THE UNIVERSITY OF SYDNEY
MATH2921

Semester 1 | Vector Calculus and Differential Equations (Advanced) | 2019

Tutorial VC-R (Vector Calculus Review): Try solve all the problems listed below. Feel free to use your favourite computer software package to help you solve the problems and get a handle on what is going on, but it is also a good idea for many of the problems to work out how you would solve the problem without a computer (as in an exam environment).

1. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in $\mathbb{R}^3$

   (a) Show that

   $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \quad \text{and} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

   (b) Prove the *Jacobi identity*

   $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0$$

2. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be two nonzero, linearly independent vectors in $\mathbb{R}^3$ i.e $\mathbf{a} \times \mathbf{b} \neq 0$.

   (a) Show that the set $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ forms a basis of $\mathbb{R}^3$.

   (b) Suppose further that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (i.e $\mathbf{b} \cdot \mathbf{a} = 0$). Show that the system of equations

   $$\mathbf{x} \times \mathbf{a} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \cdot \mathbf{a} = ||\mathbf{a}||$$

   has a unique solution $\mathbf{x} \in \mathbb{R}^3$.

3. Find the equation of the plane that contains the line

   $$(x, y, z) = (-1, 1, 2) + t(3, 2, 4)$$

   and is perpendicular to the plane $2x + y - 3z = -4$.

4. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *radial* if $f(\mathbf{x})$ depends only on $||x||$. That is $f(\mathbf{x}) = g(||\mathbf{x}||)$ where $g(r)$ is some function $\mathbb{R}^+ \to \mathbb{R}$. Assuming that g is differentiable, show that if f is radial, then $\nabla f(\mathbf{x})$ is a scalar multiple of $\mathbf{x}$.

5. Suppose you know a curve in $\mathbb{R}^2$ in polar coordinates $r(\theta) = f(\theta)$. What is the arc length formula for this curve in terms of θ?

6. Consider the surface which is the graph of the function

   $$z = f(x, y) = x^2 - y^2$$

   (a) Express the surface $z = f(x, y) = x^2 - y^2$ in cylindrical coordinates

   (b) Express the surface $z = f(x, y) = x^2 - y^2$ in spherical coordinates.

7. Compute f_x, f_y and f_z for the following functions and evaluate them at the indicated points

   (a) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}; \quad (x, y, z) = (3, 0, 4)$

   (b) $f(x, y, z) = \sin(xy^2z^3); \quad (x, y, z) = (\pi, 1, 1)$

8. A function $u = f(x, y)$ with continuous second partial derivatives satisfying *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called a *harmonic function.* Which of the following functions are harmonic

(a) $f(x, y) = x^3 - 3xy^2$

(b) $f(x, y) = x^2 + y^2$

(c) $f(x, y) = \sin x \cosh y$