On Root Systems and Automaticity of Coxeter Groups

by

Brigitte Brink

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy in Pure Mathematics at the University of Sydney.

August 1994
Statement of Authorship

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other University. The material presented here is believed to be original, with the exception of standard general results in Chapter 1, and other cases where due attribution is given.
Acknowledgements

First of all, I would like to express my gratitude towards my supervisor, Dr Bob Howlett, for his invaluable guidance over the past three years, his patience, and his ever-willingness to discuss any problems I encountered. I also wish to thank him for his extensive proof reading of this thesis, and for all the time he sacrificed to help me with \textup{T\!\!\!\!eX}.

Secondly, I would like to thank Dr Gerhard Röhrle for introducing me to PICTeX, and Matthew Drummond, John Graham, Tim Lavers and Dr Charles Zwoestine for their proof reading; special thanks to Charles for all his time spent rectifying my English and punctuation.

Furthermore, I would like to take this opportunity to thank everyone who made it possible for me to come to Australia, and everybody who made me feel so welcome here; in particular, Professors Terry Gagen, John Mack and Tim Wall. I am very grateful to have had this chance.

Finally, I wish to thank all my family and friends, both here and overseas, for their support (and for putting up with me over these last few months).

\hspace{1cm}

This work was supported with an Overseas Postgraduate Research Award and a University of Sydney Postgraduate Research Award.
Introduction

The aim of this thesis is to examine root systems associated with Coxeter groups. We introduce the notions of dominance and elementary roots, and employ them to show that the stabilizer of a root is the semidirect product of a Coxeter group and a free group, as well as to obtain an automatic structure for finitely generated Coxeter groups.

The structure of this thesis is as follows. We begin in Chapter 1 by reviewing some well known facts about Coxeter groups and root systems. In Chapter 2 a function from the root system to the integers is defined, which is an analogue of the length function on the Coxeter group, and enables us to use inductive proofs on the root system. This is then applied to give an alternative proof of the Theorem, due to Deodhar [4] and Dyer [5], that each reflection subgroup of a Coxeter group is itself a Coxeter group; furthermore, we derive certain properties relating to the coefficients which occur when roots are expressed as linear combinations of simple roots.

In Chapter 3 the concepts of dominance and elementary roots are introduced, and we prove the principal result of this thesis, namely, that the set of elementary roots is finite (provided the Coxeter group has finite rank). We also prove the important technical result that if $r_1 r_2 \cdots r_l$ is a reduced expression for an element of the Coxeter group having the property that $(r_1 r_2 \cdots r_l) \cdot \alpha = \beta$ for some simple roots $\alpha$ and $\beta$, then $(r_i r_{i+1} \cdots r_l) \cdot \alpha$ is elementary for all $i$. This is then used in the subsequent chapters to deal with the stabilizer of a root and the automaticity of Coxeter groups respectively.

Finally, in Chapter 6 we use the properties of elementary roots obtained in Chapter 3 to give an explicit description of the set of elementary roots in every case.

In the interest of keeping this thesis as self-contained as possible, proofs of various well-known properties of Coxeter groups are included in the early chapters.
Chapter 1

Preliminaries

In this chapter some well known properties of Coxeter groups are presented (see [1], [3] or [7]). We begin by introducing some notation. The set of real numbers will be denoted by \( \mathbb{R} \), the set of positive integers by \( \mathbb{N} \), and the set of nonnegative integers by \( \mathbb{N}_0 \). For any set \( M \), we denote the cardinality of \( M \) by \( |M| \); where appropriate, \( -M \) denotes the set \( \{ -x \mid x \in M \} \). For sets \( M \) and \( N \), the set difference of \( M \) and \( N \) will be denoted by \( M \setminus N \).

Throughout this thesis, \( W \) is a Coxeter group with distinguished generating set \( R \); that is, \( W \) has a presentation
\[
\langle r \in R \mid (rs)^{m_{rs}} = 1 \text{ for } r, s \in R \rangle,
\]
where \( m_{rr} = 1 \) for all \( r \in R \), and \( m_{rs} = m_{sr} \geq 2 \) or \( m_{rs} = m_{sr} = \infty \) for \( r, s \in R \) with \( r \neq s \). (Here \((rs)^\infty = 1\) is regarded as vacuously true).

The Coxeter graph of \( W \) has vertex set in one-one correspondence with \( R \), and two distinct vertices corresponding to \( r \) and \( s \) are joined by an edge or bond of weight \( m_{rs} \) if \( m_{rs} \neq 2 \). For convenience of notation we frequently identify \( r \in R \) with the vertex corresponding to \( r \). If \( r \) and \( s \) are joined by an edge, \( r \) and \( s \) are said to be adjoined, and the edge is labelled by \( m_{rs} \); if \( m_{rs} = 3 \) this label is suppressed. We say that the bond adjoining \( r \) and \( s \) is simple, non-simple, infinite if \( m_{rs} = 3 \), \( m_{rs} \neq 3 \) and \( m_{rs} = \infty \) respectively. By abuse of notation, \( S \subseteq R \) will denote both a subset of \( R \), and the subgraph of the Coxeter graph consisting of the vertices in \( S \) and the bonds adjoining them.

For \( w \in W \), define the length \( l(w) \) of \( w \) by
\[
l(w) = \min \{ l \in \mathbb{N}_0 \mid w = r_1 \cdots r_l \text{ for some } r_1, \ldots, r_l \in R \}.
\]
By definition of \( W \), all elements of \( R \) are self inverse, and hence \( R \) is closed under taking inverses. Thus \( l(w^{-1}) = l(w) \) and \( l(w) - 1 \leq l(wr) \leq l(w) + 1 \) for all \( w \in W \) and \( r \in R \); moreover, if \( w \) is an element of \( W \) with \( l(w) \geq 1 \), then there exists an \( r \in R \) with \( l(wr) < l(w) \); that is, \( l(wr) = l(w) - 1 \).
Let $V$ be an $\mathbb{R}$-vector space with basis $\Pi$ in one-one correspondence with $R$, and for $r \in R$ denote the basis element corresponding to $r$ by $\alpha_r$. The vertex set of the Coxeter graph is by construction in one-one correspondence with $\Pi$, and as above for $r$, we will frequently identify $\alpha_r$ with the vertex corresponding to $\alpha_r$; moreover, we use $\Pi' \subseteq \Pi$ both to denote a subset of $\Pi$, and the subgraph of the Coxeter graph consisting of the vertices in $\Pi'$ and the bonds adjoining them.

Next, let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be a symmetric bilinear form which satisfies $\langle \alpha_r, \alpha_s \rangle = -\cos(\pi/m_{rs})$ for all $r, s \in R$ with $m_{rs}$ finite, and $\langle \alpha_r, \alpha_s \rangle \leq -1$ for $r, s \in R$ with $m_{rs}$ infinite; (in particular, $\langle \alpha_r, \alpha_r \rangle = 1$). Observe that $\langle \cdot, \cdot \rangle$ is uniquely determined by the presentation of $W$ if and only if there are no infinite bonds in the Coxeter graph of $W$.

For $r \in R$ define $\rho_r : V \to V$ by $\rho_r(v) = v - 2\langle v, \alpha_r \rangle \alpha_r$. Then

$$\rho_r^2(v) = \rho_r(v - 2\langle v, \alpha_r \rangle \alpha_r) = (v - 2\langle v, \alpha_r \rangle \alpha_r - 2\langle v - 2\langle v, \alpha_r \rangle \alpha_r, \alpha_r \rangle \alpha_r) = v + (-2\langle v, \alpha_r \rangle - 2\langle v, \alpha_r \rangle + 4\langle v, \alpha_r \rangle) \alpha_r = v$$

for all $v \in V$; furthermore, for $v, v' \in V$,

$$\langle \rho_r(v), \rho_r(v') \rangle = \langle v - 2\langle v, \alpha_r \rangle \alpha_r, v' - 2\langle v', \alpha_r \rangle \alpha_r \rangle = \langle v, v' \rangle - 2\langle v', \alpha_r \rangle \langle v, \alpha_r \rangle - 2\langle v, \alpha_r \rangle \langle v, v' \rangle + 4\langle v, \alpha_r \rangle \langle v', \alpha_r \rangle \langle \alpha_r, \alpha_r \rangle = \langle v, v' \rangle - 4\langle v', \alpha_r \rangle \langle v, \alpha_r \rangle + 4\langle v, \alpha_r \rangle \langle v', \alpha_r \rangle = \langle v, v' \rangle.$$

So $\rho_r$ is self inverse and preserves the bilinear form, and is thus an element of $O(V)$, the orthogonal group of the bilinear form $\langle \cdot, \cdot \rangle$ on $V$.

If $r, s \in R$ are distinct, it can be easily seen that $\rho_r$ and $\rho_s$ preserve the space spanned by $\alpha_r$ and $\alpha_s$. Denote $2\langle \alpha_r, \alpha_s \rangle$ by $c$; then the matrices of $\rho_r, \rho_s$ and $\rho_r \rho_s$ on this space with respect to the basis $\alpha_r, \alpha_s$ are

$$\begin{pmatrix} -1 & -c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -c & -1 \end{pmatrix} \text{ and } \begin{pmatrix} c^2 - 1 & c \\ -c & -1 \end{pmatrix}$$

respectively.

