Curvature contraction of convex hypersurfaces

James McCoy

Joint work with Ben Andrews, Andrew Holder, Glen Wheeler, Valentina-Mira Wheeler and Graham Williams

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We begin with curvature contraction flows of the form

$$\frac{\partial X}{\partial t} (p, t) = -F (W (p, t)) \nu (p, t)$$

(1)

with a smooth, compact, strictly convex initial hypersurface $X (S^n, 0) = X_0 (S^n) = M_0$ without boundary.
Definition

The **principal curvatures** $\kappa_i, i = 1, \ldots, n$, are the eigenvalues of the Weingarten map $\mathcal{W}$ of $M_t$.

Write $\kappa = \{\kappa_1, \ldots , \kappa_n\}$.

**Basic properties of the speed:**

- $F(\mathcal{W}) = f(\kappa)$, $f$ symmetric on the positive cone

$$\Gamma_+ = \{\kappa: \kappa_i > 0 \text{ for all } i = 1, \ldots, n\}.$$

- $f > 0$ (contraction flow), $f(1, \ldots, 1) = 1$ (normalised)
- $f$ strictly increasing in each argument, everywhere on $\Gamma_+$
- $f$ is homogeneous of degree $\alpha > 0$
A function $f(\kappa) = f(\kappa_1, \ldots, \kappa_n)$ is **homogeneous of degree** $\alpha$ if for every $\kappa$ and for all $k > 0$,

$$f(k\kappa) = k^\alpha f(\kappa).$$

**Theorem (Euler’s homogeneous function theorem)**

*If $f$ is differentiable and homogeneous of degree $\alpha$ then*

$$\sum_{i=1}^{n} \dot{f}_i \kappa_i := \sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \kappa_i = \alpha f.$$

**Idea of proof:** Take $\frac{\partial}{\partial k}$ of definition then set $k = 1$. \hfill $\square$

**Note:** $f(k\kappa) = k^\alpha f(\kappa) \iff F(k\mathcal{W}) = k^\alpha F(\mathcal{W})$. 
For any smooth strictly convex initial hypersurface $M_0$ and speed $f$ smooth and homogeneous of degree 1:

- (Huisken, ’84) Under **mean curvature flow**, $F = H$
- (Chow, ’85) Under flow by $F = K^{1/n}$
- (Andrews, ’94) Under flows by $f$ either
  - convex, or
  - concave and
    - $n = 2$
    - $f \to 0$ as $\kappa \to \partial \Gamma_+$, or
    - $\sup_{M_0} \frac{H}{F} < \liminf_{\kappa \in \partial \Gamma_+} \frac{\sum_j \kappa_j}{f(\kappa)}$.
- (Andrews, ’07) Under flows by $f$ concave and inverse concave, that is, the function $f_*$ is concave, where
  \[
  f_* (x_1, \ldots, x_n) = \frac{1}{f \left( \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right)}.
  \]
- (Andrews, ’10) Under flows by $f$ when $n = 2$ (surfaces).
And more recently

- (Andrews, M., Zheng, '13) Under flows with $f$ inverse concave and either
  - $f_* \rightarrow 0$ on $\partial \Gamma_+$, or
  - matrix of inverse Weingarten map $(r_{ij})$ of $M_0$ satisfies

\[
\sup_{\nu \in T_z S^n, \|\nu\| = 1} \left( \frac{r(\nu, \nu)|_z}{F_*(r(z))} \right) < \liminf_{\partial \Gamma_+} \inf_{\nu \in T_z S^n, \|\nu\| = 1} \frac{r(\nu, \nu)}{f_*(r)}.
\]

- (M., Mofarreh, V-M Wheeler, '14) $M_0$ axially symmetric

More results for smooth speeds $f$ homogeneous of positive degree with $M_0$ suitably pointwise ‘curvature pinched’:

- (Chow, ’85) Flows by $K^\beta$, $\beta \geq \frac{1}{n}$
- (Schulze, ’06) $F = H^k$, $k \geq 1$
- (Alessandroni-Sinestrari, ’10), $F = R^k$, $k \geq \frac{1}{2}$
- (Andrews, M., ’12) $f$ homogeneous of degree $\alpha \geq 1$ (rather tight pinching).
If $n = 2$, more results are possible, without curvature pinching, or convexity of $f$, by exploiting symmetries of Codazzi equations:

