\,dx\,.$

Solution.

(a) Let $u=\log x$. Then $du=u'(x)\,dx=x^{-1}\,dx$. Substituting,

$$\int x^{-1}\log x\,dx == \int u\,du = \tfrac{1}{2}u^2+C=\tfrac{1}{2}(\log x)^2+C\,.$$

(b) Let $u=\sqrt{x}$, then $x=u^2$, and dx=2u du.
Substituting gives,

$$\int e^{\sqrt{x}}\,dx=\int e^u\,2u\,du=2\int e^u u\,du$$

Integration by parts gives

$$\begin{aligned}\int ue^u\,du &= \int u\frac{de^u}{du}\,du\\ &= ue^u-\int e^u\cdot 1\,du = ue^u-e^u+C.\end{aligned}$$

Thus the desired anti-derivative is $2e^{\sqrt{x}}(\sqrt{x}-1)+C$.

(c) Let $u=\sqrt{e^x+1}$. Then $e^x=u^2-1$, so that $x=\log(u^2-1)$. Thus $\dfrac{dx}{du}=\dfrac{2u}{u^2-1}$. The integral is

$$\int \frac{(u^2-1)^2}{u}\frac{2u}{u^2-1}\,du=2\int (u^2-1)\,du=\frac{2}{3}u^3-2u=\frac{2}{3}(e^x-2)\sqrt{e^x+1}\,.$$

(d) Let $x = \tan t$. Then $dx/dt = 1/\cos^2 t$ and also $\sqrt{1+\tan^2 t} = 1/\cos t$. So the integral becomes

$$\int \frac{1/\cos t}{(\sin t/\cos t)^4} \frac{1}{\cos t}\, dt = \int \frac{\cos t}{\sin^4 t}\, dt = -\frac{1}{3}(\sin t)^{-3} + C = -\frac{1}{3}\left(\frac{\sqrt{x^2+1}}{x}\right)^3 + C\,.$$

2. Find $\int 2 \sin x \cos x \, dx$

(a) by using the substitution $u = \sin x$,

(b) by using the substitution $v = \cos x$,

(c) by using the formula from trigonometry for $\sin 2x$.

Check that your three answers are consistent with each other.

Solution.

(a) Put $u = \sin x$. Then $du = \cos x\, dx$, and so the integral becomes $\int 2u\, du = u^2 + C = \sin^2 x + C$, where C can be any constant.

(b) $v = \cos x$ gives $dv = -\sin x\, dx$, and the integral becomes $-\int 2v\, dv = -v^2 + D = -\cos^2 x + D$, where D is a constant.

(c) $\int 2 \sin x \cos x\, dx = \int \sin 2x\, dx = -\frac{1}{2}\cos 2x + E$, where E is a constant.

The answers are consistent: $\sin^2 x + C = -\cos^2 x + D$ is true provided that $D - C = \sin^2 x + \cos^2 x = 1$. It is perfectly possible for two constants to differ by 1. Similarly, $-\cos^2 x + D = -\frac{1}{2}\cos 2x + E$ provided that $\cos 2x = 2\cos^2 x - 2(D - E)$. But $\cos 2x = 2\cos^2 x - 1$; so the requirement is that the constants D and E should differ by 1/2.

3. Show that the substitution $u = \tan x$ leads to the formula

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C\,,$$

where $\tan^{-1} u$ (or $\arctan u$) is the inverse function to the tangent.

Chapter 7

Integration Techniques: 2

Exercises

1. Find a reduction formula for the indefinite integral

$$I_n = \int \frac{dx}{(1+x^2)^n}.$$

HINT: Take $dv/dx = 1$.

Solution. The case $n = 1$ is a standard integral: $I_1 = \int \dfrac{dx}{(1+x^2)} = \tan^{-1} x + C$.

For any n integrate by parts with $u = 1/(1+x^2)^n$ and $dv/dx = 1$. Then $du/dx = -2nx(1+x^2)^{n+1}$ and we can take $v = x$. This gives

$$I_n = \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2}{(1+x^2)^{n+1}}\, dx.$$

Now $x^2 = (1+x^2) - 1$ and we can break up the last integral as

$$\int \frac{x^2}{(1+x^2)^{n+1}}\, dx = \int \frac{1+x^2}{(1+x^2)^{n+1}}\, dx - \int \frac{1}{(1+x^2)^{n+1}}\, dx = I_n - I_{n+1}.$$

This gives the relation

$$I_n = \frac{x}{(1+x^2)^n} + 2n(I_n - I_{n+1}),$$

which can be rearranged to give I_{n+1} in terms of I_n for $n \neq 0$.

2. Find the following integrals

(a) $\displaystyle\int \frac{dx}{(x-1)(x-3)}$ (b) $\displaystyle\int \frac{x^2\, dx}{(x-1)(x-3)}$.

Answers.

(a) $-\dfrac{1}{2} \ln|x-1| + \dfrac{1}{2} \ln|x-3| + C$.

(b) $x - \dfrac{1}{2} \ln|x-1| + \frac{9}{2} \ln|x-3| + C$.