The proof of the following assertion is a straightforward induction on $n$, and will be omitted.
Chapter 1  Preliminaries

(1.1) Lemma  Let $\alpha, \beta \in V$ be linearly independent, and let $A$ be a linear map on the span of $\alpha$ and $\beta$ with matrix

\[
\begin{pmatrix}
c^2 - 1 & c \\
-c & -1
\end{pmatrix}
\]

with respect to the basis $\alpha, \beta$. For $n \in \mathbb{N}$, let $A^n(\alpha) = \lambda_n \alpha + \mu_n \beta$ with $\lambda_n, \mu_n \in \mathbb{R}$. Then $A^n(\beta) = -\mu_n \alpha - \lambda_{n-1} \beta$ and

(i) if $c = -2 \cos(\theta)$ for some $\theta \in (0, \pi)$,

\[
\begin{align*}
\lambda_n &= \frac{\sin((2n+1)\theta)}{\sin(\theta)} \\
\mu_n &= \frac{\sin(2n\theta)}{\sin(\theta)};
\end{align*}
\]

(ii) if $c \leq -2$, then $\lambda_n \geq \mu_n + 1$ and $\mu_{n+1} \geq \lambda_n + 1$.

Lemma (1.1) enables us to prove the following proposition.

(1.2) Proposition  There is a representation $\rho$ of $W$ on $V$ with $\rho(r) = \rho_r$ for all $r \in R$.

Note that (1.1) yields for $r, s \in R$ that $\rho_r \rho_s$ has order at least $m_{rs}$. So when (1.2) is established, it will follow that $rs$ has order at least $m_{rs}$. If $m_{rs}$ is finite, $rs$ has order at most $m_{rs}$ by definition of $W$, therefore $m_{rs}$ is in fact the order of $rs$ for all $r, s \in R$.

Proof of (1.2). Let $F(R)$ be the free group on $R$; the map $r \mapsto \rho_r$ from $R$ to $O(V)$ extends to a homomorphism $F(R) \rightarrow O(V)$. Further, let $N$ be the normal closure in $F(R)$ of the set

\[
\{ (rs)^{m_{rs}} \mid r, s \in R \text{ with } m_{rs} < \infty \},
\]

so that $W$ is isomorphic to the quotient $F(R)/N$. It suffices to show that $N$ is in the kernel of the above homomorphism from $F(R)$ to $O(V)$. This is the case if $(\rho_r \rho_s)^{m_{rs}}$ equals the identity for all $r, s \in R$ with $m_{rs}$. By an earlier remark, $\rho_r^2$ equals the identity on $V$ for all $r \in R$. It remains to show that $(\rho_r \rho_s)^m$ is the identity for all $r, s \in R$ with $r \neq s$ and $m = m_{rs}$ finite. Since $\Pi$ is a basis for $V$, it suffices to show $(\rho_r \rho_s)^m(\alpha_t) = \alpha_t$ for all $\alpha_t \in \Pi$. If $t \in \{r, s\}$, this is true by (1.1)(i) with $\theta = \pi/m$ and $n = m$. So suppose that $\alpha_r, \alpha_s, \alpha_t$ are linearly independent. The space spanned by these vectors is obviously invariant under $\rho_r \rho_s$, and if $M$ is the matrix corresponding to
$\rho_r, \rho_s$ on this space with respect to the basis $\alpha_r, \alpha_s, \alpha_t$, it suffices to show that $M^m$ is the $3 \times 3$ identity matrix. A short calculation yields that

$$M = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix},$$

where $A$ is the $2 \times 2$ matrix corresponding to $\rho_r, \rho_s$ on the span of $\alpha_r$ and $\alpha_s$ with respect to the basis $\alpha_r, \alpha_s, \alpha_t$, and $b$ is a $2 \times 1$ vector. An easy induction yields that

$$M^n = \begin{pmatrix} A^n & b_n \\ 0 & 1 \end{pmatrix},$$

for $n \in \mathbb{N}$, where $b_n = (A^0 + \ldots + A^{n-1})b$. As $A^m$ is the matrix of $(\rho_r, \rho_s)^m$ on the span of $\alpha_r$ and $\alpha_s$ with respect to the basis $\alpha_r, \alpha_s$, (1.1)(i) for $\theta = \pi/m$ and $n = m$ gives that $A^m$ equals $I_2$, the $2 \times 2$ identity matrix. Thus

$$(A - I_2)(A^0 + \ldots + A^{m-1}) = A^m - I_2 = 0,$$

the $2 \times 2$ zero matrix. The determinant of $A - I_2$ equals $4 \sin^2(\pi/m)$, which is nonzero, and so $A - I_2$ is invertible. Hence $A^0 + \ldots + A^{m-1} = 0$ and $b_m$ is the $2 \times 1$ zero vector. This yields that $M^m$ is the $3 \times 3$ identity matrix, and we have in fact a homomorphism $\rho: W \to O(V)$.

The representation $\rho$ of $W$ on $V$ defined in (1.2) will be called a standard geometric realization of $W$. From now on we consider the action of $W$ on $V$ induced by a standard geometric realization of $W$ on $V$. We have seen that this action preserves $\langle , \rangle$. For $w \in W$ and $v \in V$, we denote the image of $v$ under $\rho(w)$ by $w \cdot v$. The set $\Phi = \{ w \cdot \alpha_r \mid w \in W, r \in R \}$ is called the root system of $W$ in $V$, and the elements of $\Pi$ are the simple roots. Note that since $\langle \alpha_r, \alpha_r \rangle = 1$ for $r \in R$, we have $\langle \alpha, \alpha \rangle = 1$ for all $\alpha \in \Phi$.

Every element $v$ of $V$ can be uniquely written as $\sum_{\alpha \in \Pi} \lambda_\alpha \alpha$ for some $\lambda_\alpha \in \mathbb{R}$, and $\lambda_\alpha$ is said to be the coefficient of $\alpha$ in $v$. Define the support of $v$ to be the set of $\alpha \in \Pi$ such that the coefficient of $\alpha$ in $v$ is nonzero, and denote this set by $\text{supp}(v)$. Furthermore, let $I(v)$ denote the set of $r \in R$ in one-one correspondence with elements of $\text{supp}(v)$. The set $PLC(\Pi)$ of positive linear combinations of $\Pi$ is the set of vectors in $V$ with all coefficients greater than or equal to 0. The sets of positive roots $\Phi^+$ and negative roots $\Phi^-$ are defined to be $\Phi^+ = \Phi \cap PLC(\Pi)$ and $\Phi^- = -\Phi^+$ respectively.
(1.3) **Lemma** Suppose \( w \in W \) and \( r \in R \) with \( l(wr) \geq l(w) \). Then \( w \cdot \alpha_r \in \Phi^+ \).

**Proof.** If \( l(w) = 0 \), then \( w = 1 \) and \( w \cdot \alpha_r = \alpha_r \) is in \( \Phi^+ \). Next, suppose that \( l(w) \geq 1 \), and let \( s \in R \) with \( l(ws) = l(w) - 1 \); then \( r \neq s \), since \( l(wr) \geq l(w) \). Set \( I = \{r, s\} \), and let \( W_I \) denote the subgroup of \( W \) generated by \( I \). Furthermore, let \( l_I \) be the length function on \( W_I \) with respect to \( I \). Since \( I \subseteq R \), it is clear that \( l(z) \leq l_I(z) \) for all \( z \in W_I \).

Consider the set

\[
A = \{ u \in W \mid u^{-1}w \in W_I \text{ and } l(u) + l_I(u^{-1}w) = l(w) \}.
\]

Then \( ws \in A \), since \( (ws)^{-1}w = s^{-1} = s \in W_I \) and

\[
l(ws) + l_I((ws)^{-1}w) = l(ws) + 1 = l(w).
\]

So \( A \) is non-empty, and if we let \( x \) be an element of \( A \) of minimal length, then \( l(x) \leq l(ws) \), and thus \( l(x) \leq l(w) - 1 < l(w) \). Assume for a contradiction that \( l(xr) < l(x) \); that is, \( l(xr) = l(x) - 1 \). Then \( xr \notin A \) by minimality of \( x \). On the other hand, \( rx^{-1}w \in W_I \) and

\[
l(w) = l((xr)(rx^{-1}w)) \leq l(xr) + l(rx^{-1}w)
\]

\[
\leq l(xr) + l_I(rx^{-1}w) \leq l(x) - 1 + l_I(x^{-1}w) + 1
\]

\[
= l(x) + l_I(x^{-1}w) = l(w),
\]

and we must have equality everywhere; hence \( l(w) = l(xr) + l_I(rx^{-1}w) \), which implies \( xr \in A \), a contradiction. So \( l(xr) \geq l(x) \) and similarly \( l(xs) \geq l(x) \), and by induction \( x \cdot \alpha_r \) and \( x \cdot \alpha_s \) are both positive roots.

Let \( y = x^{-1}w \); then

\[
l(x) + l_I(y) = l(w) \leq l(wr) = l(xy) \leq l(x) + l(yr) \leq l(x) + l_I(yr),
\]

and hence \( l_I(yr) \geq l_I(y) \). Therefore \( y \) equals \( (rs)^n \) or \( s(rs)^n \) for some \( n \in \mathbb{N} \).