- (Andrews, ’99) $F = K$ (Firey’s conjecture)
- (Schnürer, ’05) various speeds of integer homogeneity between 1 and 6
- (Schnürer-Schulze, ’06) $F = H^k$, $k = 1, 2, 3, 4, 5$
- (Andrews, ’10) $f$ homogeneous of degree $\alpha \geq 1$ ($\alpha$-dependent pinching if $\alpha > 1$)
- (Andrews-Chen, ’12) $F = K^{\frac{\alpha}{2}}$, $\alpha \in [1, 2]$

Even more results on convergence of smooth convex hypersurfaces to a point (or not) without roundness. See, eg (Andrews, M., Zheng ’13).
Definition

Let \( X : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1} \) represent a suitably smooth convex hypersurface \( M \) in \( \mathbb{R}^{n+1} \). The \textit{support function} of \( M \) is given by

\[
u (x) = \langle X (x), \nu (x) \rangle.
\]

**Note:** The image of \( M \) in \( \mathbb{R}^{n+1} \) can be reconstructed via

\[
\hat{X} (x) = u (x) x + \nabla u (x),
\]

where \( \nabla \) and \( \sigma \) are the standard gradient and metric on \( \mathbb{S}^n \).

Solutions of (1) remain convex and correspond to solutions of the scalar parabolic equation

\[
\frac{\partial u}{\partial t} = -F \left( (\nabla_i \nabla_j u + \sigma_{ij} u)^{-1} \right) = -F_* (r_{ij})^{-1} =: \Psi (r_{ij}).
\]
Suppose now $F (\mathcal{W})$ satisfies the following

- $F (\mathcal{W}) = f (\kappa)$, $f$ symmetric on $\Gamma_+$
- $f > 0$, $f (1, \ldots, 1) = 1$
- for all $\kappa \in \Gamma_+$, $f (\kappa + \delta e_i) > f (\kappa)$ for all $\delta > 0$ and each $i$
- $f$ is homogeneous of degree 1
- $f$ is convex, i.e. for all $x, y \in \Gamma_+$ and $\lambda \in [0, 1]$,

$$f (\lambda x + (1 - \lambda y)) \leq \lambda f (x) + (1 - \lambda) f (y).$$

**Properties:**

- $f$ is almost everywhere in $\Gamma_+$ twice differentiable (Aleksandrov).
- $f$ is Lipschitz on compact subsets of $\Gamma_+$. 
Example \((n = 2, \ f (\kappa) = \alpha \kappa_{\min} + \beta \kappa_{\max}, \ \beta \geq \alpha > 0, \ \alpha + \beta = 1)\)

- \(f\) defined on \(\mathbb{R}^2\) and symmetric, since
  \[
  \kappa_{\min} = \left(\kappa_1 + \kappa_2 - |\kappa_1 - \kappa_2|\right) / 2
  \]
  \[
  \kappa_{\max} = \left(\kappa_1 + \kappa_2 + |\kappa_1 - \kappa_2|\right) / 2
  \]
  We can rewrite \(f(\kappa) = \frac{1}{2}H + \left(\frac{\beta - \alpha}{2}\right) |\kappa_1 - \kappa_2|\).

- \(f > 0\) on \(\Gamma_+\), not differentiable if \(\beta > \alpha\) wherever \(\kappa_1 = \kappa_2\)
- \(f\) is everywhere increasing (by triangle inequality)
- \(f\) degree 1 homogeneous, convex (triangle inequal, \(\beta \geq \alpha\))

Example (Maxima of convex functions)

\[
F = \max \left(\frac{H}{n}, \eta |A|\right), \quad \frac{1}{n} < \eta < \frac{1}{\sqrt{n}}.
\]
For a mollifier \( j_\varepsilon (x) = \varepsilon^{-n} j \left( \frac{x}{\varepsilon} \right) \), such as

\[
    j(x) = \begin{cases} 
        c_n e^{\frac{1}{|x|^{2-1}}} & \text{for } |x| < 1 \\
        0 & \text{for } |x| \geq 1,
    \end{cases}
\]

where \( c_n \) is chosen such that \( \int_{\mathbb{R}^n} j(x) \, dx = 1 \), set

\[
    f_\varepsilon (\kappa) = \frac{H}{\hat{f}_\varepsilon^1} \int_{\mathbb{R}^n} j_\varepsilon(y) f \left( \frac{\kappa}{H} - y \right) \, dy = \frac{1}{\hat{f}_\varepsilon^1} \int_{\mathbb{R}^n} j_\varepsilon(y) f (\kappa - Hy) \, dy,
\]

where \( \hat{f}_\varepsilon^1 \) is a normalisation factor.

