Gaussian heat kernel estimates: from functions to forms

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Geometry and Analysis meets PDE, WOMASY, 1 October 2014
The setting

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Let $M$ be a connected, complete, non-compact Riemannian manifold $d$ the geodesic distance, $\mu$ the Riemannian measure $\nabla$ the gradient, $\Delta$ the (nonnegative) Laplace-Beltrami operator $E(f) := \langle \Delta f, f \rangle$ the Dirichlet form $(e^{-t\Delta})_t > 0$ is a Markov semigroup, acts on $L^p(M,\mu)$, $1 \leq p \leq +\infty$ $e^{-t\Delta}$ has a smooth kernel $p_t(x,y) = p_t(y,x) > 0$:

$$e^{-t\Delta}f(x) = \int_M p_t(x,y)f(y)d\mu(y), f \in L^2(M,\mu), \forall x \in M$$

In order to do analysis on $M$, one would like to estimate $p_t(x,y)$ from above and below No curvature assumptions, rather direct geometric properties of $M$
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No curvature assumptions, rather direct geometric properties of $M$
Uniform bounds of the heat kernel: the polynomial case

Assume \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M,\mu) (L^\infty(M,\mu))\)

\[
\sup_{x,y \in M} p_t(x,y) \leq C t^{-D/2}, \quad \forall \ t > 0, \ x \in M, \ \text{some} \ D > 0,
\]

is equivalent to:

- the Sobolev inequality:
  \[
  \|f\|_{\alpha D/(D - \alpha p)} \leq C \|L^{\alpha/2} f\|_p, \quad \forall f \in D_p(L^{\alpha/2})
  \]
  for \(p > 1\) and \(0 < \alpha p < D\) [Varopoulos 1984, C. 1990].

- the Nash inequality:
  \[
  \|f\|_2^2 + \left(\frac{4}{D}\right)^2 \leq C \|f\|^4/D E(f), \quad \forall f \in F.
  \]
  [Carlen-Kusuoka-Stroock 1987]

- the Gagliardo-Nirenberg type inequalities, for instance
  \[
  \|f\|_2^2 \leq C \|f\|_2^{2-q} \|L^2 E(f)\|_q^2, \quad \forall f \in F,
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  for \(q > 2\) such that \(q^2 - 2q - 2q D^2 < 1\) [C. 1992].
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Extrapolation

In the Sobolev and in the Gagliardo-Nirenberg case (not in the Nash case), one needs:

Lemma (C., 1990)

Assume \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\) and there exist \(1 \leq p < q \leq +\infty, \alpha > 0\) such that

\[
\|e^{-tL}\|_{p \to q} \leq Ct^{-\alpha}, \quad \forall \ t > 0.
\]

Then

\[
\|e^{-tL}\|_{1 \to \infty} \leq Ct^{-\beta}, \quad \forall \ t > 0,
\]

where \(\beta = \frac{\alpha}{\frac{1}{p} - \frac{1}{q}}\).
Real life heat kernel estimates are not uniform!

To do analysis on \((M, \mu)\), one needs estimates of \(p_t(x, x)\) and even of \(p_t(x, y)\): \(\sup_{x, y \in M} p_t(x, y)\) is not enough.

Indeed, for instance on manifolds with non-negative Ricci curvature, \(p_t(x, x) \simeq \frac{1}{V(x, \sqrt{t})}\), where \(V(x, r) = \mu(B(x, r))\), and \(V(x, r)\) does vary with \(r\). We have to assume doubling \((D)\):

\[
V(x, 2r) \leq CV(x, r), \quad \forall x \in M, r > 0 \quad (1)
\]

It follows easily that there exists \(\nu > 0\) such that

\[
V(x, r) \frac{1}{V(x, s)} \lesssim (\frac{r}{s})^\nu, \quad \forall x \in M, r \geq s > 0.
\]

\(\nu(D)\)

