1. Given a set $X$ and a subset $A \subseteq X$, define a function $f_A : X \to \{0, 1\}$ by setting $f_A(x) = 1$ if $x \in A$ and $f_A(x) = 0$ if $x \notin A$. If $B$ is also a subset of $X$, define $(f_A f_B)(x) = f_A(x) f_B(x)$.

(a) Work out $f_A f_B$ when $X = \{1, 2, 3, 4, 5\}$, $A = \{2, 4, 5\}$ and $B = \{1, 2, 5\}$.

Solution: $f_A f_B$ takes that value 1 on 2 and 5 and is 0 elsewhere.

(b) What subset, if any, does $f_A f_B$ correspond to?

Solution: $f_A f_B$ corresponds to $A \cap B$.

(c) Define $f_A'$ by $f_A'(x) = 1 - f_A(x)$. What subset, if any, does $f_A'$ correspond to?

Solution: $f_A'$ corresponds to $X \setminus A$.

(d) Form combinations of $f_A$ and $f_B$ which represent

(i) the union of $A$ and $B$;

Solution: We have $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) f_B(x)$.

(ii) the symmetric difference $A + B = (A \setminus B) \cup (B \setminus A)$ of $A$ and $B$.

Solution: Define $(f_A + f_B)(x) = f_A(x) + f_B(x) - 2 f_A(x) f_B(x)$.
Alternatively, define $(f_A + f_B)(x) = f_A(x) + f_B(x)$, where addition is taken modulo 2.

2. For the following sets $X$, $Y$ and function $X \to Y$, determine whether the function is injective or surjective.

(a) $X = \mathbb{N}$, $Y = \mathbb{N}$, $f(x) = x + 1$.

Solution: If $x, y \in \mathbb{N}$ and $x + 1 = f(x) = f(y) = y + 1$, then $x = y$. Therefore $f$ is injective. Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$. The function is not surjective because $x + 1 = 0$ has no solution with $x \in \mathbb{N}$.

(b) $X = \mathbb{Z}$, $Y = \mathbb{Z}$, $g(x) = x + 1$.

Solution: The function $g : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x + 1$ is injective for the same reason as in part (a). It is surjective because if $x \in \mathbb{Z}$, $x = (x - 1) + 1 = g(x - 1)$.

(c) $X = \mathbb{Z}$, $Y = \mathbb{Z}$, $h(x) = x^2 + 5$.

Solution: We show that the function $h : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x^2 + 5$ is neither injective nor surjective. We have $f(1) = 1^2 + 5 = 6 = (-1)^2 + 5 = f(-1)$. Furthermore, $f(x) = x^2 + 5 = 4$ is insoluble because the square of an integer is never negative.

(d) $X = \mathbb{Z}$, $Y = \mathbb{Z}$, $p(x) = x^3 + 1$.

Solution: We show that the function $p : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x^3 + 1$ is injective but not surjective. First we prove that $p$ is injective by establishing the
stronger result that \( p \) is increasing, that is, if \( x, y \in \mathbb{Z} \) and \( x < y \), then \( p(x) < p(y) \). This follows from the fact the function of real variable \( f(x) = x^3 + 1 \) is increasing. Finally we show that \( p \) is not surjective. If \( n \geq 2 \), \( p(n) \geq p(2) = 9 \). On the other hand if \( n \leq 1 \), \( p(n) \leq p(1) = 2 \). Therefore \( p(n) \neq 3 \) for any integer \( n \).

(e) \( X \) is a nonempty set, \( Y = \mathcal{P}(X) \) (the set of all subsets of \( X \)), \( h(x) = \{x\} \).

Solution: The function \( h : X \to \mathcal{P}(X) \), \( x \mapsto \{x\} \) is injective because if \( \{x\} = h(x) = h(y) = \{y\} \), then \( x = y \). However \( h \) is not surjective because the empty set \( \emptyset \in \mathcal{P}(X) \) does not satisfy \( h(x) = \{x\} = \emptyset \) for any \( x \in X \).

*3.* Given finite sets \( A \) and \( B \), let \( E \) be a subset of \( A \times B \). For \( a \in A \), let

\[ E(a) = \{ b \in B \mid (a, b) \in E \} \]

and for \( b \in B \), let

\[ E^\vee(b) = \{ a \in A \mid (a, b) \in E \} \].