**Lemma**

1. For each \( \varepsilon > 0 \), \( f_\varepsilon \) is smooth; \( f_\varepsilon \rightarrow f \) uniformly on \( \tilde{\Gamma} \subset \subset \Gamma_+ \).
2. For each \( \varepsilon \in \left( 0, \min \left( \frac{1}{n}, \varepsilon_0 \right) \right) \), \( f_\varepsilon \) satisfies the same properties as \( f \).
3. \( f(\kappa) - \varepsilon H \leq f_\varepsilon(\kappa) \leq f(\kappa) + \varepsilon H \).

*Given* $M_0$ *compact, strictly convex* $C^{1,\beta}$ *hypersurface and a convex function* $f$ *on* $\Gamma_+$ *satisfying the above properties, a solution* $u \in C^{2,\alpha}(\mathbb{S}^n \times (0, T))$ *to (3) exists for* $T < \infty$. *The* $M_t$ *contract to a point as* $t \to T$. *Under rescaling,* $\tilde{M}_t$ *approaches* $\mathbb{S}^n$ *exponentially in* $C^{2,\alpha'}$ *for* $0 < \alpha' < \alpha$.

**Remarks:**

1. Cannot estimate curvature derivatives via Schauder estimates. So to obtain contraction to a point by contradiction, we need short time existence for $C^{1,\beta}$ initial hypersurfaces. Modification of Lieberman, Chapter 14.

2. If the speed is more regular, then the solution is correspondingly more regular, by boot-strapping.
We obtain estimates independent of $\varepsilon$ for the flows

$$\frac{\partial u^\varepsilon}{\partial t} = -F^\varepsilon \left( (\tilde{\nabla}_i \tilde{\nabla}_j u^\varepsilon + \bar{\sigma}_{ij} u^\varepsilon)^{-1} \right) =: F_*^\varepsilon (r_{ij}),$$

all with initial hypersurface $M_0$. The speeds are given by

$$F^\varepsilon (\mathcal{W}) := f^\varepsilon (\kappa (\mathcal{W}));$$

we will denote by $\kappa^\varepsilon_i$ the curvatures of the $M^\varepsilon_t$, etc.

Let $\rho_-, \rho_+$ denote the inner and outer radius of $M_0$.

**Lemma (maximal time estimate)**

$$\frac{\rho_-^2}{2} \leq T \leq \frac{\rho_+^2}{2}.$$
Proof: The radius $r^\varepsilon(t)$ of a sphere evolving under (5) satisfies
\[
\frac{d}{dt} r^\varepsilon(t) = -f^\varepsilon \left( \frac{1}{r^\varepsilon}, \ldots, \frac{1}{r^\varepsilon} \right) = -f^\varepsilon(1, \ldots, 1) \frac{1}{r^\varepsilon} = -\frac{1}{r^\varepsilon}.
\]

With a condition $r^\varepsilon(t_0) = r_0$, independent of $\varepsilon$, the ODE has solution
\[
r^\varepsilon(t) = \sqrt{r_0^2 - 2(t - t_0)}.
\]
The sphere shrinks to a point at time $t = t_0 + \frac{r^2(t_0)}{2}$.

Since $M_0$ encloses $B_{\rho_-}$ and is enclosed by $B_{\rho_+}$,
\[
\frac{\rho_-^2}{2} \leq T^\varepsilon \leq \frac{\rho_+^2}{2}
\]
for all $\varepsilon$. □
Lemma (lower bound on speed and mean curvature)

Under (3), $H^\varepsilon \geq H_0 > 0$ and $F^\varepsilon \geq \frac{1}{n} H_0$ remain true, independent of $\varepsilon$.

Proof: Solutions of (5) satisfy

$$\frac{\partial}{\partial t} H^\varepsilon = \mathcal{L}^\varepsilon H^\varepsilon + \tilde{F}^{kl,rs}_\varepsilon \nabla^i \varepsilon h^\varepsilon_{kl} \nabla^i \varepsilon h^\varepsilon_{rs} + \tilde{F}^{kl}_\varepsilon h^\varepsilon_{km} h^\varepsilon_{ml} H^\varepsilon,$$

where $\mathcal{L}^\varepsilon = \tilde{F}^{ij}_\varepsilon \nabla^i \varepsilon \nabla^j \varepsilon$. Since the $F^\varepsilon$ are convex,

$$\min_{M_t} H^\varepsilon \geq \min_{M_0} H =: H_0 > 0$$

independent of $\varepsilon$.