It is known that if \(M\) is connected, non-compact, and satisfies (1), then the following reverse doubling condition holds: there exist \(0 < \nu' \leq \nu\) such that, for all \(r \geq s > 0\) and \(x \in M\),

\[
(\frac{r}{s})^{\nu'} \lesssim V(x, r) \frac{1}{V(x, s)}.
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\frac{V(x, r)}{V(x, s)} \lesssim \left( \frac{r}{s} \right)^\nu, \quad \forall x \in M, \ r \geq s > 0.
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\((VD_\nu)\)
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The basic upper on-diagonal estimate
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Self-improves into

\[(UE) \quad p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{Ct}\right), \forall \ t > 0, \ x, y \in M\]
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\[ (UE) \quad p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{Ct} \right), \forall \, t > 0, \, x, y \in M \]

which implies the on-diagonal lower Gaussian estimate

\[ (DLE) \quad p_t(x, x) \geq \frac{C}{V(x, \sqrt{t})}, \forall \, x \in M, \, t > 0 \]
Full Gaussian lower estimate

\[(LE)\quad p_t(x, y) \geq \frac{c}{V(x, \sqrt{t})} \exp \left( -C \frac{d^2(x, y)}{t} \right), \quad \forall x, y \in M, \; t > 0\]
Heat kernel estimates under volume doubling

Full Gaussian lower estimate

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Gradient upper estimate

\[(G)\quad |\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})}, \quad \forall \ x, y \in M, \ t > 0\]
Heat kernel estimates under volume doubling 2

Full Gaussian lower estimate

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All this is true on manifolds with non-negative Ricci curvature.
Theorem

(DUE) ⇔ (UE) ⇒ (DLE) ⇏ (LE) ⇏ (G)

(G) ⇒ (LE) ⇒ (DUE)


Three levels:

- (UE)
- (UE) + (LE) = (LY) = parabolic Harnack
- (G)
Application: Riesz transform

**Theorem**

Let $M$ be a complete non-compact Riemannian manifold satisfying $(D)$ and $(DUE)$. Then

$$(R_p) \quad \| |\nabla f|\|_p \leq C \|\Delta^{1/2} f\|_p, \quad \forall f \in C_0^\infty(M),$$

for $1 < p < 2$.

[Coulhon, Duong, T.A.M.S. 1999]

**Theorem**

Let $M$ be a complete non-compact Riemannian manifold satisfying $(D)$ and $(G)$. Then the equivalence

$$(E_p) \quad \| |\nabla f|\|_p \simeq \|\Delta^{1/2} f\|_p, \quad \forall f \in C_0^\infty(M),$$

holds for $1 < p < \infty$.

Joint work with Salahaddine Boutayeb and Adam Sikora, 2013.
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$(M, d, \mu)$ a metric measure space satisfying the doubling volume property $(D)$
Pointwise heat kernel upper estimates revisited 1

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\((M, d, \mu)\) a metric measure space satisfying the doubling volume property \((D)\)

Dirichlet form \(\mathcal{E}(f, f)\)
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Markov semigroup $(e^{-t\Delta})_{t>0}$ on $L^2(M, \mu)$
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Markov semigroup $(e^{-t\Delta})_{t>0}$ on $L^2(M, \mu)$

$v : M \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$(D_v) \quad v(x, 2r) \leq C v(x, r), \forall r > 0, \mu - a.e. x \in M$$

and
Joint work with Salahaddine Boutayeb and Adam Sikora, 2013.

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$$(D_v) \quad v(x, 2r) \leq C v(x, r), \forall r > 0, \mu - \text{a.e. } x \in M$$

and

$$(D'_v) \quad v(y, r) \leq C v(x, r), \forall x, y \in M, r > 0, d(x, y) \leq r$$

$v$ may NOT be the volume function $V$; in fact $v \gtrsim V$, slow decays allowed
Pointwise heat kernel upper estimates revisited 2

\((\text{DUE}^{\nu})\): \((e^{-t\Delta})_{t>0}\) has a measurable kernel \(p_t\), that is

\[ e^{-t\Delta}f(x) = \int_M p_t(x, y)f(y)d\mu(y), \quad t > 0, \ f \in L^2(M, \mu), \ \mu \text{-a.e. } x \in M \]

and

\[ p_t(x, y) \leq \frac{C}{\sqrt{\nu(x, \sqrt{t})\nu(y, \sqrt{t})}}, \quad \text{for all } t > 0, \ \mu \text{-a.e. } x, y \in M. \]
Denote

\[ v_r(x) := v(x, r), \quad r > 0, \quad x \in M. \]
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Introduce
\[ \| f \|_2^2 \lesssim \| f v_r^{-1/2} \|_1^2 + r^2 \mathcal{E}(f), \quad \forall \, r > 0, \quad \forall f \in \mathcal{F}. \]

