The Field of Moduli of a Polarized Abelian Variety

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1 Preliminaries

Let A be an abelian variety. Let $\mathrm{Div}(A)$ denote the group of divisors on A and $\mathrm{Div}^0(A)$ the subgroup of divisors algebraically equivalent to zero. One has

$$\mathrm{Div}^0(A) = \{X \in \mathrm{Div}(A) | \forall x : t_x^* X \sim_{lin} X\}$$

Let

$$\mathrm{Prin}(A) = \{\mathrm{div}(f) | f \in k(A)\}$$

be the set of divisors *linearly equivalent to zero*. We set

$$\mathrm{Pic}^0(A) = \mathrm{Div}^0(A)/\mathrm{Prin}(A).$$

Over an algebraically closed field, the group $\mathrm{Pic}^0(A)$ can be given the structure of an abelian variety. It is called the *dual abelian variety* associated to A. Let X be a divisor on A. Define

$$\varphi_X : A \to \mathrm{Pic}^0(A)$$

by

$$x \mapsto [t_x^* X - X] .$$

Definition 1.1 *The divisor X is called non-degnerate if φ_X is an isogeny.*

Proposition 1.2 *An effective divisor X is ample if and only if X is non-degenerate.*

Lemma 1.3 *Two divisors X and Y are algebraically equivalent if and only if $\varphi_X = \varphi_Y$.*

Proof. Using the above definitions,

$$\begin{aligned}
\varphi_X = \varphi_Y \quad &\Leftrightarrow \quad \varphi_X(x) = \varphi_Y(x) \ \forall x \in A \\
&\Leftrightarrow \quad [t_x^* X - X] = [t_x^* Y - Y] \\
&\Leftrightarrow \quad [t_x^* X - t_x^* Y - X + Y] = [t_x^*(X - Y) - (X - Y)] = [0] \\
&\Leftrightarrow \quad t_x^*(X - Y) \sim_{lin} (X - Y) \\
&\Leftrightarrow \quad X - Y \sim_{alg} 0 \\
&\Leftrightarrow \quad X \sim_{alg} Y .
\end{aligned}$$

□

Definition 1.4 *A polarized abelian variety is a pair $(A, \mathcal{C})$ where A is an abelian variety and there exists an ample divisor X such that*

$$\mathcal{C} = \mathcal{C}(X) = \{Y \in \mathrm{Div}(A) | \exists m, n > 0 : mX \sim_{alg} nY\}.$$

Let $(A_1, \mathcal{C}_1)$ and $(A_2, \mathcal{C}_2)$ be two polarized abelian varieties of the same dimension. A morphism $\lambda : A_1 \to A_2$ is said to be a morphism of polarized abelian varieties $\lambda : (A_1, \mathcal{C}_1) \to (A_2, \mathcal{C}_2)$ if there exists an $X_2 \in \mathcal{C}_2$ such that $\lambda^{-1}(X_2) \in \mathcal{C}_1$.

Proposition 1.5 *Suppose $(A, \mathcal{C})$ is a polarized abelian variety. There exists a $Y \in \mathcal{C}$ such that every divisor in $\mathcal{C}$ is algebraically equivalent to mY for some $m > 0$.*

Such a Y is called a *basic polar divisor.*

Proof. For each $X \in \mathcal{C}$ consider $\varphi_X : A \to \mathrm{Pic}^0(A)$ as above. By Proposition 1.2 every $X \in \mathcal{C}$ is nondegenerate which means that φ_X is an isogeny, so $\deg(\varphi_X)$ is a positive integer. Choose a divisor $Y \in \mathcal{C}$ such that $\deg(\varphi_X)$ is minimal. We proceed to show that Y is a basic polar divisor. By definition of $\mathcal{C}$, there exist integers $a, b > 0$ such that $aX \sim_{alg} bY$. By Lemma 1.3 we have $\varphi_{aX} = \varphi_{bY}$. Note that $\varphi_{aX} = [a] \circ \varphi_X$ and $\varphi_{X+Y} = \varphi_X + \varphi_Y$. Applying the division algorithm, there exist integers q, r such that $b = aq + r$ with $0 \leq r < a$. Suppose (by way of contradiction) that $r > 0$. Then, putting $Z = X - qY$ we have

$$[a] \circ \varphi_Z = \varphi_{aZ} = \varphi_{aX-aqY} = \varphi_{bY-aqY} = \varphi_{rY} = [r] \circ \varphi Y\ .$$

By Lemma 1.3, $aZ \sim_{alg} rY$ and so $Z \in \mathcal{C}$. Taking degrees of the equation $a\varphi_Z = r\varphi_Y$ where $0 < r < a$, shows that $\deg(\varphi_Z) < \deg(\varphi_Y)$. This contradicts the definition of Y. This proves that X is algebraically equivalent to qY. □

2 Field of Moduli

Let A be an abelian variety over the complex numbers and $\mathcal{C}$ a polarization of A.