Prove that

\[ \sum_{a \in A} |E(a)| = \sum_{b \in B} |E^\vee(b)|. \]

Solution: Find the cardinality \(|E|\). For each \( a \in A \) there are \(|E(a)|\) pairs \((a, y) \in E\) and so \(|E| = \sum_{a \in A} |E(a)|\). Similarly, for each \( b \in B \) there are \(|E^\vee(b)|\) pairs \((x, b) \in E\) and therefore \(|E| = \sum_{b \in B} |E^\vee(b)|\). Equating these two expressions for \(|E|\) gives the result.

4. If there are 40 students in the class, their surnames can't all start with different letters (because there are only 26 letters). What can be said if there are 80 students in the class?

Solution: The Pigeonhole Principle says that there must be \( \lceil \frac{80}{26} \rceil = 4 \) students whose surnames begin with the same letter. This doesn't mean that there is necessarily a letter which 'has' exactly 4 students, but there must be a letter which has at least 4. Recall the reasoning: if each of the 26 letters were to have at most 3 students, then there could be at most \( 26 \times 3 = 78 \) students.

5. How many six-digit numbers are there (not starting with 0)? How many of these have six different digits? Explain your answers.

Solution: The six-digit numbers are all the numbers greater than or equal to 100000 and less than 1000000, so there are 900000 of them. Another way to count them is to note that there are 9 possibilities for the first digit, and then choosing the other digits is an ordered selection of 5 from 10 possibilities with repetition allowed, so the total is \( 9 \times 10^5 \). The number of six-digit numbers which have no repeated digits is \( 9 \times 9(5) = 9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136080 \), because there are 9 choices for the first digit, and then choosing the other digits is an ordered selection of 5 from 9 possibilities with repetition not allowed.

6. A permutation of \( \{1, 2, 3, 4, 5\} \) can be thought of as a number with these five digits in some order, such as 21453 or 41352. There are \( 5! = 120 \) such numbers.
(a) How many of these numbers start with 5 and end with 2?

**Solution:** To specify such a number we just need to specify the order of 1, 3, 4 in the middle three digits; there are $3! = 6$ possible orders.

(b) How many have a first digit which is larger than the second digit?

**Solution:** The smart answer is that the first digit is just as likely to be larger than the second digit as it is to be smaller (and they can’t be equal), so exactly half of the 120 numbers have the required property. Another way to get the answer 60 is to note that there are ten possibilities for the first two digits (21, 31, 32, 41, 42, 43, 51, 52, 53, or 54), and for each of these there are six possibilities for the last three digits, as in the previous part.

(c) How many have the property that the first digit is larger than the second, which is smaller than the third, which is larger than the fourth, which is smaller than the fifth? *(Hint: what positions can the digit 1 occupy in such a number?)*

**Solution:** Clearly the digit 1 must be in either second place or fourth place, because otherwise it would have to be larger than some other digit. Because the condition is symmetric, the numbers satisfying the condition where 1 is the fourth digit are just the reversals of the numbers satisfying the condition where 1 is the second digit. So we only need to consider one of these cases: say that 1 is the second digit. Then the requirements that the first digit is larger than the second, and the second smaller than the third, are automatically satisfied. There are four possible choices for the first digit. Whatever first digit is chosen, there are three out of {2, 3, 4, 5} remaining to fill the last three digits, and the requirements mean that we have to choose the smallest of these three as the fourth digit; the other two can be the third and fifth digits in either order. So there are $4 \times 2 = 8$ numbers satisfying the condition where 1 is the second digit, and another 8 where 1 is the fourth digit, so $8 + 8 = 16$ is the answer to the question.

*7. Use the Pigeonhole Principle to prove that for any positive integer $n$, there is a multiple of $n$ which has no digits other than 0’s and 1’s (in its usual decimal expression).*

**Solution:** Consider the set of numbers $X = \{1, 11, 111, 1111, \ldots, 111\cdots1\}$, where the last number is a string of $n$ ones. If any of the elements of $X$ is a multiple of $n$, then we are done, so suppose none of them is a multiple of $n$. We then have a function from $X$ to the set $\{1, 2, \cdots, n - 1\}$, given by taking the remainder after division by $n$. By the Pigeonhole Principle, this function cannot be injective, since $|X| > |\{1, 2, \cdots, n - 1\}|$. So there must be two elements of $X$ which have the same remainder after division by $n$. Subtracting the smaller from the larger gives a multiple of $n$ which has the form $11\cdots100\cdots0$, so we are done.