If \( rs \) has infinite order, \( (rs)^n \cdot \alpha_r = \lambda \alpha_r + \mu \alpha_s \) for some \( \lambda \geq 0 \) by (1.1)(ii), and hence \( s(rs)^n \cdot \alpha_r = \lambda \alpha_r + \mu' \alpha_s \) for some \( \mu' \geq \lambda \geq 0 \). If \( rs \) has order \( m \), then \( l_I(yr) \geq l_I(y) \) yields that \( 2n+1 \leq m \) if \( y = (rs)^n \), and that \( 2(n+1) \leq m \) if \( y = s(rs)^n \), and we deduce from (1.1)(i) that \( y \cdot \alpha_r = \lambda \alpha_r + \mu \alpha_s \) for some \( \lambda, \mu \geq 0 \). So in any case,

\[
w \cdot \alpha_r = (xy) \cdot \alpha_r = x \cdot (\lambda \alpha_r + \mu \alpha_s) = \lambda (x \cdot \alpha_r) + \mu (x \cdot \alpha_s)
\]

for some \( \lambda, \mu \geq 0 \). Since \( x \cdot \alpha_r, x \cdot \alpha_s \in PLC(\Pi) \), it follows that \( w \cdot \alpha_r \) is in \( PLC(\Pi) \), and thus in \( \Phi^+ \), as required. \( \square \)
Chapter 1 Preliminaries

It is clear that $\Phi^+$ is a subset of $\Phi$, and since $r \cdot \alpha_r = -\alpha_r$ for $r \in R$, also $\Phi^- \subseteq \Phi$. In fact,

$$\Phi = \Phi^+ \cup \Phi^-.$$ 

For if $\alpha \in \Phi$, then by definition of $\Phi$ there exists a $w \in W$ and an $r \in R$ with $\alpha = w \cdot \alpha_r$. If $l(wr) \geq l(w)$, then $\alpha = w \cdot \alpha_r \in \Phi^+$ by the previous lemma, while if $l(wr) < l(w) = l((wr)r)$, then $(wr) \cdot \alpha_r \in \Phi^+$ by (1.3) with $wr$ in place of $w$; hence $w \cdot \alpha_r = -(wr) \cdot \alpha_r \in \Phi^-.$

A trivial, but very useful, consequence of this is the following result.

(1.4) Corollary \hspace{0.5cm} $r \cdot (\Phi^+ \setminus \{\alpha_r\}) \subseteq \Phi^+$ for $r \in R$.

Next, let $w \in W$ and $r \in R$. By definition of the root system, $w \cdot \alpha_r$ is a root, and since $\Phi$ equals the union of $\Phi^+$ and $\Phi^-$, we find that $w \cdot \alpha_r$ is either positive or negative. If $w \cdot \alpha_r \in \Phi^-$, then $l(wr) \geq l(w)$ by (1.3); hence $l(wr) < l(w)$, that is, $l(wr) = l(w) - 1$. If $w \cdot \alpha_r \in \Phi^+$, then $(wr) \cdot \alpha_r \in \Phi^-$, and thus

$$l(w) = l((wr)r) = l(wr) - 1$$

by the previous case with $wr$ in place of $w$; whence $l(wr) = l(w) + 1$.

(1.5) Lemma \hspace{0.5cm} For all $w \in W$ and $r \in R$,

$$l(wr) = \begin{cases} 
  l(w) + 1 & \text{if } w \cdot \alpha_r \in \Phi^+, \\
  l(w) - 1 & \text{if } w \cdot \alpha_r \in \Phi^-.
\end{cases}$$

We can now deduce that the standard geometric realization of $W$ on $V$ induces a faithful action of $W$ on $V$. For if $w \in W$ such that $w \cdot v = v$ for all $v \in V$, then in particular, $w \cdot \alpha_r = \alpha_r \in \Phi^+$ for all $r \in R$; that is, $l(wr) > l(w)$ for all $r \in R$, and thus $w = 1$.

It is clear that $\Phi$ is finite if $W$ is finite. Faithfulness of the action of $W$ on $V$ yields the converse, and so $\Phi$ is finite if and only if $W$ is finite.
Next, let \( w \in W \) and \( r, s \in R \) with \( w \cdot \alpha_r = \alpha_s \); then for all \( v \in V \),

\[
(wrw^{-1}s) \cdot v = (wrw^{-1}) \cdot (v - 2\langle v, \alpha_s \rangle \alpha_s) \\
= (wr) \cdot (w^{-1} \cdot v - 2\langle v, \alpha_s \rangle w^{-1} \cdot \alpha_s) \\
= (wr) \cdot (w^{-1} \cdot v - 2\langle v, \alpha_s \rangle \alpha_r) \\
= w \cdot (w^{-1} \cdot v - 2\langle w^{-1} \cdot v, \alpha_r \rangle \alpha_r + 2\langle v, \alpha_s \rangle \alpha_r) \\
= w \cdot (w^{-1} \cdot v - 2\langle v, w \cdot \alpha_r \rangle \alpha_r + 2\langle v, \alpha_s \rangle \alpha_r) \\
= w \cdot (w^{-1} \cdot v) \\
= v.
\]

Since the action of \( W \) on \( V \) is faithful, this implies \( wrw^{-1}s = 1 \); that is, \( wrw^{-1} = s \).

Now let \( \alpha \in \Phi \); by definition of the root system, there exist \( w \in W \) and \( r \in R \) with \( \alpha = w \cdot \alpha_r \). If \( \alpha \) also equals \( u \cdot \alpha_s \) for some \( u \in W \) and \( s \in R \), then \( (u^{-1}w) \cdot \alpha_r = \alpha_s \), and hence \( (u^{-1}w)r(u^{-1}w)^{-1} = s \) by the above; that is, \( wrw^{-1} = usu^{-1} \). So without ambiguity we may define the reflection \( r_\alpha \) to be \( wrw^{-1} \). Then for \( v \in V \),

\[
r_\alpha \cdot v = (wvw^{-1}) \cdot v = (wr) \cdot (w^{-1} \cdot v) \\
= w \cdot (w^{-1} \cdot v - 2\langle w^{-1} \cdot v, \alpha_r \rangle \alpha_r) \\
= v - 2\langle w^{-1} \cdot v, \alpha_r \rangle w \cdot \alpha_r \\
= v - 2\langle v, w \cdot \alpha_r \rangle w \cdot \alpha_r \\
= v - 2\langle v, \alpha \rangle \alpha.
\]

Observe that this yields for roots \( \alpha \) and \( \beta \) with \( r_\alpha = r_\beta \) that \( \alpha = \pm \beta \); for then

\[
-\alpha = r_\alpha \cdot \alpha = r_\beta \cdot \alpha = \alpha - 2\langle \alpha, \beta \rangle \beta,
\]

and thus \( \alpha = \langle \alpha, \beta \rangle \beta \), which leaves us with \( \alpha = \pm \beta \), since \( \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1 \).

\textbf{(1.6) Proposition} \hspace{1em} \textit{(Strong Exchange Condition)}

Let \( r_1, r_2, \ldots, r_n \in R \) and \( \alpha \in \Phi^+ \) such that \( (r_1r_2 \cdots r_n) \cdot \alpha \in \Phi^- \). Then there exists an \( i \in \{1, \ldots, n\} \) with

\[
(r_1r_2 \cdots r_n)r_\alpha = r_1r_2 \cdots r_i r_{i+1} \cdots r_n.
\]
Proof. Let \( i \in \{1, \ldots, n\} \) be maximal such that \((r_i \cdots r_n) \cdot \alpha \in \Phi^-\); then \((r_{i+1} \cdots r_n) \cdot \alpha \) must be positive, and thus \((r_{i+1} \cdots r_n) \cdot \alpha = \alpha_{r_i} \) by (1.4). So \( \alpha = (r_n \cdots r_{i+1}) \cdot \alpha_{r_i} \) and \( r_\alpha = (r_n \cdots r_{i+1}) r_i (r_{i+1} \cdots r_n) \). This implies

\[
(r_1 r_2 \cdots r_n) r_\alpha = r_1 r_2 \cdots r_{n-1} r_n (r_n r_{n-1} \cdots r_{i+1} r_{i+1} \cdots r_{n-1} r_n)
= r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_n,
\]
as required.

Note that (1.6) together with (1.5) imply the Exchange Condition:

Let \( r_1, \ldots, r_n, s \in R \) such that \( l(r_1 r_2 \cdots r_n) = n \) and \( l(r_1 r_2 \cdots r_n s) < n + 1 \). Then there exists an \( i \in \{1, \ldots, n\} \) such that

\[
(r_1 r_2 \cdots r_n s) = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n.
\]

For a set of roots \( \Gamma \), we denote the corresponding set of reflections by \( S_\Gamma \), and the subspace of \( V \) spanned by \( \Gamma \) by \( V_\Gamma \). The subgroup generated by \( S = S_\Gamma \) is denoted by \( W_S \) or \( W_\Gamma \), and \( \Phi_\Gamma \) or \( \Phi_S \) is defined to be the set of roots of the form \( w \cdot \gamma \) with \( w \in W_\Gamma \) and \( \gamma \in \Gamma \). It is clear that \( W_\Gamma \subseteq W_S \) if \( \Gamma' \subseteq \Phi_\Gamma \).

