More difficult questions are marked with either * or **. Those marked * are at the level which MATH2069 students will have to solve in order to be sure of getting a Credit, or to have a chance of a Distinction or High Distinction. Those marked ** are mainly intended for MATH2969 students. Some numerical answers are at the end of the sheet.

1. A factory makes jelly beans of 15 different flavours, which it sells in bags of 10. If the jelly beans in a bag must all have different flavours, how many possible kinds of bag are there? What about if this restriction is removed?

2. If there are 10 chairs in a row and 8 students who want to sit down, you would normally say there are $10^8$ possible outcomes (a case of ordered selection with repetition not allowed). What unusual circumstances would make you change this answer to $10^8$? How about $\binom{10}{8}$ or $\binom{17}{8}$?

3. Suppose that $A$ and $B$ are subsets of some set $X$, and $|A| = 140$, $|B| = 92$.
   (a) Find $|A \cup B|$, given that $|A \cap B| = 36$.
   (b) Find $|A \cap B|$, given that $|A \cup B| = 150$.
   *(c) Prove that it is impossible to have another subset $C$ of $X$ such that $|C| = 58$, $|A \cap B| = 32$, $|A \cap B \cap C| = 10$, $|A \cup B \cup C| = 250$.

4. If you are dealt 5 cards from a standard deck of 52 (4 suits, each containing $A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2$), the number of possible hands is $\binom{52}{5} = 2598960$.
   (a) How many of these hands contain the ace of spades?
   (b) How many of the hands contain exactly two aces?
   (c) How many of the hands contain at least two aces?
   (d) How many of the hands are a full house (three of one value and two of another, with no condition on the suits)?
   (e) How many of the hands are a flush (all of the same suit), but not a straight flush (all of the same suit and five consecutive values, counting $J = 11$, $Q = 12$, $K = 13$, $A = 14$)?
   *(f) By replacing “at least two” with “at least zero” in part (c), show the equality
   $$\binom{52}{5} = \binom{4}{0}\binom{48}{5} + \binom{4}{1}\binom{48}{4} + \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}.$$ 
   Of what general identity is this a special case?
5. The 4 players in a bridge game (North, South, East, and West) are each dealt 13 cards from the same deck of 52.

(a) How many possible deals are there (assuming that it matters who gets which cards)?

*(b) In how many of these deals do North and South end up with all the spades between them?

6. In this question you should use the Stirling numbers $S(5, 2) = 15$, $S(5, 3) = 25$, $S(5, 4) = 10$ computed in lectures.

(a) Count the surjective functions $\{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4\}$.

(b) Count the ways of assigning five students to five tutors so that exactly one of the tutors is not assigned any students.

(c) Count the ways of assigning five students to four tutors so that at most two of the tutors are assigned students.

(d) Write $n^5$ as a linear combination of binomial coefficients $\binom{n}{k}$.

*7. The **Bell number** $B(n)$ is defined to be the total number of partitions of the set $\{1, 2, \cdots, n\}$ (or any set with $n$ elements). Thus $B(n) = \sum_{k=0}^{n} S(n, k)$. Prove the following recurrence relation for the Bell numbers:

$$B(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} B(n-i), \text{ for all } n \geq 1.$$  

*(Hint: say that the first step in constructing a partition of $\{1, 2, \cdots, n\}$ is to choose the block containing $n$.)

*8. Suppose that $m, k \in \mathbb{N}$ with $m \geq k$. Prove that the number of surjective functions $f : \{1, 2, \cdots, m\} \rightarrow \{1, 2, \cdots, k\}$ equals

$$\sum_{m_1, m_2, \cdots, m_k \geq 1 \atop m_1 + m_2 + \cdots + m_k = m} \binom{m}{m_1, m_2, \cdots, m_k}.$$  

How many terms are there in this sum?

**9.** Prove the formula

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

by induction on $n$, using the recurrence relation for the Stirling numbers.

**Selected numerical answers:**
1. 3003, 1961256. 3. 196, 82. 4. 249900, 103776, 108336, 3744, 5112.