3. Show that the Wallis product formula for $\pi/2$ can also be expressed as

$$\lim_{n\to\infty} \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-2} = \pi/2,$$

where the round brackets signify the *binomial coefficient*

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}\,.$$

Hint. Show first

$$2\times 4\times 6\times\cdots\times 2n = 2^n n!, \quad \text{and then} \quad 1\times 3\times 5\times\cdots\times 2n-1 = \frac{(2n)!}{2^n n!}.$$

4. For a continuous function $f(x)$ defined for $x \geq a$ the integral from a to ∞ is defined to be the limit

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

(assuming the limit exists). By obtaining a suitable reduction formula, show that

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

for all integers $n \geq 0$. This suggests the possibility of extending the definition of the factorial function to non-integer values of the argument. This is one motivation for the definition of the *Gamma Function* $\Gamma(s)$ by the formula

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx\,,$$

where s is any real number > 1. Although x^{s-1} is not defined at $x = 0$ when $s < 1$ we can still set

$$\int_0^\infty x^{s-1}e^{-x}\,dx = \lim_{a\to 0}\int_a^\infty x^{s-1}e^{-x}\,dx$$

to extend the definition of $\Gamma(s)$ to all $s > 0$. Then it is possible to check that $\Gamma(n+1) = n!$ (for integer n) and $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$.

Solution. The integral

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}(1-e^{-b}) = 1$$

Recall that 0!=1. So for $n = 0$,

$$\int_0^\infty x^n e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = 0!\,.$$

For $n \geq 1$, integration by parts with $u = x^n$ and $dv/dx = e^{-x}$ gives

$$\begin{aligned}\int_0^b x^n e^{-x}\,dx &= \big[-x^n e^{-x}\big]_0^b + n\int_0^b x^{n-1}e^{-x}\,dx \\ &= b^n e^{-b} + n\int_0^b x^{n-1}e^{-x}\,dx.\end{aligned}$$

As $n \to \infty$, $b^n e^{-b} \to 0$. Hence we get the reduction formula,

$$\int_0^\infty x^n e^{-x}\,dx = n\int_0^\infty x^{n-1}e^{-x}\,dx$$

valid for $n \geq 1$.

For $n \geq 1$, successive applications of the reduction formula give

$$\begin{aligned}\int_0^\infty x^n e^{-x}\,dx &= n \times \int_0^\infty x^{n-1}e^{-x}\,dx \\ &= n \times (n-1) \times \int_0^\infty x^{n-2}e^{-x}\,dx \\ &= n \times (n-1) \times (n-2) \times \int_0^\infty x^{n-3}e^{-x}\,dx \\ &\vdots \\ &= n \times (n-1) \times (n-2) \times \cdots \times 2 \times \int_0^\infty xe^{-x}\,dx \\ &= n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \times \int_0^\infty e^{-x}\,dx \\ &= n \times (n-1) \times \cdots \times 2 \times 1, \quad \left(\int_0^\infty e^{-x}\,dx = 1\right) \\ &= n!\end{aligned}$$

5. Verify the formula

$$\int_0^1 x^m(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}\,,$$

where m, n are integers ≥ 0.

Hint.

Check the formula holds with $n = 0$, $m \geq 0$.

Establish the reduction formula

$$\int_0^1 x^m(1-x)^n\,dx = \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}\,dx \qquad m \geq 0, n \geq 1.$$

Chapter 8

Models and Differential Equations

Exercises

1. Find the general solutions and sketch the solution curves of the differential equations:

(a) $\dfrac{dy}{dx} = e^x$, (b) $\dfrac{dy}{dx} = \sin x$, (c) $\dfrac{dy}{dx} = \sinh x$.

Answers. General solutions are:

(a) $y = e^x + C$, (b) $y = -\cos x + C$, (c) $y = \cosh x + C$.

2. Find the particular solutions satisfying the given conditions:

(a) $\dfrac{dy}{dx} = 1 + \cos x + \cos^2 x$, $y = 2$ when $x = 0$.

(b) $\dfrac{dy}{dx} = 1 - 2x - 3x^2$, $y = -1$ when $x = 1$.

(c) $e^{2x}\dfrac{dy}{dx} + 1 = 0$, $y \to 2$ as $x \to \infty$.

(d) $\dfrac{dy}{dx} = \left(x + \dfrac{1}{x}\right)^2$, $y = 1$ when $x = 1$.

(e) $\dfrac{dy}{dx} = x\sqrt{1 + x^2}$, $y = -3$ when $x = 0$.

(f) $x^2 e^{x^2}\dfrac{dy}{dx} = x^3$, $y(0) = 1$.

Answer. Particular solutions are

(a) $y = \frac{3}{2}x + \sin x + \frac{1}{4}\sin 2x + 2$, (b) $y = x - x^2 + 3x^{-1} - 4$,

(c) $y = \frac{1}{2}e^{-2x} + 2$, (d) $y = \frac{1}{3}x^3 + 2x - x^{-1} - \frac{1}{3}$,

(e) $y = \frac{1}{3}(1 + x^2)^{3/2} - \frac{10}{3}$, (f) $y = \frac{3}{2} - \frac{1}{2}e^{-x^2}$.

3. Find the general solution of

(a) $x^3\dfrac{dy}{dx} = 2x^2 + 5$, (b) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 + 3x}}$, (c) $\cos^2 x\dfrac{dy}{dx} = \sin x$.