Suppose now that \( \Gamma \) is a set of positive roots such that for \( \alpha, \beta \in \Gamma \) with \( \alpha \neq \beta \), either \( \langle \alpha, \beta \rangle \leq -1 \) or \( \langle \alpha, \beta \rangle = -\cos(\pi/m_{\alpha,\beta}) \) for some integer \( m_{\alpha,\beta} \geq 2 \). Let \( \tilde{S} \) be a set in one-one correspondence with \( \Gamma \), and for \( \gamma \in \Gamma \) denote the element of \( \tilde{S} \) corresponding to \( \gamma \) by \( \sigma_\gamma \). Further, let \( \tilde{W} \) denote the Coxeter group with distinguished generating set \( \tilde{S} \) and defining relations

\[
(\sigma_\alpha \sigma_\beta)^{m_{\alpha,\beta}} = 1 \quad \text{for all } \alpha, \beta \in \Gamma \text{ such that } \langle \alpha, \beta \rangle > -1,
\]
and let \( \tilde{V} \) be an \( \mathbb{R} \)-vector space with basis \( \tilde{\Pi} = \{ \tilde{\gamma} \mid \gamma \in \Gamma \} \). If the order of \( \sigma_\alpha \sigma_\beta \) equals \( m \), define \( (\tilde{\alpha}, \tilde{\beta}) = -\cos(\pi/m) \), while if \( \sigma_\alpha \sigma_\beta \) is of infinite order, define \( (\tilde{\alpha}, \tilde{\beta}) = \langle \alpha, \beta \rangle \leq -1 \). This determines a bilinear form on \( \tilde{V} \), and we get a standard geometric realization of \( \tilde{W} \) on \( \tilde{V} \) with

\[
(\tilde{\alpha}, \tilde{\beta}) = \langle \alpha, \beta \rangle \quad \text{for all } \alpha, \beta \in \Gamma.
\]

We show now that \( \pi: \sigma_\gamma \mapsto r_\gamma \) for \( \gamma \in \Gamma \) defines a group homomorphism from \( \tilde{W} \) to \( W_\Gamma \). Since clearly \( r_\alpha^2 = 1 \) for \( \alpha \in \Gamma \), it suffices to show that
\[(r_\alpha r_\beta)^m \text{ equals the identity for } \alpha, \beta \in \Gamma \text{ with } \alpha \neq \beta \text{ such that } \sigma_\alpha \sigma_\beta \text{ is of order } m. \text{ So let } \alpha, \beta \in \Gamma \text{ with } \langle \alpha, \beta \rangle = -\cos(\pi/m). \text{ Then (1.1)(i) yields that } (r_\alpha r_\beta)^m \text{ acts as the identity on the space spanned by } \alpha \text{ and } \beta. \text{ In particular, } (r_\alpha r_\beta)^m \cdot \alpha = \alpha, \text{ and thus } (r_\alpha r_\beta)^m r_\alpha (r_\beta r_\alpha)^m = r_\alpha; \text{ that is, } (r_\alpha r_\beta)^{2m} = 1. \text{ Now let } v \in V; \text{ then there exist } \lambda, \mu \in \mathbb{R} \text{ such that } (r_\alpha r_\beta)^m \cdot v \text{ equals } v + \lambda \alpha + \mu \beta, \text{ and }
\]
\[
v = (r_\alpha r_\beta)^{2m} \cdot v \\
= (r_\alpha r_\beta)^m \cdot (v + \lambda \alpha + \mu \beta) \\
= v + \lambda \alpha + \mu \beta + \lambda (r_\alpha r_\beta)^m \cdot \alpha + \mu (r_\alpha r_\beta)^m \cdot \beta \\
= v + 2(\lambda \alpha + \mu \beta).
\]

So \(\lambda \alpha + \mu \beta\) equals the zero vector, and thus \((r_\alpha r_\beta)^m \cdot v = v\) for all \(v \in V\).

It follows by faithfulness of the standard geometric realization that \((r_\alpha r_\beta)^m\) equals 1, as required. Note that \(\pi\) is certainly surjective since \(\pi(S) = S_\Gamma\).

Denote the root system of \(\tilde{W}\) in \(\tilde{V}\) by \(\tilde{\Phi}\), and define \(\psi: \tilde{V} \to V_\Gamma\) by linear extension of \(\psi(\tilde{\gamma}) = \gamma\) for \(\gamma \in \Gamma\). We show now that \(\psi\) maps \(\tilde{\Phi}\) onto \(\Phi_\Gamma\). First, let \(\alpha, \beta \in \Gamma\); then

\[
\psi(\sigma_\alpha \cdot \tilde{\beta}) = \psi(\tilde{\beta} - 2(\tilde{\alpha}, \tilde{\beta})\tilde{\alpha}) \\
= \psi(\tilde{\beta}) - 2(\tilde{\alpha}, \tilde{\beta}) \psi(\tilde{\alpha}) \\
= \beta - 2(\alpha, \beta) \alpha \\
= r_\alpha \cdot \beta.
\]

Since \(\tilde{\Pi}\) forms a basis for \(\tilde{V}\), this yields \(\psi(\tilde{s} \cdot \tilde{v}) = \pi(\tilde{s}) \cdot \psi(\tilde{v})\) for all \(\tilde{s} \in \tilde{S}\) and \(\tilde{v} \in \tilde{V}\), and an easy induction on the length of \(w\) gives \(\psi(w \cdot \tilde{v}) = \pi(w) \cdot \psi(\tilde{v})\) for all \(w \in \tilde{W}\) and \(v \in \tilde{V}\). As \(\pi\) is surjective and \(\psi(\tilde{\Pi}) = \Gamma\), it follows that

\[
\psi(\tilde{\Phi}) = \psi(\tilde{W} \cdot \tilde{\Pi}) = \pi(\tilde{W}) \cdot \psi(\tilde{\Pi}) = W_\Gamma \cdot \Gamma = \Phi_\Gamma.
\]

Denote the set of roots in \(\Phi_\Gamma\) which can be written as positive linear combinations of elements of \(\Gamma\) by \(\Phi_\Gamma^+\); since \(\Gamma\) consists of positive roots, it follows easily that \(\Phi_\Gamma^+\) is a subset of \(\Phi^+\). Furthermore, \(\psi(\Phi_\Gamma^+) \subseteq \Phi_\Gamma^+\) as \(\psi(\tilde{\Pi}) = \Gamma\), and symmetrically \(\psi(\Phi_\Gamma^-) \subseteq -\Phi_\Gamma^+\). Since \(\Phi_\Gamma = \psi(\tilde{\Phi}) = \psi(\Phi_\Gamma^+) \cup \psi(\Phi_\Gamma^-)\), this yields that \(\psi(\tilde{\Phi}) = \Phi_\Gamma^+\); therefore \(\Phi_\Gamma^+ = \Phi_\Gamma \cap \Phi^+, \text{ and } \Phi_\Gamma\) is the disjoint union of \(\Phi_\Gamma^+\) and \(-\Phi_\Gamma^+\).
Now let \( w \in \tilde{W} \setminus \{1\} \). Then there exists an \( \tilde{\alpha} \in \tilde{\Phi}^+ \) such that \( w \cdot \tilde{\alpha} \) is in \( \tilde{\Phi}^- \). Thus \( \psi(\tilde{\alpha}) \in \Phi^+_\Gamma \subseteq \Phi^+ \), and \( \pi(w) \cdot \psi(\tilde{\alpha}) = \psi(w \cdot \tilde{\alpha}) \in \Phi^+_\Gamma \subseteq \Phi^- \); hence \( \pi(w) \neq 1 \), and this shows that \( \pi \) is injective. Since \( \pi \) is also surjective, \( \pi \) is a group isomorphism, and thus \( W_\Gamma \) is a Coxeter group with distinguished generating set \( S_\Gamma \).

We have proved the following theorem, which is due to M.J.Dyer.

\begin{equation}
(1.7) \text{ THEOREM} \quad \text{Let } \Gamma \subseteq \Phi^+ \text{ such that for } \alpha, \beta \in \Gamma \text{ with } \alpha \neq \beta \text{ either } \langle \alpha, \beta \rangle \leq -1 \text{ or } \langle \alpha, \beta \rangle = -\cos(\pi/m) \text{ for some integer } m. \text{ Then } W_\Gamma \text{ is a Coxeter group with distinguished generating set } S_\Gamma.
\end{equation}

Note furthermore that \( \psi \) induces a bijection between \( \tilde{\Phi} \) and \( \Phi_\Gamma \). For, as we have already seen, \( \psi \) maps \( \tilde{\Phi} \) onto \( \Phi_\Gamma \), and it remains to show that \( \psi \) restricted to \( \tilde{\Phi} \) is one-one. Since \( \Psi(\tilde{\Phi}^+) \cap \psi(\tilde{\Phi}^-) = \emptyset \), and by symmetry of \( \tilde{\Phi}^+ \) and \( \tilde{\Phi}^- \), it suffices to show that \( \psi \) restricted to \( \tilde{\Phi}^+ \) is one-one. So let \( w, u \in \tilde{W} \) and \( \alpha, \beta \in \Gamma \) such that \( w \cdot \tilde{\alpha} \) and \( u \cdot \tilde{\beta} \) are positive and \( \psi(w \cdot \tilde{\alpha}) = \psi(u \cdot \tilde{\beta}) \); that is, \( \pi(w) \cdot \psi(\tilde{\alpha}) = \pi(u) \cdot \psi(\tilde{\beta}) \). Then

\[ \pi(wr_\alpha w^{-1}) = \pi(w)r_\alpha \pi(w)^{-1} = \pi(u) = \pi(u) \pi(w)^{-1} = \pi(u \sigma w^{-1}). \]

Since \( \pi \) is injective, this yields \( w \sigma w^{-1} = u \sigma u^{-1} \), and thus \( w \cdot \tilde{\alpha} = \pm u \cdot \tilde{\beta} \) by an earlier remark. As \( w \cdot \tilde{\alpha} \) and \( u \cdot \tilde{\beta} \) are both positive, we deduce that \( w \cdot \tilde{\alpha} = u \cdot \tilde{\beta} \), as required.

\begin{equation}
(1.8) \text{ PROPOSITION} \quad \text{Let } \Gamma \text{ be a set of positive roots as in (1.7), and define } \Phi_\Gamma \text{ to be the set of roots of the form } w \cdot \gamma \text{ with } w \in W_\Gamma \text{ and } \gamma \in \Gamma. \text{ Furthermore, let } \Phi^+_\Gamma \text{ denote the set of roots in } \Phi_\Gamma \text{ which can be written as positive linear combinations of roots in } \Gamma. \text{ Then } \Phi^+_\Gamma = \Phi_\Gamma \cap \Phi^+, \text{ and there exists a standard geometric realization of } W_\Gamma \text{ with root system } \tilde{\Phi} \text{ and bilinear form } (, ), \text{ and a bijection } \psi: \tilde{\Phi} \to \Phi_\Gamma \text{ such that}
\end{equation}

\begin{enumerate}
\item[(i)] \( \psi(\tilde{\Phi}^+) = \Phi^+_\Gamma \),
\item[(ii)] \( \psi(w \cdot \tilde{\alpha}) = w \cdot \psi(\tilde{\alpha}) \) for all \( w \in W_\Gamma \) and \( \tilde{\alpha} \in \tilde{\Phi} \), and
\item[(iii)] \( \langle \psi(\tilde{\alpha}), \psi(\tilde{\beta}) \rangle = (\tilde{\alpha}, \tilde{\beta}) \) for all \( \tilde{\alpha}, \tilde{\beta} \in \tilde{\Phi} \).
\end{enumerate}

Note that \( \Gamma \) does not necessarily have to be linearly independent. For example, suppose that \( W \) has the following Coxeter graph:
Then
\[ \Gamma = \{ \alpha_a + \alpha_b, \alpha_a + \alpha_b + 2\alpha_c, \alpha_d + \alpha_e, 2\alpha_c + \alpha_d + \alpha_e \} \]
is certainly not linearly independent, but it can be easily checked that for
distinct roots \( \alpha, \beta \in \Gamma \), either \( \langle \alpha, \beta \rangle = 0 \) or \( \langle \alpha, \beta \rangle \leq -1 \).