(equivalent to Nash if \( v(x, r) \simeq r^D \)) and

\[ \text{Theorem} \]
Assume that \((M, d, \mu, L)\) satisfies \((D)\) and Davies-Gaffney and that \(v\) satisfies \((D_v)\) and \((D'_v)\). Then \((\text{DUE}_v)\) is equivalent to \((N_v)\) and to \((\text{GN}_v^q)\) for \(q > 2\) small enough.
Denote
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(N^v) \quad \| f \|_2^2 \lesssim \| f v_r^{-1/2} \|_1^2 + r^2 \mathcal{E}(f), \quad \forall r > 0, \quad \forall f \in F.
\]
(equivalent to Nash if \( v(x, r) \simeq r^D \)) and for \( q > 2 \) (not too big)
\[
(GN^v_q) \quad \| f v_r^{1/2 - 1/q} \|_q^2 \lesssim \| f \|_2^2 + r^2 \mathcal{E}(f), \quad \forall r > 0, \quad \forall f \in F,
\]
(equivalent to Gagliardo-Nirenberg if \( v(x, r) \simeq r^D \))
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(equivalent to Gagliardo-Nirenberg if \( v(x, r) \simeq r^D \))

**Theorem**

Assume that \((M, d, \mu, L)\) satisfies \((D)\) and Davies-Gaffney and that \( v \) satisfies \((D^v)\) and \((D^v_\lambda)\). Then \((DUE^v)\) is equivalent to \((N^v)\) and to \((GN^v_q)\) for \( q > 2 \) small enough.
Idea of the proof

Introduce weighted $L^p - L^q$ inequalities: $1 \leq p \leq q \leq +\infty$, $\gamma, \delta$ real numbers such that $\gamma + \delta = \frac{1}{p} - \frac{1}{q}$

\[ \sup_{t > 0} \| \nabla^\gamma e^{-t\Delta} \nabla^\delta \|_{p\to q} < +\infty. \]  

$(vEv_{p,q,\gamma})$
Idea of the proof

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$$\sup_{t>0} \| v^{\gamma} e^{-t\Delta} v^{\delta} \|_{p \to q} < +\infty. \quad (vEv_{p,q,\gamma})$$

$$(DUE^v) = v^{1/2}(x)p_t(x,y)v^{1/2}(y) \leq C \text{ is equivalent to } (vEv)_{1,\infty,1/2} \text{ or}$$

$$(vE_{2,\infty}) \sup_{t>0} \| v^{1/2} e^{-t\Delta} \|_{2 \to \infty} < +\infty$$
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$$\sup_{t > 0} \| v^{\gamma} e^{-t\Delta} v^{\delta} \|_{p \rightarrow q} < +\infty. \quad (vEv_{p,q,\gamma})$$

$$(DUE^v) = v^{1/2}(x) p_t(x,y) v^{1/2}(y) \leq C$$ is equivalent to $(vEv)_{1,\infty,1/2}$ or

$$(vE_{2,\infty}) \sup_{t > 0} \| v^{\frac{1}{2}} e^{-t\Delta} \|_{2 \rightarrow \infty} < +\infty$$

$(GN^v_q)$ is equivalent to

$$(vE_{2,q}) \sup_{t > 0} \| v^{\frac{1}{2} - \frac{1}{q}} e^{-t\Delta} \|_{2 \rightarrow q} < +\infty$$

Finite propagation speed of the associated wave equation $\Rightarrow$ commutation between the semigroup and the volume: for $p$, $q$ fixed, equivalence between $(vEv_{p,q,\gamma}) \Rightarrow$ extrapolation: pass from $q$ to $\infty$. 
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$$\sup_{t>0} \| v^{\gamma} \sqrt{t} e^{-t\Delta} v^{\delta} \|_{p \to q} < +\infty. \quad (vEv_{p,q,\gamma})$$

$$(DUE^v) = v^{1/2}(x)p_t(x,y)v^{1/2}(y) \leq C$$ is equivalent to $(vEv)_{1,\infty,1/2}$ or

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Conclusion: $(GN^v_q) \Rightarrow (DUE^v)$
Heat kernel on one-forms 1

$d, \delta$

\[ \tilde{\Delta} = d\delta + \delta d \]
Heat kernel on one-forms 1

\[ \tilde{\Delta} = d\delta + \delta d \]

Bochner's formula:

\[ \tilde{\Delta} = \nabla^* \nabla + \text{Ric}. \]

\[ |\tilde{p}_t(x, y)| \lesssim \frac{1}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{Ct} \right), \quad \forall \ t > 0, \ a.e. \ x, y \in M, \quad (U\tilde{E}) \]

for some \( C > 0 \). Here \( \tilde{p}_t(x, y) \) is a linear operator from \( T^*_yM \) to \( T^*_xM \), endowed with the Riemannian metrics at \( y \) and \( x \), and \( |\cdot| \) is its norm.
Heat kernel on one-forms 1

\[ d, \delta \]

\[
\tilde{\Delta} = d\delta + \delta d
\]

