The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 10

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
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Questions marked with * are more difficult questions.

Material covered

☐ linear first order equations.
☐ method of integrating factors for solving inhomogeneous linear first order equations.
☐ change of variables.

Outcomes

After completing this tutorial you should

☐ be able to solve linear first order differential equations using the method of integrating factors
☐ be able to do simple change of variables in differential equations

Summary of essential material

First order linear homogeneous equations. These are equations of the form

$$a(x)y'(x) + b(x)y(x) = 0.$$

The equation is called *linear* since the dependence on y and y' is linear and homogeneous since its right hand side is zero. The fact that it is linear means that any sum and scalar multiples of solutions are also solutions, a property often referred to as the *superposition principle*. If $a(x) \neq 0$ the equation can be written in *standard form*

$$y'(x) + p(x)y(x) = 0.$$

The equation is separable with solution

$$y(x) = Ae^{-\int p(x)\,dx} \quad A \text{ is a constant, } \int p(x)\,dx \text{ is some anti-derivative of } p\text{, no constant required.}$$

(You can just use this formula, or derive it by separation of variables.)

First order linear inhomogeneous equations. These are equations of the form

$$a(x)y'(x) + b(x)y(x) = f(x).$$

The equation is called *linear* since the dependence on y and y' is linear and inhomogeneous since its right hand side f is non-zero. Such equations are solved by division by $a(x)$ to bring them into *standard form*

$$y'(x) + p(x)y(x) = q(x)$$

and then multiplying with an *integrating factor*. An integrating factor is an (arbitrary) non-zero solution of the homogeneous $w'(x) - p(x)w(x) = 0$ (note the changed sign!), that is,

$$w(x) = e^{\int p(x)\,dx}.$$

Then

$$(yw)' = qw \qquad \text{and} \qquad y = \frac{1}{w}\int qw\,dx,$$

where the last integral involves an integration constant. When initial conditions are given it is convenient (but not necessary) to take a definite integral:

$$y(x) = y(x_0)\frac{w(x_0)}{w(x)}\int_{x_0}^{x} q(\xi)w(\xi)\,d\xi.$$

The solution exists as long as $a(x) \neq 0$ and $w(x) \neq 0$.

Questions to do before the tutorial

1. Find the general solution to the following differential equations.

(a) $\dfrac{dy}{dx} - y\tan x = x$

(b) $\dfrac{dx}{dt} + 2tx = 2t^3$

Questions to complete during the tutorial

2. Which of the following differential equations are linear? Can any of the nonlinear cases be transformed into a linear differential equation by a simple change of variables?

(a) $(x-1)^3\dfrac{dy}{dx} + 4(x-1)^2 y = x+1$

(b) $(x-y)\dfrac{dy}{dx} + y = e^x$

3. Solve the following differential equations.

(a) $dx - (\sec y + 2x\tan y)dy = 0$

(b) $\dfrac{dy}{dx} = \dfrac{2y}{y - x - y^3}$

(c) $(1+x)\dfrac{dy}{dx} + y = 3x^2$, given $y(0) = 2$.

(d) $2dx + (2x+3y)dy = 0$, given $y(2) = 0$.

4. Consider a logistic equation $\dfrac{dP}{dt} = a(t)P - b(t)P^2$, where P models the size of a population and $a, b\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. We assume that a and b are T-periodic, that is, $a(t+T) = a(t)$ for all $t \in \mathbb{R}$ and likewise for b. We further assume that $b(t) > 0$ for all $t \in \mathbb{R}$.

The periodicity assumption is natural if we consider seasonal changes in the environment.

(a) Show that $v = 1/P$ satisfies the linear differential equation $v' + a(t)v = b(t)$.

(b) Hence, deduce that if $P(0) = P_0$, then

$$P(t) = \frac{P_0 \exp\left(\int_0^t a(\tau)\,d\tau\right)}{1 + P_0 \displaystyle\int_0^t \exp\left(\int_0^s a(\tau)\,d\tau\right) b(s)\,ds}$$

(c) A solution is T-periodic if $P_0 = P(0) = P(T)$. Find the initial condition of the T-periodic solution. Show that the periodic solution is positive if and only if $\int_0^T a(\tau)\,d\tau > 0$.