If \( \Gamma \) is linearly dependent, the bilinear form \( (, ) \) on \( \tilde{V} \) is not positive
definite. For assume that there exist \( n > 0, m \geq 0 \) and pairwise distinct
roots \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \) in \( \Gamma \) as well as \( \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m > 0 \) such
that \( \sum_{i=1}^{n} \lambda_i \alpha_i - \sum_{j=1}^{m} \mu_j \beta_j \) equals the zero vector. Define \( \tilde{v} = \sum_{i=1}^{n} \lambda_i \tilde{\alpha}_i \);
this is a nonzero vector since \( \alpha_1, \ldots, \alpha_n \) are linearly independent and \( n \geq 1 \)
with \( \lambda_1 > 0 \). Then \( \psi(\tilde{v}) = \sum_{i=1}^{n} \lambda_i \alpha_i = \sum_{j=1}^{m} \mu_j \beta_j \), and thus
\[ \langle \tilde{v}, \tilde{v} \rangle = \langle \psi(\tilde{v}), \psi(\tilde{v}) \rangle = \sum_{i,j=1}^{n,m} \lambda_i \mu_j \langle \alpha_i, \beta_j \rangle \leq 0, \]
since \( \langle \alpha_i, \beta_j \rangle \leq 0 \) for all \( i \) and \( j \).

The elements of \( R = \{ r_\alpha \mid \alpha \in \Pi \} \) are called simple reflections. If \( J \)
is a set of simple reflections, \( W_J \) is called a parabolic subgroup of \( W \). This is
a Coxeter group with the obvious standard geometric realization on \( V_J \), the
space spanned by \( \alpha_r \) with \( r \in J \).

Denote the length function on \( W_J \) with respect to \( J \) by \( l_J \); we show
now that \( l_J(w) = l(w) \) for all \( w \in W_J \). For \( w \in W \) define
\[ N(w) = \{ \alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^- \}, \]
and observe that for \( u \) and \( w \) in \( W \),
\[ N(uw) \subseteq N(w) \cup w^{-1} \cdot N(u). \]
For if \( \gamma \in N(uw) \setminus N(w) \), then \( w \cdot \gamma \in \Phi^+ \) and \( u \cdot (w \cdot \gamma) = (uw) \cdot \gamma \in \Phi^- \);
thus \( w \cdot \gamma \in N(u) \), that is \( \gamma \in w^{-1} \cdot N(u) \). Note also that
\[ N(w) \cap w^{-1} \cdot N(u) = (w^{-1}w) \cdot (N(w) \cap w^{-1} \cdot N(u)) \]
\[ = w^{-1} \cdot (w \cdot N(w) \cap (ww^{-1}) \cdot N(u)) \]
\[ = w^{-1} \cdot (w \cdot N(w) \cap N(u)) \]
\[ \subseteq w^{-1}(\Phi^- \cap \Phi^+) = \emptyset, \]
whence the above union is disjoint.

If \( J \subseteq R \) and \( w \in W_J \), then \( N_J(w) \subseteq N(w) \), where \( N_J(w) \) denotes the set of all roots \( \alpha \in \Phi_J^+ \) such that \( w \cdot \alpha \in \Phi_J^- \). The next lemma yields that \( l_J(w) = |N_J(w)| \leq |N(w)| = l(w) \); since \( J \subseteq R \), the reverse inequality is certainly true, and so \( l_J(w) = l(w) \).

1.9 Lemma \(|N(w)| = l(w)\) for all \( w \in W \).

Proof. We use induction on \( l(w) \). If \( l(w) = 0 \), then \( w = 1 \) and \( N(1) = \emptyset \). So suppose that \( l(w) \geq 1 \), and let \( u \in W \) and \( r \in R \) such that \( w = ur \) and \( l(u) = l(w) - 1 \); then \( |N(u)| = l(u) \) by induction. By the above remark,

\[
N(w) = N(wr) \subseteq N(r) \cup r \cdot N(u) = \{ \alpha_r \} \cup r \cdot N(u),
\]

and this union is disjoint. In order to prove that \( |N(w)| = l(w) \), it suffices to show that \( \{ \alpha_r \} \) and \( r \cdot N(u) \) are subsets of \( N(w) \). By (1.5) we know that \( w \cdot \alpha_r \) is negative and \( u \cdot \alpha_r \) is positive, and thus \( \{ \alpha_r \} \subseteq N(w) \) and \( \alpha_r \notin N(u) \).

The latter together with (1.4) yield that \( r \cdot N(u) \) contains only positive roots, and since \( w \cdot (r \cdot N(u)) = u \cdot N(u) \subseteq \Phi^- \), we conclude that \( r \cdot N(u) \) is a subset of \( N(w) \), as required. \( \square \)

Now let \( w \in W \) and \( \alpha \in \Phi^+ \) with \( w \cdot \alpha \in \Phi^- \). Then \( -w \cdot \alpha \in \Phi^+ \) and \( w^{-1} \cdot (-w \cdot \alpha) = -\alpha \in \Phi^- \); that is, \( -w \cdot \alpha \in N(w^{-1}) \). So \( -w \cdot N(w) \subseteq N(w^{-1}) \), and since \( l(w) = l(w^{-1}) \), we deduce the following:

1.10 Corollary \( -w \cdot N(w) = N(w^{-1}) \) for all \( w \in W \).

Suppose now that we have \( u, w \in W \) with \( l(uw) = l(u) + l(w) \). Then \( N(uw) \) is a subset of \( N(w) \cup w^{-1} \cdot N(u) \); since \( l(uw) = |N(uw)| \), and this equals \( |N(u)| + |N(w)| \), it follows that \( N(uw) = N(w) \cup w^{-1} \cdot N(u) \). In particular, \( w^{-1} \cdot N(u) \subseteq \Phi^+ \), and thus necessarily \( N(u) \cap N(w^{-1}) = \emptyset \).

Conversely, suppose that \( N(u) \cap N(w^{-1}) = \emptyset \). Then \( w^{-1} \cdot N(u) \subseteq \Phi^+ \), and

\[
(uw) \cdot (w^{-1} \cdot N(u)) = u \cdot N(u) \subseteq \Phi^-;
\]
whence \( w^{-1} \cdot N(u) \subseteq N(uw) \). Further, \( N(w^{-1}) \subseteq \Phi^+ \setminus N(u) \), and so (1.10) yields that

\[
(uw) \cdot N(w) = u \cdot (-N(w^{-1})) = -u \cdot N(w^{-1}) \subseteq -u \cdot (\Phi^+ \setminus N(u)) \subseteq \Phi^-;
\]
hence also \( N(w) \subseteq N(uw) \). Therefore \( N(uw) = N(w) \cup w^{-1} \cdot N(u) \), and since this union is disjoint, we have \( l(uw) = l(u) + l(w) \).
Chapter 1  Preliminaries

(1.11) **Lemma**  Let \( u, w \in W \). Then the following are equivalent:

(i) \( l(uw) = l(u) + l(w) \)

(ii) \( N(uw) = N(w) \cup w^{-1} \cdot N(u) \)

(iii) \( N(u) \cap N(w^{-1}) = \emptyset \).

We conclude this chapter with a proposition whose proof is outlined in [1], Ch.V, §4.

(1.12) **Proposition**  Suppose \( H \) is a finite subgroup of \( W \). Then there exist a finite parabolic subgroup \( W_J \) of \( W \) such that \( H \) is conjugate to a subgroup of \( W_J \).

**Proof.** Since \( H \) is finite, we may assume without loss of generality that \( R \) is finite. If \( |W| < \infty \), the assertion is true with \( J = R \). So suppose from now on that \( W \) is infinite; then \( \Phi \) is infinite and \( |R| > 1 \), and we proceed by induction on \( |R| \).

Let \( V^* \) denote the dual space of \( V \), and let \( \{ \delta_\alpha \mid \alpha \in \Pi \} \) be the basis dual to \( \Pi \); that is, \( \delta_\alpha(v) \) equals the coefficient of \( \alpha \) in \( v \). For \( f \in V^* \) and \( w \in W \), we define \( f(w) \in V^* \) by \( f(w): v \mapsto f(w \cdot v) \); this determines a right action of \( W \) on \( V^* \). For \( f \in V^* \) define further \( S(f) = \{ \gamma \in \Phi^+ \mid f(\gamma) < 0 \} \).