Bochner’s formula:

\[
\tilde{\Delta} = \nabla^* \nabla + \text{Ric.}
\]

\[
|\tilde{\rho}_t(x, y)| \lesssim \frac{1}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{Ct} \right), \quad \forall \ t > 0, \ \text{a.e.} \ x, y \in M, \quad (U\tilde{E})
\]

for some \( C > 0 \). Here \( \tilde{\rho}_t(x, y) \) is a linear operator from \( T^*_y M \) to \( T^*_x M \), endowed with the Riemannian metrics at \( y \) and \( x \), and \( |\cdot| \) is its norm.

Implies \((G)\)
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Implies \( (G) \)

Manifolds with non-negative Ricci:

\[ |\tilde{p}_t(x, y)| \leq p_t(x, y) \]

\[ |e^{-t\tilde{\Delta}}\omega| \leq e^{-t\Delta} |\omega| \]
In general, problem: no positivity, no maximum principle, no Dirichlet form, $e^{-t\Delta}$ is a priori not bounded on $L^1$ or $L^\infty$
In general, problem: no positivity, no maximum principle, no Dirichlet form, 
e^{−t\Delta} is a priori not bounded on $L^1$ or $L^\infty$
Joint work with Baptiste Devyver and Adam Sikora, in preparation.
A potential $\mathcal{V} \in L^\infty_{loc}$ is said to belong to the Kato class at infinity $K^\infty(M)$ if

$$\lim_{R \to \infty} \sup_{x \in M} \int_{M \setminus B(x_0, R)} G(x, y) |\mathcal{V}(y)| \, d\mu(y) = 0,$$

for some (all) $x_0 \in M$.

**Theorem**

Let $M$ be a complete non-compact connected manifold satisfying (D) and (DUE) and such that $|\text{Ric}_-| \in K^\infty(M)$. Let $\nu'$ be the reverse doubling exponent. If $\nu' > 4$, the heat kernel of $\tilde{\Delta}$ satisfies (UE) if and only if $\text{Ker}_{L^2}(\tilde{\Delta}) = \{0\}$. 
Recall the Gaussian lower bound

\[ p_t(x, y) \gtrsim \frac{1}{V(x, \sqrt{t})} \exp \left( - \frac{d^2(x, y)}{ct} \right), \quad \forall \ t > 0, \ a.e. \ x, y \in M \]  

(LE)

**Corollary**

*Under the above assumptions, (LE) holds.*

**Corollary**

*Under the above assumptions, \((E_p)\) holds for all \(p \in (1, +\infty)\).*
Sketch of proof 1

Since $\text{Ric}_- \in K^\infty(M)$, there is a compact subset $K_0$ of $M$ such that

$$\sup_{x \in M} \int_{M \setminus K_0} G(x, y) |\text{Ric}_-|(y) \, d\mu(y) < \frac{1}{2}. \quad (3)$$
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\]
(3)

Let \( R \) be the section of the vector bundle \( \mathcal{L}(T^*M) \) given by
\[
x \mapsto R(x) = \text{Ric}_-(x)1_{K_0}(x).
\]

We shall also denote by \( R \) the associated operator on one-forms. Set
\[
H = \nabla^* \nabla + \text{Ric}_+ - (\text{Ric}_-)1_{M \setminus K_0},
\]
so that
\[
\tilde{\triangle} = H - R.
\]
Lemma

\((e^{-tH})_{t>0}\) satisfies Gaussian estimates.
Sketch of proof 2

Lemma

\((e^{-tH})_{t>0}\) satisfies Gaussian estimates.