Answers. General solutions are

(a) $y = 2\ln|x| - \frac{5}{2}x^{-2} + C$, (b) $y = \frac{2}{3}\sqrt{1+3x} + C$,

(c) $y = \sec x + C$.

4. If $y = 3$ when $x = 3$ and $dy/dx = (x^2+1)/x^2$, find the value of y when $x = 1$.

Answer. $y(1) = \frac{1}{3}$.

5. The graph of $y = f(x)$ passes through the point $(9, 4)$. Also, the line tangent to the graph at any point (x, y) has the slope $3\sqrt{x}$. Find $f(x)$.

Answer. The curve is $y = 2x^{3/2} - 50$.

6. Solve

(a) $\dfrac{dy}{dx} = y^2$, (b) $\dfrac{dy}{dx} = \dfrac{4(1+y^2)^{3/2}}{y}$.

Answers. General solutions are

(a) $x = -y^{-1} + C$, which we can solve for y in terms of x, $y = 1/(C - x)$.

(b) $x = -\frac{1}{4}(1+y^2)^{-1/2} + C$ defining y implicitly as a function of x.

7. Find the general solution of the differential equation

$$y^{\nu}\frac{dy}{dx} = 1,$$

where ν is a real number. Write out the solutions explicitly in the cases when $\nu = -1$, $\nu = -3$, $\nu = -\frac{1}{2}$.

Answer. All solutions have the same form, $y^{\nu+1}/(\nu+1) = x + C$, except when $\nu = -1$. The case $\nu = 1$ has to be considered separately.

The explicit solutions are:

$$y = Ae^{x}, \quad y^2 = 1/(A - 2x), \quad y^{1/2} = \tfrac{1}{2}x + A.$$

8. Use partial fractions to find the particular solutions of

(a) $\dfrac{dy}{dx} = y^2 - 1,\ y(0) = 2;$

(b) $\dfrac{dy}{dx} = 4 - y^2,\ y(0) = 6;$

(c) $\dfrac{dy}{dx} = y^2 + 3y + 2,\ y(0) = 0.$

Answers. The particular solutions are

(a) $y = (3 + e^{2x})/(3 - e^{2x})$,

(b) $y = (4 + 2e^{4x})/(2e^{4x} - 1)$,

(c) $y = (2e^x - 2)/(2 - e^x)$.

9. Find the family of curves which describe the general solution of

$$\frac{dy}{dx} = 2xy\,.$$

Sketch them.

Answer. The family of curves is $y = Ae^{x^2}$.

10. Find the particular solutions of

(a) $y - x\dfrac{dy}{dx} = 1 + x^2\dfrac{dy}{dx}\,,\qquad y = 3/2$ when $x = 1$.

(b) $(1 + x^2)^2\dfrac{dy}{dx} + xy^2 = 0\,,\qquad y \to \infty$ as $x \to 0$.

(c) $y\cos^2 x\dfrac{dy}{dx} = \tan x + 2\,,\qquad y = 2$ when $x = \pi/4$.

(d) $x(x + 1)\dfrac{dy}{dx} = y(y + 1)\,,\qquad y(1) = 2\,.$

Answers. The particular solutions are

(a) $y - 1 = x/(x + 1)\,,$ (b) $y = 2(1 + 1/x^2)$

(c) $y^2 = \sec^2 x + 4\tan x - 2\,,$ (d) $y = -4x/(x - 3)\,.$

11. Solve

(a) $\dfrac{dy}{dx} = \cot x \cot y,$ (b) $(x^2 + 1)\dfrac{dy}{dx} + xy = xy^2,$

(c) $(1 + x)^2\dfrac{dy}{dx} + y^2 = 1\,.$

Answers. The general solutions are

(a) $\sin x \cos y = C\,,$ (b) $(y - 1)/y = A\sqrt{x^2 + 1}\,,$

(c) $(1 - y)/(1 + y) = Ae^{2/(1+x)}\,.$

Chapter 9

Applications of Separable Equations

Exercises

1. Fechner's law states that
$$\frac{dR}{dS} = \frac{n}{S}\,.$$
Determine $R(S)$.

(An example of its application to describe the response of the eye to brightness is given in Table 1.)

Answer. $R = n\ln S + C\,.$ The visual magnitude scale for stars developed in antiquity was a logarithmic function of brightness.

2. A generalization of Stevens' law (which includes Fechner's law as a special case) is
$$\frac{dR}{dS} = n\frac{R^p}{S^q}\,,$$
where n, p and q are all positive constants. Find $R(S)$ given that $R = 0$ when $S = 0$.

Answer. $R = \left[\dfrac{n(1-p)S^{1-q}}{(1-q)}\right]^{1/(1-p)} \quad \text{if} \quad p, q < 1\,.$

3. The Loewenstein model of physiological reaction is
$$\frac{dR}{dS} = \frac{R}{(1+\alpha S)S}\,,$$
where α is a positive constant. Find $R(S)$.

Answer. $R = \dfrac{AS}{1+\alpha S}\,.$

4. (a) If there are initially N_0 atoms of radioactive material with decay constant k in a sample, find $N(t)$.

(b) A sample of lunar rock was found to contain equal numbers of potassium and argon atoms. Assuming that all the argon is the result of the radioactive decay of potassium (which has a half-life of 1.28×10^9 years) and that one in every nine potassium atom disintegrations produces one argon atom, determine the age of the lunar rock.