Let \( f = \sum_{\alpha \in \Pi} \delta_\alpha \) and \( F = \sum_{h \in H} fh \). Then \( f(\alpha) = 1 \) for all \( \alpha \in \Pi \), and thus

\[
(\gamma) > 0 \text{ for all } \gamma \in \Phi^+. \tag{*}
\]

Now let \( A = \bigcup_{h \in H} N(h) \). Since \( H \) is finite and \( N(h) \) is finite for all \( h \in H \), \( A \) is finite. Furthermore, for all \( \gamma \in \Phi^+ \setminus A \),

\[
F(\gamma) = \sum_{h \in H} (fh)(\gamma) = \sum_{h \in H} f(h \cdot \gamma) > 0 \tag{**}
\]

by (*), since \( h \cdot \gamma \in \Phi^+ \) for all \( h \in H \). So \( S(F) \cap (\Phi^+ \setminus A) = \emptyset \); that is, \( S(F) \subseteq A \), and in particular, \( S(F) \) must be finite.

Let \( x \in W \) such that \( |S(Fx)| \) is minimal, and assume for a contradiction that \( S(Fx) \neq \emptyset \). Then \( (Fx)(\gamma) < 0 \) for some \( \gamma \in \Phi^+ \), and it follows that \( (Fx)(\alpha_r) < 0 \) for some \( r \in R \); that is, \( \alpha_r \in S(Fx) \). Now

\[
(Fx)(\alpha_r) = (Fx)(r \cdot \alpha_r) = (Fx)(-\alpha_r) = -(Fx)(\alpha_r) > 0,
\]


and hence $\alpha_r \not\in S(Fxr)$. So $\gamma \mapsto r \cdot \gamma$ maps $S(Fxr)$ into $\Phi^+ \setminus \{\alpha_r\}$, and if $\gamma \in S(Fxr)$, then $(Fx)(r \cdot \gamma) = (Fxr)(\gamma) < 0$; therefore $\gamma \mapsto r \cdot \gamma$ maps $S(Fxr)$ into $S(Fx) \setminus \{\alpha_r\}$. Since the above map is clearly one-one, this yields $|S(Fxr)| \leq |S(Fx) \setminus \{\alpha_r\}| < |S(Fx)|$, contrary to the choice of $x$. Thus $S(Fx) = \emptyset$.

Since $Fh = F$ for $h \in H$, we deduce that

$$Fx(x^{-1}hx) = (Fx x^{-1})hx = Fhx = Fx;$$

so $x^{-1}Hx \subseteq \{y \in W \mid (Fx)y = Fx\}$, and we show now that

$$\{y \in W \mid (Fx)y = Fx\} \subseteq W_I,$$

where $I = \{r \in R \mid (Fx)(\alpha_r) = 0\}$. Let $y \in W$ with $(Fx)y = Fx$. If $l(y) = 0$, then $y = 1 \in W_I$. Proceeding by induction, suppose $l(y) \geq 1$, and let $z \in W$ and $s \in R$ such that $y = zs$ and $l(z) = l(y) - 1$. Then $z \cdot \alpha_s \in \Phi^+$ by (1.5), and

$$(Fx)(\alpha_s) = (Fx y)(\alpha_s) = (Fx)(y \cdot \alpha_s) = -(Fx)((ys) \cdot \alpha_s) = -(Fx)(z \cdot \alpha_s);$$

since $S(Fx) = \emptyset$, it follows that $(Fx)(\alpha_s) = 0$, and thus $s \in I$. So

$$(Fx s)(v) = (Fx)(v - 2\langle v, \alpha_s \rangle \alpha_s) = (Fx)(v) - 2\langle v, \alpha_s \rangle (Fx)(\alpha_s) = (Fx)(v)$$

for all $v \in V$, which yields $(Fx)z = (Fx y)s = (Fx)s = Fx$; hence $z \in W_I$ by induction, and $y = zs \in W_I$, as desired.

Since $\Phi^+$ is infinite by construction, and $A$ is finite, it follows that $\Phi^+ \setminus A$ is nonempty; so $F \neq 0$ by (**), which yields $Fx \neq 0$ and $I \neq R$. By induction there exist $J \subseteq I$ and $u \in W_I$ such that $W_J$ is finite and $u^{-1}(x^{-1}Hx)u \subseteq W_J$, as desired. □

Chapter 2

The Depth of a Root
To facilitate inductive proofs of facts about root systems, it is convenient for us to introduce a concept which, in some sense, measures how far a root is from being simple. For a positive root $\alpha$, we define the depth of $\alpha$ to be
\[
dp(\alpha) = \min \{ l \in \mathbb{N}_0 \mid w \cdot \alpha \in \Phi^- \text{ for some } w \in W \text{ with } l(w) = l \}.
\]

Observe that such an integer always exists, since by definition of the root system every root $\alpha$ has the form $u \cdot \alpha_r$ for some $u \in W$ and $r \in R$, and then $(ru^{-1}) \cdot \alpha = -\alpha_r \in \Phi^-$. Suppose now that $\alpha \in \Phi^+$, and let $w \in W$ with $l(w) = \dp(\alpha)$ such that $w \cdot \alpha \in \Phi^-$. Then $w \neq 1$, and hence there exists an $r \in R$ and a $u \in W$ with $w = ru^{-1}$ and $l(w) = l(u^{-1}) + 1 = l(u) + 1$. Minimality of $w$ yields that $u^{-1} \cdot \alpha$ is positive, and since $r \cdot (u^{-1} \cdot \alpha) = w \cdot \alpha$ is negative, (1.4) gives $u^{-1} \cdot \alpha = \alpha_r$; that is, $\alpha = u \cdot \alpha_r$. Thus every positive root can be written as $u \cdot \alpha_r$ with $l(u)$ equal to the depth of the root minus 1. Moreover, the depth of $\alpha$ equals the minimal integer $l$ such that $w \cdot \alpha \in -\Pi$ for some $w \in W$ of length $l$. For a negative root $\beta$, we define the depth of $\beta$ to be
\[
dp(\beta) = -\min \{ l \in \mathbb{N}_0 \mid w \cdot \beta \in -\Pi \text{ for some } w \in W \text{ with } l(w) = l \}.
\]

Note that for $r \in R$ and $\alpha \in \Phi$, clearly $\dp(\alpha) - 1 \leq \dp(r \cdot \alpha) \leq \dp(\alpha) + 1$. Furthermore, if $\alpha$ is a positive root, then there exists an $r \in R$ such that $\dp(r \cdot \alpha) = \dp(\alpha) - 1$. Also, for $w \in W$ and $r \in R$ clearly $\dp(w \cdot \alpha_r) \leq l(w) + 1$. Finally, note that $\dp(\alpha) \leq l(w)$ if $w \cdot \alpha$ is negative for some root $\alpha$ and $w \in W$; for if $\alpha$ is positive, this follows by definition of the depth of $\alpha$, and if $\alpha$ is negative, then $\dp(\alpha) \leq 0 \leq l(w)$.

Now let $\alpha \in \Phi^+$ and $w \in W$, $r \in R$ such that $\alpha$ equals $w \cdot \alpha_r$ and $l(w) = \dp(\alpha) - 1$. Then $-\alpha = -w \cdot \alpha_r$, and so $w^{-1} \cdot (-\alpha) = -\alpha_r$ and $l(w) = l(w^{-1}) \geq -\dp(-\alpha)$. On the other hand, if $u \in W$ and $s \in R$ such that $l(u) = -\dp(-\alpha)$ and $u \cdot (-\alpha) = -\alpha_s$, then $(su) \cdot \alpha = -\alpha_s$; thus
\[
dp(\alpha) \leq l(su) \leq l(u) + 1 = -\dp(-\alpha) + 1 \leq l(w) + 1 = \dp(\alpha),
\]
and we must have equality everywhere. In particular, $\dp(\alpha) + \dp(-\alpha) = 1$. Since for $\beta \in \Phi^-$ clearly $\alpha = -\beta \in \Phi^+$, this proves the following result:

(2.13) **Lemma** \( \dp(\alpha) + \dp(-\alpha) = 1 \) for all $\alpha \in \Phi$.

(2.14) **Proposition** Let $r \in R$ and $\alpha \in \Phi$. Then
\[
\dp(r \cdot \alpha) = \begin{cases} 
\dp(\alpha) - 1 & \text{if } \langle \alpha, \alpha_r \rangle > 0, \\
\dp(\alpha) & \text{if } \langle \alpha, \alpha_r \rangle = 0, \\
\dp(\alpha) + 1 & \text{if } \langle \alpha, \alpha_r \rangle < 0.
\end{cases}
\]
Chapter 2  The depth of a root

Proof. If the assertion is proved for positive roots, (2.13) yields for \( \alpha \in \Phi^− \),
\[
dp(r \cdot \alpha) = -dp(-(r \cdot \alpha)) + 1 = -dp(r \cdot (-\alpha)) + 1
\]
\[
= \begin{cases} 
-\left(dp(-\alpha) - 1\right) + 1 & \text{if } \langle \alpha, \alpha_r \rangle > 0, \\
-\left(dp(-\alpha) + 1\right) & \text{if } \langle \alpha, \alpha_r \rangle = 0, \\
-\left(dp(-\alpha) + 1\right) + 1 & \text{if } \langle \alpha, \alpha_r \rangle < 0,
\end{cases}
\]
\[
= \begin{cases} 
dp(\alpha) & \text{if } \langle \alpha, \alpha_r \rangle = 0, \\
dp(\alpha) - 1 & \text{if } \langle \alpha, \alpha_r \rangle > 0.
\end{cases}
\]

Hence it suffices to prove the proposition for positive roots. If \( \langle \alpha, \alpha_r \rangle = 0 \), then \( r \cdot \alpha = \alpha - 2\langle \alpha, \alpha_r \rangle \alpha_r = \alpha \), and trivially \( dp(r \cdot \alpha) = dp(\alpha) \).