Any convex $F$ satisfies $F \geq \frac{1}{n} H$, the result follows. \qed
Using Hamilton’s maximum principle for tensors,

**Lemma (preservation of convexity, curvature pinching)**

**Under** (3),

1. \( h^i_\varepsilon j > 0 \),

2. \( h^i_\varepsilon j - \eta F^\varepsilon \delta^i_j \geq 0 \), for any constant \( \eta \leq \min_{M_0} \frac{n \kappa_{\min}}{n f(\kappa) + H} \).

**Proof:** For any constant \( \eta \),

\[
\frac{\partial}{\partial t} \left( h^i_\varepsilon j - \eta F^\varepsilon \delta^i_j \right) = \mathcal{L}^\varepsilon \left( h^i_\varepsilon j - \eta F^\varepsilon \delta^i_j \right) + \tilde{F}^{k_1,r_1}_{\varepsilon} \nabla^i_\varepsilon h^j_\varepsilon \nabla^j_\varepsilon h^{k_2}_\varepsilon
\]

\[
+ \tilde{F}^{k_1}_{\varepsilon} h^j_\varepsilon h^m_\varepsilon l \left( h^i_\varepsilon j - \eta F^\varepsilon \delta^i_j \right) .
\]

so if the inequality holds initially, then it is preserved under the flow (5). Choice of \( \eta \) follows from estimate of \( f_\varepsilon \) in terms of \( f \) and \( H \).
Remark: Taking \( \varepsilon \leq \varepsilon_0 = \frac{n}{n} \), the argument of the convolution \( f_\varepsilon \) remains within \( \Gamma_+ \).

Lemma (upper speed bound, while inradius remains positive)

Fix \( t_0 < T \). Then for any \( r > 0 \) such that \( u \geq r \) at time \( t_0 \), we have on \([0, t_0]\):

\[
F_\varepsilon (x, t) \leq 2\rho_+ \max \left\{ \frac{\max_{M_0} F}{r}, \frac{2}{r^2} \right\}.
\]

Idea of proof: A method of Chou ('85); choose origin such that \( B_r(O) \) is enclosed by \( M_{t_0}^\varepsilon \). Then \( u_\varepsilon (x, t) \geq r \) for all \( x \in S^n \) and \( t \in [0, t_0] \). The function \( Q_\varepsilon = \frac{F_\varepsilon}{2u_\varepsilon - r} \) satisfies

\[
\frac{\partial}{\partial t} Q_\varepsilon \leq \mathcal{L}_\varepsilon Q_\varepsilon + 4\dot{F}_\varepsilon^{kl} \frac{\nabla_k u_\varepsilon}{2u_\varepsilon - r} \nabla_l Q_\varepsilon + Q_\varepsilon^2 \left( 2 - r^2 Q_\varepsilon \right).
\]
Lemma (equation (5) is uniformly parabolic)

\[ C \delta \leq f(\kappa + \delta e_i) - f(\kappa) \leq \overline{C} \delta. \]

Idea of proof: Upper and lower bounds on \( F \) and curvature pinching give that \( \kappa \) remains within a compact \( K \subset \Gamma_+ \).

We have now shown that the solutions \( u^\varepsilon \) to (5) are bounded in \( C^{2,\alpha} \), independent of \( \varepsilon \), as long as the inradius is positive. Taking \( \varepsilon \to 0 \) we obtain a \( C^2 \) solution to (3).

Specifically, in view of our estimates independent of \( \varepsilon \), using the method of continuity and mollification as in Lieberman Chapter 14, we obtain

Theorem (short-time existence of solution to (1))

For any \( u_0 \in C^{1,\alpha}(\mathbb{S}^n) \) there exists a \( \delta > 0 \) and a unique solution \( u \in C^{2,1}(\mathbb{S}^n \times (0, \delta)) \cap C(\mathbb{S}^n \times [0, \delta]). \)
Theorem (contraction to a point)

The images $M_t$ shrink to a point as $t \to T < \infty$.