Answer. 4.25×10^9 years, a good estimate of the age of the solar system.

5. Charcoal from Stonehenge was found to contain C^{14}/C^{12} at a level of 63% that of the present day value. [1]. The half-life of C^{14} is 5730 years. Show that the sample is 3820 years old.

6. The number N of bacteria in a culture grew according to the Malthusian law. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2}$ hours?

Answer. $N = 605$.

7. A model for bacterial growth was proposed by Smith in 1963. His observations suggested that bacterial populations demonstrate that growth is inhibited by an extra term proportional to the rate of population change. Thus

$$\frac{dN}{dt} = \gamma N \left(1 - \frac{N}{M} - \frac{\mu}{M}\frac{dN}{dt}\right),$$

where μ is another positive constant.

Solve the differential equation, given that the initial population size is N_0. What is the limiting population size?

Answer. $N(1 - N/M)^{\gamma\mu+1} = \dfrac{N_0 e^{\gamma t}}{(1 - N_0/M)^{\gamma\nu+1}}$.

The limiting population size (as $t \to \infty$) is M.

8. A model for growth which is influenced by seasonal factors is provided by writing

$$\frac{dN}{dt} = (\gamma + \mu \cos \omega t) N,$$

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[1]The most commonly used means of absolute dating in archaeology measures the radioactive decay of C^{14}. The basis of this method is the following.

Radioactive carbon (C^{14}) is produced in the Earth's atmosphere by cosmic rays (high-energy protons) colliding with atoms of the atmosphere. The atomic fragments produced on impact include some neutrons. These neutrons can be captured by nitrogen (N^{14}) atoms in the atmosphere. In the process a proton is ejected. The overall mass remains the same (14) but the number of protons decreases by one so that the atom slips one place in the chemical table from nitrogen to carbon. The radioactive carbon in the atmosphere then decays slowly, with a half-life of 5730 years, as the neutrons emit a positron and antineutrino in order to revert to a proton. The atoms then become N^{14} again.

The amount of C^{14} in the atmosphere is established by balancing the cosmic ray production rate against the spontaneous decay rate. The ratio of radioactive carbon to stable carbon (C^{12}) is therefore a function of the amount of N^{14} and C^{12} in the atmosphere, as well as the cosmic ray flux. These quantities are assumed to have remained unchanged over historic time, maintaining the C^{14}/C^{12} ratio in the atmosphere at its present day value of about 3.6×10^{-13}.

Radioactive carbon is chemically identical to normal carbon and combines with oxygen to form carbon dioxide in the same proportion. Living beings then respire and ingest radioactive carbon and accumulate the same fraction in their tissues. After death, however, carbon is no longer assimilated and the radioactive carbon which decays is not renewed. The C^{14}/C^{12} ratio in the material then drops exponentially from the atmospheric value. The amount of time which has elapsed since the death of the animal or plant can then be found quite simply by comparing the two values.

where γ, μ and ω are constants. Find $N(t)$ given that the initial population is N_0.

Describe the behaviour of the solutions when $|\gamma| < |\mu|$ and $|\gamma| > |\mu|$.

Answer. $N = N_0 e^{\gamma t} e^{(\mu \sin \omega t)/\omega}$.

If $\gamma > 0$ and $\gamma < |\mu|$ the curve oscillates up and down between limits which increase exponentially with time. If $\gamma > |\mu|$ the curve shows just ripples as it increases monotonically.

If $\gamma < 0$ and $|\gamma| < |\mu|$ the curve oscillates up and down between limits with decay to zero exponentially with time (N is always positive though). If $|\gamma| > |\mu|$, the curve shows ripples as it decays monotonically.

9. An equation governing the growth of fish was proposed in 1951 by von Bertalanffy. It is based on metabolic balance. Chemical energy is gained by the body through the walls of the gut. The rate is proportional to the surface area, that is, roughly to $V^{2/3}$ where V is the volume of the fish. Chemical energy is used by cells throughout the body, so that the rate of loss is proportional to V. Any excess chemical energy goes into cell growth, dV/dt. Since the mass m is itself proportional to V, the rate of mass change can be written as

$$\frac{dm}{dt} = \alpha m^{2/3} - \beta m \,,$$

where α and β are positive constants. Find $m(t)$ given that the initial mass is zero.

[Hint: use the substitution $x = m^{1/3}$ to perform the integration.]

Answer.

$$m = \left(\tfrac{\alpha}{\beta}\right)^3 \left(1 - e^{-\beta t/3}\right)^3 \,.$$

10. If interest is compounded continuously at a rate of 10% per year, what will an initial investment P_0 be worth after one year? What is the effective yearly rate of interest?

Answers. $1.105 P_0$, 10.5%.

11. An investor has \$1000 with which to open an account and plans to add \$1000 per year. All funds in the account will earn 10% interest per year, continuously compounded.

(a) Assuming that the added deposits are also made continuously, find the differential equation governing the sum in the account x at time t.

(b) How many years would it take for the account to reach \$100000?

Answer.

(a) $\dfrac{dx}{dt} = 0.10x + 1000,$

(b) 23 years.

12. The spread of innovation (in agriculture and industry) has been successfully modelled by assuming that the rate of spread is proportional to both the number of people already having adopted the new system and to the number of people who have not yet adopted it.