Definition 2.1 *We say that $\sigma : k \hookrightarrow \mathbb{C}$ is a field of definition for A if there exists an abelian variety A' over k such that $A' \otimes_\sigma \mathbb{C} \cong A$. The pair (A', σ) is called a k-model. Similarly, we say that $\sigma : k \hookrightarrow \mathbb{C}$ is a field of definition for $(A, \mathcal{C})$ if there exists a polarized abelian variety $(A', \mathcal{C}')$ defined over k such that $(A', \mathcal{C}') \otimes_\sigma \mathbb{C} \cong (A, \mathcal{C})$.*

Proposition 2.2 *Let A be an abelian variety which can be defined over k. Then every polarization $\mathcal{C}$ of A can be defined over a finite algebraic extension of k.*

Proposition 2.3 *Let A be an abelian variety over $\mathbb{C}$. If A can be defined over k, (ie. there exists an abelian variety A' over k such that $A' \otimes_\sigma \mathbb{C} \cong A$) and A is polarizable, then there exists a polarization on A'*

Proposition 2.4 *Every abelian variety over an algebraically closed field is polarizable.*

Corollary 2.5 *Every abelian variety is polarizable.*

Proof. Follows from Propositions 2.2, 2.3 and 2.4. □

Theorem 2.6 *Let $(A, \mathcal{C})$ be a polarized abelian variety over k. There exists a unique field $k_0 \hookrightarrow k$ having the property that for all $\sigma \in \mathrm{End}(k)$ we have*

$$(A^\sigma, \mathcal{C}^\sigma) \cong (A, \mathcal{C}) \Leftrightarrow \sigma|_{k_0} = \mathrm{id}$$

where $(A^\sigma, \mathcal{C}^\sigma) = (A, \mathcal{C}) \otimes_\sigma \mathbb{C}$

The field k_0 is called the *field of moduli* of $(A, \mathcal{C})$.

Proof. Proof of Theorem 2.6 We restrict to the case where $(A, \mathcal{C})$ can be defined over a number field. A general proof can be found in Shimura. By Proposition 2.2, $(A, \mathcal{C})$ can be defined over a finite algebraic extension k' of k. Take a Galois extension k'' of $\mathbb{Q}$ containing k' and write $G = \mathrm{Gal}(k''/\mathbb{Q})$. Let

$$H = \{\sigma \in G | (A^\sigma, \mathcal{C}^\sigma) \cong (A, \mathcal{C})\} .$$

Then the field of moduli k_0 is given by the fixed field

$$(k'')^H = \{x \in k'' | \sigma(x) = x \;\; \forall \sigma \in H\} .$$

□

3 Example

In the following we will work on the following example. Let E be the complex elliptic curve given by

$$y^2 = f(x)$$

where $f(x) \in \mathbb{C}[x]$ is of degree 3 and the discriminant of f is non-zero. The field of moduli of E will turn out to be the 'minimal' field of definition of E. Let j_E (or $j(E)$) denote the j-invariant of E. Consider the divisor $X = (0_E) \in \mathrm{Div}(E)$ where 0_E is the identity element of the elliptic curve. The divisor X is ample since $\varphi_X(P) = [(P) - (0_E)]$ is non-zero for any point $P \neq 0_E$. The divisor X defines a polarization on E. We claim that $k_0 = \mathbb{Q}(j_E)$ is the field of moduli of $(E, \mathcal{C})$. Clearly E can be defined over k if and only if $(E, \mathcal{C})$ can be defined over k. In other words k is a field of definition for E if and only if k is a field of definition for $(E, \mathcal{C})$. Every isomorphism $\lambda : E_1 \to E_2$ induces an isomorphism of polarized abelian varieties $\lambda : (E_1, (0_{E_1})) \to (E_2, (0_{E_1}))$. We have $(E, \mathcal{C}) \cong (E^\sigma, \mathcal{C}^\sigma)$ being equivalent to $E \cong E^\sigma$ over k, which is equivalent to $j(E) = j(E^\sigma) = \sigma(j(E))$. This means that j_E is fixed by σ, and so $\mathbb{Q}(j_E)$ is fixed by σ. Hence $\mathbb{Q}(j_E)$ with the induced embedding $\mathbb{Q}(j_E) \hookrightarrow \mathbb{C}$ is the field of moduli for the pair $(E, \mathcal{C})$.