Proof:

1. Suppose $\rho_- \not\to 0$ as $t \to T$.
2. Speed bounds and curvature pinching imply bounds above and below on the principal curvatures.
3. These imply convergence to $u(\cdot, T)$, generating a $C^{1,1}$ hypersurface $\tilde{M}_T$.
4. $\tilde{M}_T$ could then be used as initial data in the short-time existence theorem, contradicting the maximality of $T$.
5. Therefore $\rho_- \to 0$ as $t \to T$ and, via curvature pinching, $\rho_+ \to 0$ also.

Moreover, by standard arguments we get $u \in C^{2,\alpha}(\mathbb{S}^n \times (0, T))$. □
The natural rescaling of the solution to (1) is

\[ \tilde{X}(x, t) = \frac{1}{\sqrt{2(T - t)}} (X(x, t) - p), \]

where \( M_t \) contracts to the point \( p \in \mathbb{R}^{n+1} \) at time \( T \). The rescaled time parameter is

\[ \tau = -\frac{1}{2} \ln \left( 1 - \frac{t}{T} \right) \in [0, \infty). \]

The rescaled immersions \( \tilde{M}_\tau \) evolve according to

\[ \frac{\partial}{\partial \tau} \tilde{X}(x, \tau) = -F(\tilde{\nabla} \tilde{X}(x, \tau)) \tilde{\nu}(x, \tau) + \tilde{X}(x, \tau), \quad (6) \]

with initial condition

\[ \tilde{X}(x, 0) = \frac{1}{\sqrt{2T}} (X_0 - p). \]
By standard arguments, solutions to (6) have
- uniform positive lower and upper bounds on $F$;
- curvature pinching (homogeneous degree zero estimate).

Consequently, (6) is uniformly parabolic and $C^{2,\alpha}$ regularity of $\tilde{X}$ then follows by standard arguments for fully nonlinear equations (Krylov, Safanov).

For **asymptotic sphericity**, we need a geometric quantity whose extremum characterises a sphere, and whose monotonicity survives in the limit $\varepsilon \to 0$. Consider the family of flows

$$
\frac{\partial}{\partial \tau} \tilde{X}^\varepsilon (x, \tau_\varepsilon) = -F^\varepsilon (\nabla \tilde{V}^\varepsilon (x, \tau_\varepsilon)) \tilde{V}^\varepsilon (x, \tau_\varepsilon) + \tilde{X}^\varepsilon (x, \tau_\varepsilon),
$$

with initial condition

$$
\tilde{X}^\varepsilon (x, 0) = \frac{1}{\sqrt{2T_\varepsilon}} (X_0 - p_\varepsilon).
$$
For $\tilde{G}_0^\varepsilon := \tilde{G}_0(\tilde{W}^\varepsilon)$ smooth, positive, increasing, concave, degree-one homogeneous and normalised, set

$$\tilde{G}^\varepsilon := \tilde{G}_0^\varepsilon + k\tilde{H}^\varepsilon \iff \tilde{g}^\varepsilon := \tilde{g}_0^\varepsilon + k \sum \tilde{\kappa}_i^\varepsilon$$

and $\tilde{Q}^\varepsilon_\alpha := \tilde{G}^\varepsilon - \alpha \tilde{F}^\varepsilon \iff \tilde{q}^\varepsilon_\alpha := \tilde{g}^\varepsilon - \alpha f^\varepsilon$, for numbers $k$ and $\alpha$.

Since $\tilde{G}$ is concave and $\tilde{F}$ is convex, the function $\frac{\tilde{G}}{\tilde{F}}$ has only one local maximum on $\Gamma^+ \cap \{\parallel\tilde{A}^\varepsilon\parallel = 1\}$, at $(1, \ldots, 1)$, implying the structural bound $\frac{\tilde{G}}{\tilde{F}} \leq 1$.