If M is the potential market for the innovation and $N(t)$ the number who have already adopted it, then

$$\frac{dN}{dt} = aN(M - N)\,,$$

where a is a constant. This has the same form as the Verhulst population equation.

Now suppose that the effect of advertising is added. Suppose further that it contributes to the rate of adoption in proportion to the number of people at whom the advertising is aimed, that is, the number $M - N$ who have still to change to the new system. The new model equation then reads

$$\frac{dN}{dt} = aN(M - N) + b(M - N)\,,$$

where b is another constant.

Solve this equation to find $N(t)$, given that $N(0) = N_0$.

Answer. $N(t) = \dfrac{(aM + b)N_0}{(aN + b) + a(M - N)e^{-(aM+b)t}}\,.$

13. According to Galileo's law the motion of a freely falling object is governed by the equation[2]

$$\frac{dv}{dt} = g\,,$$

where g is the constant 'acceleration due to gravity' and position x is measured downwards. Show that for an object falling from rest at the origin,

$$v = gt\,, \quad x = \frac{gt^2}{2}\,, \quad v^2 = 2gx\,.$$

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[2]Galileo Galilei lived from 1564 to 1642 in an era when many of the principles of mechanics now taught in high school were beginning to emerge from a mass of observations of objects in motion—falling bodies, projectiles and planets, for example.

The story that Galileo himself studied the motion of falling objects by dropping them from the top of the Leaning Tower of Pisa is certainly apocryphal because the law governing the motion of an object dropped from rest was well known by 1600. Observation had shown that objects fall the same distance in the same time and that the distance fallen is proportional to the square of the time elapsed.

Galileo set out to develop a model for this motion based upon the assumption of uniform acceleration. He worked on the problem for over 30 years and finally published his results in 1638.

14. What is Galileo's law of motion for a freely falling object when position x is measured *upwards*? If the object is initially at the point x_0 with velocity u, find $v(t)$, $x(t)$ and $v(x)$.

Answer. $dv/dt = -g\,;\, v = -gt\,, \quad x = x_0 - gt^2/2\,, \quad v^2 = 2g(x_0 - x)\,.$

15. The equation of motion of an object thrown upwards through air producing a resistance of $mk|v|$ is

$$m\frac{dv}{dt} = -mg - mkv\,,$$

where position x is measured upwards.

(a) If the object is initially at $x = 0$ with velocity u, find $v(t)$, $x(t)$ and $x(v)$.

(b) Find the maximum height reached and the time taken to reach it.

Answers.

(a) $$v(t) = \left(\frac{g}{k} + u\right) e^{-kt} - \frac{g}{k}\,,$$
$$x(t) = \frac{1}{k}\left(\frac{g}{k} + u\right)\left(1 - e^{-kt}\right) - \frac{gt}{k}\,,$$
$$x(v) = \frac{u - v}{k} + \frac{g}{k^2}\ln\left(\frac{1 + kv/g}{1 + ku/g}\right).$$

(b) $$x_{max} = \frac{u}{k} - \frac{g}{k^2}\ln\left(1 + ku/g\right),$$
$$t_{max} = \frac{1}{k}\ln\left(1 + ku/g\right).$$

Chapter 10

Linear Differential Equations

Exercises

1. Find the general solutions of

(a) $\dfrac{dy}{dx} - 2y = 3\,,$ (b) $\dfrac{dx}{dt} - tx = t\,,$

(c) $\dfrac{dy}{dx} = \dfrac{4x^3 - y}{x}\,,$ (d) $\dfrac{dy}{dx} + 2y = e^{-x}\,,$

(e) $\dfrac{dx}{dt} + 2tx = 2t^3\,,$ (f) $x^2\dfrac{dy}{dx} + (1-2x)y = x^2\,.$

Answers. General solutions are

(a) $y = -3/2 + Ce^{2x}\,,$ (b) $x = -1 + Ce^{t^2/2}\,,$

(c) $y = x^3 + C/x\,,$ (d) $y = e^{-x} + Ce^{-2x}\,,$

(e) $x = t^2 - 1 + Ce^{-t^2}\,,$ (f) $y = x^2 + Cx^2e^{1/x}\,.$

2. Solve $\dfrac{dy}{dx} = \dfrac{2y}{y - x - y^3}.$

Answer. $x = \frac{1}{3}y - \frac{1}{7}y^3 + Cy^{-1/2}\,.$

3. Find the particular solutions of

(a) $\dfrac{dy}{dx} + y\tan x = \sec x\,, \qquad y = 2 \quad \text{when} \quad x = 0.$

(b) $\dfrac{dy}{dx} = \dfrac{2y}{x} + x^4\,, \qquad y = 1 \quad \text{when} \quad x = 1.$

(c) $\dfrac{dx}{dt} + 4x = e^{-4t}\sin 2t\,, \qquad x = \dfrac{1}{2} \quad \text{when} \quad t = 0.$

(d) $\cos x\dfrac{dy}{dx} + y = \cos^3 x\,, \qquad y = \dfrac{1}{2} \quad \text{when} \quad x = 0.$

(e) $(1+x)\dfrac{dy}{dx} + y = 3x^2\,, \qquad y = 2 \quad \text{when} \quad x = 0$

(f) $(1+x^2)\dfrac{dy}{dx} + 2xy = 4 + 2x\,, \qquad y = 4 \quad \text{when} \quad x = 0\,.$

Answers. Particular solutions are

(a) $y = \sin x + 2\cos x$.