It follows that

\[
\sup_{t>0} \left\| (I + tH)^{-1} V^{1/p} \sqrt{\frac{1}{t}} \right\|_{p \to \infty} < +\infty
\]
Sketch of proof 2

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It follows that

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\sup_{t>0} \| (I + tH)^{-1} V^{1/p} \|_{p \to \infty} < +\infty
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(4)

We would like a similar estimate for \(\tilde{\Delta}\)
Sketch of proof 2

Lemma

\[(e^{-tH})_{t>0} \text{ satisfies Gaussian estimates.}\]

It follows that

\[\sup_{t>0} \left\| (I + tH)^{-1} \frac{V^{1/p}}{\sqrt{t}} \right\|_{p \to \infty} < +\infty \quad (4)\]

We would like a similar estimate for \(\tilde{\Delta}\)

\[ (1 + t\tilde{\Delta})^{-1} = (I - (1 + tH)^{-1}tR)^{-1}(1 + tH)^{-1}, \]
Sketch of proof 2

**Lemma**

\[(e^{-tH})_{t>0} \text{ satisfies Gaussian estimates.}\]

It follows that

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\sup_{t>0} \|(I + tH)^{-1} V^{1/p} \sqrt{t} \|_p \to \infty < +\infty
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We would like a similar estimate for \(\tilde{\Delta}\)

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(1 + t\tilde{\Delta})^{-1} = (I - (1 + tH)^{-1} tR)^{-1} (1 + tH)^{-1},
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Want

\[
\|(I - (H + \lambda)^{-1} R)^{-1} \|_{\infty \to \infty} \leq C.
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Sketch of proof 2

Lemma

\((e^{-tH})_{t>0}\) satisfies Gaussian estimates.

It follows that

\[ \sup_{t>0} \left\| (I + tH)^{-1} V^{1/p} \sqrt{t} \right\|_p \to \infty < +\infty \]  \hspace{1cm} (4)

We would like a similar estimate for \(\tilde{\Delta}\)

\[ (1 + t\tilde{\Delta})^{-1} = (I - (1 + tH)^{-1} tR)^{-1} (1 + tH)^{-1}, \]

Want

\[ \|(I - (H + \lambda)^{-1} R)^{-1}\|_{\infty \to \infty} \leq C. \]

For \(\lambda > 0\), we introduce the two operators

\[ A_\lambda = R^{1/2}(H + \lambda)^{-1} R^{1/2} \]

and

\[ B_\lambda = (H + \lambda)^{-1} R. \]
Lemma

For any $\lambda \in [0, \infty)$, $B_\lambda$ is compact on $L^\infty$, $\sup_{\lambda \geq 0} \|B_\lambda\|_{\infty \to \infty} < \infty$, and the map $\lambda \mapsto B_\lambda \in \mathcal{L}(L^\infty, L^\infty)$ is continuous on $[0, \infty)$.

Lemma

For every $\lambda \geq 0$, the operator $A_\lambda$ is self-adjoint and compact on $L^2$. Furthermore, $\text{Ker}_{L^2}(\Delta) = \{0\}$ if and only if there is $\eta \in (0, 1)$ such that for all $\lambda \geq 0$,

$$\|A_\lambda\|_{2 \to 2} \leq 1 - \eta.$$

Lemma

Assume that $\text{Ker}_{L^2}(\Delta) = \{0\}$. If $\eta \in (0, 1)$ is as above then the spectral radius of $B_\lambda$ on $L^\infty$ satisfies

$$r_\infty(B_\lambda) \leq 1 - \eta, \forall \lambda \geq 0.$$
Weighted $L^p - L^q$ inequalities again

Start from

$$\sup_{t>0} \| (I + t\Delta)^{-1} V_1^1/p_0 \|_{p_0 \to \infty} < +\infty \quad (RV_{p,\infty})$$

By duality and interpolation,

$$\sup_{t>0} \| V_{\sqrt{t}}^\gamma (I + t\Delta)^{-1} V_{\sqrt{t}}^\delta \|_{p \to q} < +\infty \quad (VRV_{p,q,\gamma})$$

for any $p, q$ such that $1 \leq p \leq p_0$, $\frac{1}{p} - \frac{1}{q} = \gamma + \delta = \frac{1}{p_0}$, $\gamma = \frac{1}{(p_0-1)q}$, and $\gamma + \delta = \frac{1}{p} - \frac{1}{q}$. Use the finite propagation speed to iterate (instead of extrapolating).
Weighted $L^p - L^q$ inequalities again

Start from

$$\sup_{t>0} \left\| (I + t\vec{\Delta})^{-1} V^{1/p_0} \right\|_{p_0 \to \infty} < +\infty \quad (RV_{p,\infty})$$

By duality and interpolation,

$$\sup_{t>0} \left\| V^{\gamma} (I + t\vec{\Delta})^{-1} V^{\delta} \right\|_{p \to q} < +\infty \quad (VRV_{p,q,\gamma})$$

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Use the finite propagation speed to iterate (instead of extrapolating)