**Lemma**

*For any $\bar{\alpha}$ there is an absolute constant $k_0 = k_0(\bar{\alpha})$ such that, under the rescaled flow (7), for any $\alpha \leq \bar{\alpha}$ and $k \geq k_0 > 0$,*

$$\frac{\partial}{\partial \tau^\varepsilon} \tilde{Q}^\varepsilon_\alpha \geq \tilde{L}^\varepsilon \tilde{Q}^\varepsilon_\alpha + \left(\tilde{F}^\varepsilon_{pq} \tilde{h}^\varepsilon_p m \tilde{h}^\varepsilon_{mq} - 1\right) \tilde{Q}^\varepsilon_\alpha. \quad (8)$$
We compute

\[
\frac{\partial}{\partial \tau_\varepsilon} \tilde{Q}_\alpha^\varepsilon = \tilde{\mathcal{L}}_\varepsilon \tilde{Q}_\alpha^\varepsilon + \left( \hat{Q}_{\alpha}^{\varepsilon \ ij} \hat{F}_{\varepsilon}^{\ pq, mn} - \hat{F}_{\varepsilon}^{\ ij} \tilde{Q}_{\alpha}^{\varepsilon \ pq, mn} \right) \tilde{\nabla}^\varepsilon_i \tilde{h}_{pq} \tilde{\nabla}^\varepsilon_j \tilde{h}_{mn} \\
\hspace{2cm} + \left( \hat{F}_{\varepsilon}^{\ pq} \tilde{h}_{\ v}^\varepsilon m \tilde{h}_{\ mq}^\varepsilon - 1 \right) \tilde{Q}_\alpha^\varepsilon. \tag{9}
\]

We wish to choose \( k > 0 \) such that the whole \( \tilde{\nabla} A^\varepsilon \) term in (9) is nonnegative. This requires the matrix inequality \( \hat{Q}_\alpha^\varepsilon \geq 0 \).

We have in coordinates that diagonalise the Weingarten map

\[
\frac{\partial \tilde{q}_\alpha^\varepsilon}{\partial \tilde{\kappa}_i} = \frac{\partial \tilde{g}_0^\varepsilon}{\partial \tilde{\kappa}_i} + k - \alpha \frac{\partial \tilde{f}^\varepsilon}{\partial \tilde{\kappa}_i} > k_0 - \overline{\alpha C}.
\]

Taking \( k_0 = \overline{\alpha C} \) meets the requirement. \( \square \)
Using the properties of $\tilde{G}^\varepsilon$ and the lower speed bound, we have

**Lemma**

There exists an absolute constant $\hat{C} > 0$ such that

$$\left(\hat{C} - \alpha\right)\tilde{F}^\varepsilon \leq \tilde{Q}_\alpha^\varepsilon \leq (1 - \alpha)\tilde{F}^\varepsilon.$$  (10)

**Completion of proof of asymptotic sphericity**

Consider $\tilde{Q}_{\alpha_m}^\varepsilon$ on the time interval $[m, m+1]$ for $m \geq 1$. Fix $\bar{\alpha} = 1$ and choose $\alpha = \alpha_0$ such that

$$\min_{M_0} \tilde{Q}_{\alpha_0}^\varepsilon = 0.$$

The lower bound in (10) implies $\alpha_0 > 0$, moreover, there is an upper bound on $\alpha_0$ beyond which

$$\min_{M_t, t \in [m, m+1]} \tilde{Q}_{\alpha_m}^\varepsilon < 0.$$
The sequence \( \{ \alpha_m \}_{m \in \mathbb{N}_0} \) is now generated by choosing on each interval \([m, m + 1]\) the corresponding \( \alpha = \alpha_m \) such that

\[
\min_{M_{tm}} \tilde{Q}^\varepsilon_{\alpha_m} = 0.
\]

For all \( m \) we have \( \alpha_m \leq 1 \) since otherwise, by (10),

\[
\max_{M_t, t \in [m, m+1]} \tilde{Q}^\varepsilon_{\alpha_m} < 0.
\]

The evolution equation (8) implies that on \([m, m + 1]\) the quantity \( \tilde{Q}^\varepsilon_{\alpha_m} \) is non-negative.

We show that in fact \( \min_{M_t} \tilde{Q}^\varepsilon_{\alpha_m} \) increases using the parabolic Harnack inequality.
First rewrite (8) in a local coordinate system around $B_{\rho}(x)$ for any $x \in M$,

$$
\frac{\partial}{\partial \tau_{\epsilon}} \tilde{Q}^{\epsilon}_{\alpha_{m}} \geq \tilde{F}^{ij}_{\epsilon} \left( \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} \tilde{Q}^{\epsilon}_{\alpha_{m}} - \Gamma^{k}_{ij} \frac{\partial}{\partial x_{k}} \tilde{Q}^{\epsilon}_{\alpha_{m}} \right) + \left( \tilde{F}^{pq}_{\epsilon} \tilde{h}^{\epsilon}_{p} \tilde{h}^{\epsilon}_{mq} - 1 \right) \tilde{Q}^{\epsilon}_{\alpha_{m}}.
$$