(b) $y = x^5/3 + \frac{2}{3}x^2$.

(c) $y = -\frac{1}{2}e^{-4t}\cos 2t + e^{-4t}$.

(d) $y = \cos x(\sin x - \frac{1}{4}\cos 2x + \frac{3}{4})/(1+\sin x)$.

(e) $y = (x^3+2)/(1+x)$.

(f) $y = (2+x)^2/(1+x^2)$.

4. Solve the differential equation

$$x\frac{dy}{dx} + y = e^x - xy\,.$$

Show that $y = (e^x + e^{-x})/2x$ is a particular solution provided $x \neq 0$.

Answer. The general solution is $y = (e^x + Ce^{-x})/2x$.

5. Find the general solution of $\dfrac{dy}{dx} + 2y\tan x = \sin x$.

Answer. The general solution is $y = \cos x + C\cos^2 x$.

6. Repeat the dam problem supposing that the water level was a fraction f of the initial level after T_0 days. Show that the dam the empties after $T_0/(1-\sqrt{f})$ days.

7. Modify the previous problem to find the height of water in the dam as a function of time if there is a constant flow of water into the dam. Suppose that the inflow occurs at a rate r and that the initial height of the water level is h_0.

Answer. $t = 2Ak(\sqrt{h_0} - \sqrt{h}) + 2Ark^2 \ln k\sqrt{h_0} - rk\sqrt{h} - r$.

8. Liquid runs out of the hole at the bottom of an upright funnel whose angle at the apex is 2α at a rate governed by Torricelli's law. If the hole has area a and the level of the liquid is initially a height H above the apex, find the time taken for the liquid to all run out.

Answer. $(2\pi\tan^2\alpha H^2/5a)\sqrt{H/2g}$.

9. Drugs such as penicillin are gradually removed from the bloodstream at a rate proportional to the mass present in the blood at any instant.

(a) If m is the mass of the drug and V the volume of blood, find the equation governing the concentration $c = m/V$ of the drug in the blood.

(b) Solve the equation, given that the initial concentration is c_0.

Answer.

(a) $\dfrac{dc}{dt} = -kc$.

(b) $c(t) = c_0 e^{-kt}$.

10. In a dilute sugar solution the rate of decrease of concentration is proportional to the concentration c. If $c = 0.01\,\mathrm{g/cm^3}$ at time $t = 0$ and $c = 0.005\,\mathrm{g/cm^3}$ at time $t = 4$ hours, show that the concentration after 10 hours will $0.00177\,\mathrm{g/cm^3}$.

11. The steady infusion of a substance such as glucose into the bloodstream produces a constant term in the equation governing the amount of that substance in the blood,

$$\frac{dm}{dt} = s - km\,.$$

Show that the amount of substance present a time t after infusion commences, assuming that there is an initial mass m_0 already in the blood, is

$$m = \frac{s}{k} - \left(\frac{s}{k} - m_0\right) e^{-kt}\,.$$

What happens when t becomes large?

Answer. $m \to s/k$.

12. (a) What is the limiting concentration achieved after a long time in the polluted lake of the second example?

(b) If the chemical plant were to shut down and the source of pollution eliminated, show that it would take a time $(V/r)\ln 10$ for the pollution level to fall by a factor of 10.

(c) Calculate this time for the case of Lake Ontario, given that $V = 1600\,\mathrm{km^3}$ and $r = 209\,\mathrm{km^3/year}$. The cleaning-up of the Great Lakes was a long-term project!

Answer.

(a) $c(\infty) = c_1$, (b) $t_{1/10} = (V/r)\ln 10$, (c) 17.6 years.

13. Solve the equation for the mass of pollutant in the lake as function of time and verify that it leads to the same result for the concentration.

14. A swimming pool holds 40000 litres of water with a chlorine concentration of 0.01%. An overflow maintains the volume of water constant. Water with a chlorine concentration of 0.001% is now pumped in at a rate of 20 L/min.

(a) What is the chlorine concentration in the pool after one hour?

(b) When will the concentration have dropped to 0.002%?

Answers.

(a) 0.097%, (b) 73.24 hours.

15. A tank initially contains 500 L of fresh water. A pipe is opened which admits polluted water at 4 L/min. At the same time, a drain is opened to allow 3 L/min of the mixture to leave the tank. If the inflowing polluted water contains 0.01 kg/L of impurity, what is the mass of impurity in the tank after 100 minutes?

Answer. 3.11 kg.

Hint: work in terms of concentration to obtain a separable equation.

16. The loss of heat from an object to its surroundings is commonly governed by what is known as Newton's law of cooling. This states that the rate of loss of heat is proportional to the temperature difference between the object and the surroundings. Thus

$$dTdt = -K[T - T_0(t)]\,,$$

where T is the temperature of the object, T_0 is the temperature of the surroundings (and may vary with time) and K is a constant specific to the object in question.

An object in room at a constant temperature of 15°C is initially at a temperature of 75°C. It is then cooling at a rate of 2°C/min. What will be its temperature after 15 minutes? What time will elapse before the temperature falls to 35°C?