Suppose next that \( \langle \alpha, \alpha_r \rangle > 0 \). It suffices to show that \( dp(r \cdot \alpha) < dp(\alpha) \); to do so, we construct a \( w \in W \) with \( w \cdot (r \cdot \alpha) \in \Phi^− \) and \( l(w) < dp(\alpha) \). Choose \( u \in W \) such that \( u \cdot \alpha \in \Phi^− \) and \( l(u) = dp(\alpha) \). If \( u \cdot \alpha_r \) is negative, set \( w = ur \); then \( l(w) = l(u) - 1 \) by (1.5), and \( w \cdot (r \cdot \alpha) = u \cdot \alpha \in \Phi^− \), as required. Hence we may assume from now on that \( u \cdot \alpha_r \) is positive. Clearly \( u \neq 1 \) (as \( u \cdot \alpha \) is negative, while \( \alpha \) is positive), and thus there exist \( s \in R \) and \( w \in W \) with \( u = sw \) and \( l(u) = l(w) + 1 \). Now
\[
u \cdot (r \cdot \alpha) = u \cdot (\alpha - 2\langle \alpha, \alpha_r \rangle \alpha_r) = u \cdot \alpha - 2\langle \alpha, \alpha_r \rangle u \cdot \alpha_r
\]
is negative, and since \( u \cdot \alpha \) and \( -2\langle \alpha, \alpha_r \rangle u \cdot \alpha_r \) are both negative linear combinations of simple roots and not scalar multiples of each other (since \( \alpha \) and \( \alpha_r \) are linearly independent), \( u \cdot (r \cdot \alpha) \) cannot be equal to \( -\alpha_s \). It follows by (1.4) that \( w \cdot (r \cdot \alpha) = s \cdot (u \cdot (r \cdot \alpha)) \in \Phi^− \).

Finally, suppose that \( \langle \alpha, \alpha_r \rangle < 0 \). Then \( \langle r \cdot \alpha, \alpha_r \rangle = -\langle \alpha, \alpha_r \rangle > 0 \), and the preceding paragraph shows that \( dp(\alpha) = dp(r \cdot (r \cdot \alpha)) = dp(r \cdot \alpha) - 1 \).

\[\Box\]

(2.15) Lemma Let \( \alpha \in \Phi^+ \) and \( w \in W \), \( r \in R \) such that \( \alpha = w \cdot \alpha_r \) and \( l(w) = dp(\alpha) - 1 \). Then \( w \in W_{I(\alpha)} \) and \( r \in I(\alpha) \), where \( I(\alpha) \) denotes the set of simple reflections in one-one correspondence with the support of \( \alpha \).

Proof. If \( dp(\alpha) = 1 \), the assertion is trivially true. Suppose next that \( \alpha \) is of depth greater than 1, and let \( u \in W \) and \( s \in R \) with \( w = su \) and \( l(w) = l(u) + 1 \). Then \( u \cdot \alpha_r = s \cdot \alpha \), and thus
\[
dp(s \cdot \alpha) = dp(u \cdot \alpha_r) \leq l(u) + 1 = l(w) = dp(\alpha) - 1;
\]

\[\Box\]
this yields that \( dp(s \cdot \alpha) = dp(\alpha) - 1 \), and therefore \( \langle \alpha, \alpha_s \rangle > 0 \). Since \( \alpha \) is a positive root and \( \langle \alpha_s, \alpha_t \rangle \leq 0 \) for \( t \in R \setminus \{ s \} \), we deduce that \( s \in I(\alpha) \).

Further, as \( s \cdot \alpha = u \cdot \alpha_r \) and \( l(u) = dp(s \cdot \alpha) - 1 \), induction yields that \( u \in W_I(s \cdot \alpha) \) and \( r \in I(s \cdot \alpha) \). By definition of the action of \( s \) on \( V \), it follows that \( I(s \cdot \alpha) \subseteq I(\alpha) \cup \{ s \} = I(\alpha) \), and thus \( w = su \) is in \( W_I(\alpha) \) and \( r \in I(\alpha) \).

\[
\tag{2.16}
\]

(2.16) **Corollary**  Let \( J \subseteq R \). Then \( \Phi_J = \Phi \cap V_J \) and \( dp_J(\alpha) = dp(\alpha) \) for all \( \alpha \in \Phi_J \), where \( dp_J \) denotes the depth function on \( \Phi_J \) with respect to \( W_J \) and \( l_J \).

It is clear that the part of the previous assertion concerning the depth in parabolic subsystems is not in general true for reflection subgroups \( W_\Gamma \); for if \( \Gamma = \{ \alpha \} \), then \( \alpha \) has depth 1 with respect to \( \Gamma \), independent of \( dp(\alpha) \).

Note furthermore that for \( \Gamma \subseteq \Phi \), we do not have in general \( \Phi_\Gamma = \Phi \cap V_\Gamma \). For example, suppose that \( W \) has the following Coxeter graph

\[
\begin{array}{cccc}
\infty & 6 \\
r & s & t & u
\end{array}
\]

with \( \langle \alpha_r, \alpha_s \rangle = -1 \). Define \( \alpha = tust \cdot \alpha_s \) and \( \Gamma = \{ \alpha, \alpha_r, \alpha_u \} \); then

\[
\begin{align*}
t \cdot \alpha_s &= \alpha_s + \sqrt{3} \alpha_t, \\
st \cdot \alpha_s &= 2 \alpha_s + \sqrt{3} \alpha_t, \\
ust \cdot \alpha_s &= 2 \alpha_s + \sqrt{3} \alpha_t + \sqrt{3} \alpha_u, \\
tust \cdot \alpha_s &= 2 \alpha_s + 2 \sqrt{3} \alpha_t + \sqrt{3} \alpha_u,
\end{align*}
\]

and thus \( \alpha = 2 \alpha_s + 2 \sqrt{3} \alpha_t + \sqrt{3} \alpha_u \). Now \( \langle \alpha, \alpha_r \rangle = -2 \leq -1 \), \( \langle \alpha, \alpha_u \rangle = 0 \) and \( \langle \alpha_r, \alpha_u \rangle = 0 \). It follows by (1.7) that \( W_\Gamma \) is a Coxeter group with distinguished generating set \( S_\Gamma \), and by (1.8) we know further that \( \Phi_\Gamma \) is the union of \( \Phi_\Gamma^+ \) and \( \Phi_\Gamma^- \); where \( \Phi_\Gamma^+ \) denotes the set of the roots in \( \Phi_\Gamma \) that can be written as positive linear combinations of elements of \( \Gamma \). Now let \( \beta = ts \cdot \alpha_r \); then

\[
\beta = t \cdot (\alpha_r + 2 \alpha_s) = \alpha_r + 2 \alpha_s + 2 \sqrt{3} \alpha_t,
\]

and this is equal to \( \alpha + \alpha_r - \sqrt{3} \alpha_u \). Hence \( \beta \) is an element of \( V_\Gamma \); but since \( \Gamma \) is linearly independent, \( \beta \) is neither in \( \Phi_\Gamma^+ \) nor in \( \Phi_\Gamma^- \), and thus \( \beta \) cannot be an element of \( \Phi_\Gamma \).
Chapter 2  

The depth of a root

18

Note that the explicit calculation of $\alpha$ above together with the previous proposition imply that $\alpha$ is of depth 5; for, from each step to the next, application of a simple reflection increases the corresponding coefficient, and this indicates that the depth increases by 1. Similarly, $\gamma = u\alpha \cdot \alpha_r$ is of depth 4, since

$$s \cdot \alpha_r = \alpha_r + 2\alpha_s,$$

$$t_\alpha \cdot \alpha_r = \alpha_r + 2\alpha_s + 2\sqrt{3}\alpha_t,$$

$$uts \cdot \alpha_r = \alpha_r + 2\alpha_s + 2\sqrt{3}\alpha_t + 2\sqrt{3}\alpha_u.$$

Observe that although $\alpha$ is of depth greater than $dp(\gamma)$, all coefficients in $\alpha$ are less than or equal to the corresponding coefficients in $\gamma$.

In general it is a rather tedious task to calculate the depth of a root using (2.14). In order to find a noninductive formula for the depth of a root, we define for $v \in V$:

$$V_+(v) = \{v' \in V \mid \langle v, v' \rangle > 0\}, \quad V_-(v) = \{v' \in V \mid \langle v, v' \rangle < 0\}$$

and $V_0(v) = \{v' \in V \mid \langle v, v' \rangle = 0\}$.

Then $V$ is the disjoint union of $V_+(v)$, $V_-(v)$ and $V_0(v)$, and it is clear that $V_+(v) = V_+(v) = V_-(v)$ and $V_0(v) = V_0(-v) = V_0(v)$. Furthermore,

$$\langle w^{-1} \cdot v', v \rangle = \langle w \cdot (w^{-1} \cdot v'), w \cdot v \rangle = \langle v', w \cdot v \rangle$$

for all $v' \in V$, and thus

$$V_+(w \cdot v) = w \cdot V_+(v), \quad V_-(w \cdot v) = w \cdot V_-(v) \quad \text{and} \quad V_0(w \cdot v) = w \cdot V_0(v).$$

Next, define for $v \in V$ and $w \in W$,

$$N_+(w, v) = N(w) \cap V_+(v), \quad N_-(w, v) = N(w) \cap V_-(v)$$

and $N_0(w, v) = N(w) \cap V_0(v)$.

By the above, $N(w)$ is the disjoint union of $N_+(w, v)$, $N_-(w, v)$ and $N_0(w, v)$ and, furthermore, $N_+(w, v) = N_-(w, -v)$ and $N_0(w, v) = N_0(w, -v)$.