*(d) If $\int_0^T a(\tau)\,d\tau < 0$, then the differential equation has no positive periodic solution. Show that $P(t) \to 0$ for all positive initial values $P_0 > 0$. What does this mean for the population?

5. A tank initially contains 700 litres of fresh water. A pipe is opened which admits salty water at a rate of 10 litres/min. At the same time, a drain is opened to allow 8 litres/min of the mixture to leave the tank. If the inflowing salty water contains 0.01 kg of salt per litre, what is the mass of salt in the tank after 60 minutes? What is the concentration of the salt?

Extra questions for further practice

6. Some rocks contain a radioactive isotope of radium, Ra^{226}, which has a half-life of 1590 years and decays into an isotope of lead, Pb^{210}. This lead isotope is itself radioactive, and decays with a half-life of 22 years. Let $R(t)$ be the amount of radium in the rock and $L(t)$ be the amount of lead. Then the rate of change of L is the rate at which lead is produced by the decay of radium, minus the rate at which the lead decays; so $dL/dt = \lambda R - \mu L$ where λ and μ are the decay constants of radium and lead respectively. Given that $R = R_0 e^{-\lambda t}$ and that $L = 0$ at $t = 0$, solve this equation to show that

$$L(t) = \frac{\lambda R_0}{\mu - \lambda}(e^{-\lambda t} - e^{-\mu t}).$$

What are the values of λ and μ?

7. Obtain first-order differential equations that govern the following one-parameter families of curves (this means $y(x)$ is a solution to some first order differential equation):

(a) $y = Cx^4$;

(b) $\dfrac{x^2}{C} + \dfrac{y^2}{C-1} = 1.$

8. Find the general solutions of the following differential equations.

(a) $\dfrac{dx}{dt} - tx = t$

(b) $\dfrac{dy}{dx} = \dfrac{4x^3 - y}{x}$

(c) $\dfrac{dy}{dx} + 2y = e^{-x}$

(d) $x^2\dfrac{dy}{dx} + (1-2x)y = x^2$

9. Find the particular solutions of the following differential equations under the given conditions.

(a) $\dfrac{dy}{dx} + y\tan x = \sec x, \quad y = 2$ when $x = 0$

(b) $\dfrac{dy}{dx} = \dfrac{2y}{x} + x^4, \quad y = 1$ when $x = 1$

(c) $\dfrac{dx}{dt} + 4x = e^{-4t}\sin 2t, \quad x(0) = 1/2$

(d) $(1+x^2)\dfrac{dy}{dx} + 2xy = 4 + 2x, \quad y(0) = 4$

10. The Howard family borrows \$176 000 to buy a house, and plans to make frequent regular repayments of increasing amounts so that the rate of repayment t years after the start of the loan will be $\$R(t)$ per year, where $R(t) = R_0\big(1 + t^2/80\big)$ and R_0 is the initial repayment rate. The interest rate is fixed at 5% per annum, and interest charges are added to the loan amount at frequent regular intervals.

(a) Assuming repayments and interest charges are so frequent that they are effectively continuous, show that the loan amount L varies with time according to the differential equation,

$$\frac{dL}{dt} = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

(b) Solve this equation, and hence obtain an expression for the amount still owed after t years.

(c) Show that the initial repayment rate R_0 must exceed \$800/year or else the debt will eventually grow out of control.

(d) If $R_0 = \$1000$/year, what is the remaining debt after 20 years?

11. Which of the following differential equations are linear? Can any of the nonlinear cases be transformed into a linear differential equation by a simple change of variables? Try to solve the equation.

(a) $\dfrac{dy}{dx} + \dfrac{3y}{x} = \sin x$

(b) $\dfrac{dy}{dx} = \dfrac{y^2+1}{2xy+1}$