Answers. 51.4°C, 33 minutes.

17. When a condenser of capacity C is charged, the potential difference across the condenser at time t is given by

$$CR\frac{dV}{dt} = E - V\,,$$

where E is the emf of the battery and R is a non-inductive resistance. If $V = 0$ at $t = 0$, find $V(t)$ and $V(\infty)$.

Answer. $V(t) = E(1 - e^{-t/RC})\,, \qquad V(\infty) = E\,.$

Chapter 11

Second-Order Differential Equations

Exercises

1. Find the general solution of

$$\frac{d^4y}{dx^4} = 0\,.$$

What is the particular solution satisfying $y(0) = y'(0) = y''(0) = y'''(0) = 1$?

Answer. General solution: $y = Ax^3 + Bx^2 + Cx + D$.

Particular solution: $y = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$.

2. Find the general solution of

$$\frac{d^2y}{dx^2} = e^{2x}\,.$$

What is the particular solution satisfying the boundary conditions $y(0) = 0$, $y(1) = 2$?

Answer. General solution: $y = \frac{1}{4}e^{2x} + Ax + B$.

Particular solution: $y = \frac{1}{4}\big(e^{2x} + (9 - e^2)x - 1\big)$.

3. Find the solution of

$$\frac{d^3y}{dx^3} = \cos 2x + x^2 - 1$$

satisfying $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.

Answer. General solution: $y = -\frac{1}{8}\sin 2x\frac{1}{60}x^5 - \frac{1}{6}x^3 + Ax^2 + Bx + C$.

Particular solution: $y = -\frac{1}{8}\sin 2x + \frac{1}{60}x^5 - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x$.

4. Solve the following equations, giving the general solution and then the particular solution satisfying the given boundary or initial conditions.

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 15y = 0, \quad y(0) = y'(0) = 1.$

(b) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0 \quad y(0) = 0,\ y(1) = 3.$

(c) $\dfrac{d^2y}{dx^2} + 25y = 0 \quad y(0) = y(\pi/2) = 1$

(d) $3\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 2y = 0, \quad y(0) = 1,\ y(2) = 1.$

(e) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 26y = 0 \quad y(0) = 0,\ y'(0) = 2.$
(Give the general and particular solutions in both real and complex exponential form.)

(f) $\dfrac{d^2y}{dx^2} - 99\dfrac{dy}{dx} - 100y = 0, \quad y(0) = 1,\ y'(0) = -1.$

(g) $\dfrac{d^2y}{dx^2} - 99\dfrac{dy}{dx} - 100y = 0, \quad y(0) = 1,\ y'(0) = -1.01.$
(Compare these last two cases; if you attempt to plot the particular solutions for x small and positive in each instance you will find that the first decays steadily to zero but the second will 'explode' on you. The difference in behaviour stems from the slight variation in the initial conditions. Equations like this are called *stiff*; they are hard to solve numerically, because slight errors get grotesquely magnified!)

Answers.

(a) General solution $y = Ce^{-5x} + De^{3x}$.
Particular Solution $y = (e^{-5x} + 3e^{3x})/4$.

(b) General solution $y = (C + Dx)e^{-3x}$.
Particular Solution $y = 3xe^{3(1-x)}$.

(c) General solution $y = C\cos 5x + D\sin 5x$.
Particular Solution $y = \cos 5x + \sin 5x$.

(d) General solution $y = Ce^{x/3} + De^{-2x}$.
Particular Solution $\left[(1 - e^{-4})e^{x/3} + (e^{2/3} - 1)e^{-2x}\right] / \left(e^{2/3} - e^{-4}\right)$.

(e) General solution
$y = e^{-x}(Ce^{5ix} + De^{-5ix})$ (complex form),
$y = e^{-x}(E\cos 5x + F\sin 5x)$ (real form).
Particular Solution
$y = e^{-x}(e^{5ix} - e^{-5ix})/5i$ (complex form).
$y = (2e^{-x}\sin 5x)/5$ (real form).

(f) General solution $y = Ce^{-x} + De^{100x}$. Particular Solution $y = e^{-x}$.

(g) General solution same as last question.
Particular Solution $y = 1.00009901e^{-x} + 0.00009901e^{100x}$.

5. Alternative derivation of the general solution of homogeneous second-order equations with constant coefficients. Show that the homogeneous second-order equation with constant coefficients

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

can be rewritten as

$$\frac{d}{dx}\Big(\frac{dy}{dx} - m_1 y\Big) - m_2\Big(\frac{dy}{dx} - m_1 y\Big) = 0\,,$$

where the constants m_1, m_2 are the roots of

$$m^2 + Am + B = 0\,.$$

Now, defining $p = \dfrac{dy}{dx} - m_1 y$, the second-order equation becomes a pair of first-order equations

$$\begin{aligned} \frac{dp}{dx} - m_2 p &= 0 \\ \frac{dy}{dx} - m_1 y &= p\,. \end{aligned}$$

Solve the first equation for p, then substitute in the right-hand side of the second equation and solve for y.