Using the ellipticity constants, we derive, for $\lambda$ to be chosen,

$$
\frac{\partial}{\partial \tau_{\epsilon}} \left( e^{\lambda t} \sqrt{\tilde{Q}^{\epsilon}_{\alpha_{m}}} \right) \geq \tilde{F}^{ij}_{\epsilon} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} \left( e^{\lambda t} \sqrt{\tilde{Q}^{\epsilon}_{\alpha_{m}}} \right) + \frac{1}{2} \left( \tilde{F}^{pq}_{\epsilon} \tilde{h}^{\epsilon}_{p} \tilde{h}^{\epsilon}_{mq} - \frac{1}{2} \frac{C^{2}}{C} |\Gamma|^{2} - 1 + \lambda \right) \left( e^{\lambda t} \sqrt{\tilde{Q}^{\epsilon}_{\alpha_{m}}} \right).
$$
Since the rescaled curvatures are bounded, there is a positive $\lambda = \lambda_0$ such that for

$$\tilde{Z}^{\varepsilon}_{\alpha_m} = \left( e^{\lambda_0 t} \sqrt{\tilde{Q}^{\varepsilon}_{\alpha_m}} \right),$$

$$\left( \frac{\partial}{\partial \tau^{\varepsilon}} - \tilde{F}^{ij}_{\varepsilon} \frac{\partial^2}{\partial x_i \partial x_j} \right) \tilde{Z}^{\varepsilon}_{\alpha_m} \geq 0$$

in the local coordinate chart. The weak parabolic Harnack inequality implies, for each $x \in M$,

$$\min_{B_{\rho/2}(x) \times [m+1, m+2]} \tilde{Z}^{\varepsilon}_{\alpha_m} \geq c \int_{m-1}^{m} \left( \frac{1}{|B_{\rho}(x)|} \int_{B_{\rho}(x)} |\tilde{Z}^{\varepsilon}_{\alpha_m}|^\sigma \, dx \right)^{\frac{1}{\sigma}} d\tau,$$

for absolute positive $\sigma$ and bounded $c$, independent of $x$. 
Since $\tilde{X}_\varepsilon \in C^{2,\alpha}, \tilde{Z}_{\alpha m} \in C^{0,\alpha}$ and therefore

$$
\min_{B_{\rho/2}(x) \times [m+1,m+2]} \tilde{Z}_\varepsilon \geq c \max_{B_\rho(x) \times [m-1,m]} \tilde{Z}_{\alpha m},
$$

where $c$ depends on the absolute constants $\sigma$ and $\alpha$.

A parabolic chaining argument gives

$$
\min_{M \times [m+1,m+2]} \tilde{Z}_\varepsilon \geq c \max_{M \times [m-1,m]} \tilde{Z}_{\alpha m}.
$$

Absorbing the exponential factor and squaring gives

$$
\min_{M \times [m+1,m+2]} \tilde{Q}_\varepsilon \geq c \max_{M \times [m-1,m]} \tilde{Q}_{\alpha m},
$$

(11)

where $c > 0$ is an absolute constant.
Estimating the maximum in (11) implies the recurrence relation

\[ 1 - \alpha_{m+1} \leq -c(1 - \alpha_m) + (1 - \alpha_m) \leq (1 - c)(1 - \alpha_m). \]

Since \( \alpha_m < 1 \) observe that \( 1 - c > 0 \) and so \( 1 - c \in (0, 1) \).

Iterating the recurrence relation, we have, for \( C = (1 - \alpha_0) \in (0, 1) \) and \( \gamma = -\log(1 - c) \in (0, \infty) \),

\[ 0 < 1 - \alpha_{m+1} \leq (1 - c)^m(1 - \alpha_0) = (1 - \alpha_0)e^{m \log(1 - c)} \leq Ce^{-\gamma m}. \]

This holds for \( t \in [m, m + 1] \) and implies, for all \( t \geq 0 \),

\[ 0 < 1 - \min_{\tilde{M}_t} \frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon} \leq Ce^{-\gamma t}. \]

This implies, in turn, for all \( \varepsilon \), uniform exponential convergence of \( \frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon} \) to 1, a value of the ratio that is obtained only on a sphere.