Hence show that the general solution of the second-order equation is

$$y = \begin{cases} Ce^{m_1 x} + De^{m_2 x} & \text{if } m_1 \neq m_2 \\ (C + Dx)\, e^{mx} & \text{if } m_1 = m_2 = m. \end{cases}$$

6. Show the solution for simple harmonic motion may also be written as

$$y = a\cos(\omega_0 t + \phi)\,,$$

where $a = \sqrt{C^2 + D^2}$ and $\tan\phi = -D/C$. Write down the corresponding form for $\dot{y}$, and plot a graph of both for the case $a = 1, \omega_0 = \pi, \phi = \pi/4$. What is the time before this solution repeats itself?

Chapter 12

Systems of Differential Equations

Exercises

1. Two species with populations x and y compete for their food supply. The equations describing the evolution of x and y are

$$\dot{x} = ax - by$$
$$\dot{y} = -cx + dy\,,$$

where a, b, c, d are positive constants. Explain why this is a reasonable model, and eliminate y to deduce that x satisfies the second-order equation

$$\ddot{x} - (a+d)\dot{x} + (ad - bc)x = 0\,.$$

Show that the solution for x is

$$x = Ce^{m_1 t} + De^{m_2 t}\,,$$

where you are to give expressions for m_1 and m_2; show that of necessity they are real. Find a similar solution for y.

The values of the parameters for the two species are estimated as $a = d = 2$ and $b = c = 1$. Find the particular solutions valid for the initial conditions $x = 100, y = 200$ at $t = 0$. Determine the time until one species is eliminated.

(Note that in this special case, the only difference between the species is that one starts out with more members than the other.)

Answer. The roots are $(a + d) \pm \Delta$ where $\Delta = (a + d)^2 - 4(ad - bc)$ is the discriminant of the auxiliary quadratic. The roots are real (and unequal) because the discriminant of the quadratic is positive:

$$\Delta = (a+d)^{-}4(ad - bc) = (a - d)^2 + 4bc \geq 4bc > 0.$$

The particular solution asked for is

$$x(t) = 150e^t - 50e^{3t}, \qquad y(t) = 150e^t + 50e^{3t}.$$

The first species is eliminated at $t = \frac{1}{2}\ln 3$.

2. Two species cooperate to maintain each other against their natural death rates according to the equations

$$\dot{x} = -ax + by$$
$$\dot{y} = cx - dy\,.$$

Proceeding as in the last example, show that unless $bc \geq ad$, both populations tend to extinction. If $a = 2, b = 4, c = 4$ and $d = 2$, determine the particular solutions for x and y if initially $x = 100, y = 200$. Deduce that as $t \to \infty$, the populations become equal.

Answer. By the argument as the previous question the discrminant of the auxiliary quadratic is positive, and so its roots are real.

The product of the roots is $ad - bc$. So if $bc < ad$ the product of the roots is positive. So they are of the same sign. The sum of the roots $-(a+d) < 0$. So when they are of the same sign they are both negative. The solutions to the system of equations in this case will be a linear combination of negative exponetials. So we will have both $x(t), y(t) \to 0$ as $t\infty$. For these populations once one population becomes extinct the model no longer applies, but then the reamining population will head to extiction even more quickly. Thsu unless $bc \geq ad$ both popualtions are headed for extiction.

Particular Solution

$$x(t) = 150e^{t} - 50e^{2t}, \qquad y(t) = 150e^{t} + 50e^{-6t}.$$

As $t \to \infty$ the term $\pm - 50e^{2t} \to 0$ and the term $150e^{t} \to \infty$. The model predicts for large t the populations are large both $\approx 150e^{t}$.

3. The path of a longjumper is very flat and the vertical velocity is never very large. Therefore the speed which gives the deceleration due to air resistance may be approximated by the horizontal component of the velocity,

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} - kv_x\mathbf{v}\,.$$

Solve these equations for the motion in the horizontal and vertical. Hence find the equation of the path.

Answer. Suppose the longjumper leaves the ground with intial velocity u at an angle α to the horizontal. Let x be horizontal displacement and y be vertical displacement.

Resolving horizontal and vertical components gives

$$\frac{d^2x}{dt^2} = -k\left(\frac{dx}{dt}\right)^2, \qquad \frac{d^2y}{dt^2} = -g - k\frac{dx}{dt}\frac{dy}{dt}.$$

This pair of equations is partially decoupled.

The first leads to

$$\frac{dx}{dt} = \frac{u\cos\alpha}{1+ku(\cos\alpha)t}, \quad x = \frac{\ln(1+ku(\cos\alpha)t)}{k}.$$

Substuting in the second leads to

$$\frac{dy}{dt} = -\frac{g}{2ku\cos\alpha}\left[1+\left(ku\cos\alpha\right)t\right] + \frac{g+ku^2\sin 2\alpha}{2ku\cos\alpha}\frac{1}{1+ku(\cos\alpha)t},$$

$$y = -\frac{g}{(2ku\cos\alpha)^2}\left[(1+ku\cos\alpha)t)^2-1\right] + \frac{g+ku^2\sin 2\alpha}{2(ku\cos\alpha)^2}\ln(1+ku(\cos\alpha)t).$$

We can eliminate t to describe the equation of the path in x, y coosrdinates.

$$y = \frac{g\left[1-e^{-2kx}\right]}{(2ku\cos\alpha)^2} + \frac{(g+ku^2\sin 2\alpha)x}{2k(u\cos\alpha)^2}.$$