Since $N(uw)$ is contained in $N(w) \cup w^{-1} \cdot N(u)$, it follows that

$$N_+(uw, v) = N(uw) \cap V_+(v)$$

$$\subseteq (N(w) \cup w^{-1} \cdot N(u)) \cap V_+(v)$$

$$= (N(w) \cap V_+(v)) \cup (w^{-1} \cdot N(u) \cap V_+(v))$$

$$= N_+(w, v) \cup w^{-1} \cdot (N(u) \cap w \cdot V_+(v)).$$
Now \( w \cdot V_+(v) = V_+(w \cdot v) \), and thus
\[
N_+(uw, v) \subseteq N_+(w, v) \cup w^{-1} \cdot (N(u) \cap V_+(w \cdot v))
= N+(w, v) \cup w^{-1} \cdot N_+(u, w \cdot v).
\]
Since by (1.11) further \( N(uw) = N(w) \cup w^{-1} \cdot N(u) \) if \( l(uw) = l(u) + l(w) \), this and similar arguments yield the next lemma, which will enable us to give a formula for the depth of \( w \cdot \alpha \).

\[\text{(2.17) Lemma} \quad \text{Let} \ v \in V \text{ and } u, w \in W. \text{ Then}
\]
\[
N_+(uw, v) \subseteq N_+(w, v) \cup w^{-1} \cdot N_+(u, w \cdot v),
\]
\[
N_-(uw, v) \subseteq N_-(w, v) \cup w^{-1} \cdot N_-(u, w \cdot v),
\]
\[
N_0(uw, v) \subseteq N_0(w, v) \cup w^{-1} \cdot N_0(u, w \cdot v),
\]
and these unions are disjoint. Moreover, if \( l(uw) = l(u) + l(w) \) we have equality; hence in particular, \( N_+(w, v) \subseteq N_+(uw, v) \), \( N_-(w, v) \subseteq N_-(uw, v) \) and \( N_0(w, v) \subseteq N_0(uw, v) \), and furthermore, \( N_+(u, w \cdot v) \subseteq w \cdot N_+(uw, v) \), \( N_-(u, w \cdot v) \subseteq w \cdot N_-(uw, v) \) and \( N_0(u, w \cdot v) \subseteq w \cdot N_0(uw, v) \).

\[\text{(2.18) Proposition} \quad \text{Let} \ \alpha \in \Phi \text{ and } w \in W. \text{ Then}
\]
\[
\text{dp}(w \cdot \alpha) = \text{dp}(\alpha) + |N_-(w, \alpha)| - |N_+(w, \alpha)|.
\]

**Proof.** The assertion is trivial if \( l(w) = 0 \). So suppose \( l(w) \geq 1 \), and let \( u \in W \) and \( r \in R \) with \( w = ru \) and \( l(w) = l(u) + 1 \). Then by (2.17),
\[
N_+(w, \alpha) = N_+(u, \alpha) \cup u^{-1} \cdot N_+(r, u \cdot \alpha) = N_+(u, \alpha) \cup u^{-1} \cdot \{\alpha_r\} \cap V_+(u \cdot \alpha)
\]
and
\[
N_-(w, \alpha) = N_-(u, \alpha) \cup u^{-1} \cdot N_-(r, u \cdot \alpha) = N_-(u, \alpha) \cup u^{-1} \cdot \{\alpha_r\} \cap V_+(u \cdot \alpha),
\]
and these unions are disjoint. So
\[
|N_+(w, \alpha)| - |N_+(u, \alpha)| = \begin{cases} 0 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle \leq 0, \\ 1 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle > 0, \end{cases}
\]
and
\[
|N_-(w, \alpha)| - |N_-(u, \alpha)| = \begin{cases} 1 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle < 0, \\ 0 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle \geq 0. \end{cases}
\]
Using (2.14), we deduce that
\[
\text{dp}(w \cdot \alpha) = \text{dp}(u \cdot \alpha) + (|N_-(w, \alpha)| - |N_-(u, \alpha)|) - (|N_+(w, \alpha)| - |N_+(u, \alpha)|),
\]
and induction finishes the proof.
Chapter 2

The depth of a root

If \( \alpha \) and \( \beta \) are roots, and \( w \in W \) with \( w \cdot \alpha = \beta \), Proposition (2.18) yields that \( l(w) \geq \text{dp}(\beta) - \text{dp}(\alpha) \), with equality if and only if \( N(w) \subseteq V_-(\alpha) \). In particular, if \( \alpha \) is positive and \( \beta \) equals \( -\alpha \), then

\[
l(r_\alpha) \geq \text{dp}(\alpha) - \text{dp}(-\alpha) = 2 \text{dp}(\alpha) - 1.
\]

On the other hand, if \( w \in W \) and \( r \in R \) with \( \alpha = w \cdot \alpha_r \) and \( l(w) = \text{dp}(\alpha) - 1 \), then \( r_\alpha = wrw^{-1} \), and so \( l(r_\alpha) \leq l(w) + 1 + l(w) = 2 \text{dp}(\alpha) - 1 \).

(2.19) COROLLARY \( l(r_\alpha) = 2 \text{dp}(\alpha) - 1 \) for all \( \alpha \in \Phi^+ \).

We can now generalize (2.14).

(2.20) LEMMA \( \) Let \( \alpha \in \Phi \) and \( \beta \in \Phi^+ \). Then

\[
\text{dp}(r_\beta \cdot \alpha) \begin{cases} < \text{dp}(\alpha) & \text{if } \langle \alpha, \beta \rangle > 0, \\ = \text{dp}(\alpha) & \text{if } \langle \alpha, \beta \rangle = 0, \\ > \text{dp}(\alpha) & \text{if } \langle \alpha, \beta \rangle < 0. \end{cases}
\]

Proof. First, suppose that \( \langle \alpha, \beta \rangle = 0 \). Then \( r_\beta \cdot \alpha = \alpha - 2\langle \alpha, \beta \rangle \beta = \alpha \), and hence trivially \( \text{dp}(r_\beta \cdot \alpha) = \text{dp}(\alpha) \).

Now let \( \langle \alpha, \beta \rangle > 0 \). Corollary (1.10) states that \( \gamma \mapsto -r_\beta \cdot \gamma \) defines a one-one correspondence on \( N(r_\beta) \). If \( \gamma \in N_-(r_\beta, \alpha) \), then

\[
\langle \alpha, -r_\beta \cdot \gamma \rangle = -\left( \langle \alpha, \gamma \rangle - 2\langle \gamma, \beta \rangle \langle \alpha, \beta \rangle \right) = -\langle \alpha, \gamma \rangle + 2\langle \gamma, \beta \rangle \langle \alpha, \beta \rangle,
\]

and this is greater than 0; for \( \langle \gamma, \beta \rangle > 0 \) as \( \gamma \in N(r_\beta) \), \( \langle \alpha, \beta \rangle > 0 \) by hypothesis and \( \langle \alpha, \gamma \rangle < 0 \) by choice of \( \gamma \). So \( \gamma \mapsto -r_\beta \cdot \gamma \) embeds \( N_-(r_\beta, \alpha) \) into \( N_+(r_\beta, \alpha) \). But \( \beta \) is in \( N_+(r_\beta, \alpha) \) by hypothesis, and since \( \beta = -r_\beta \cdot \beta \) clearly \( \beta \notin -r_\beta \cdot N_-(r_\beta, \alpha) \). So \( |N_-(r_\beta, \alpha)| < |N_+(r_\beta, \alpha)| \), and hence by (2.18),

\[
\text{dp}(r_\beta \cdot \alpha) = \text{dp}(\alpha) + |N_-(r_\beta, \alpha)| - |N_+(r_\beta, \alpha)| < \text{dp}(\alpha).
\]

Finally, suppose that \( \langle \alpha, \beta \rangle < 0 \). Then \( \langle r_\beta \cdot \alpha, \beta \rangle = -\langle \alpha, \beta \rangle > 0 \), and thus by the preceding paragraph, \( \text{dp}(\alpha) = \text{dp}(r_\beta \cdot (r_\beta \cdot \alpha)) < \text{dp}(r_\beta \cdot \alpha) \). \( \square \)
We conclude this chapter with some first applications of inductive proofs on the depth. First we give an alternative proof of the next theorem, which was proved independently by V. V. Deodhar [4] and M. J. Dyer [5]. Our proof is closely related to Dyer’s proof, translating his ideas into the context of root systems.

(2.21) Theorem Let $\Gamma \subseteq \Phi$, and let $\Psi$ be the set of all roots $\alpha$ in $\Phi_\Gamma \cap \Phi^+$ such that

$$N(r_\alpha) \cap \Phi_\Gamma = \{\alpha\}.$$ 

Then $W_\Gamma = W_\Psi$, and for $\alpha, \beta \in \Psi$ with $\alpha \neq \beta$, either $\langle \alpha, \beta \rangle \leq -1$ or $\langle \alpha, \beta \rangle = -\cos(\pi/m)$ for some integer $m$. Thus $W_\Gamma$ is a Coxeter group with distinguished generating set $S_\Psi$.

The following technical lemma is a vital tool in our proof of (2.21).

(2.22) Lemma Suppose $\alpha$ and $\beta$ are two positive roots with $\alpha \neq \beta$ and $\text{dp}(\alpha) \leq \text{dp}(\beta)$. Then $\text{dp}(-r_\alpha \cdot \beta) < \text{dp}(\alpha)$.

Proof. Let $w \in W$ and $r \in R$ with $\alpha = w \cdot \alpha_r$ and $l(w) = \text{dp}(\alpha) - 1$. Then $w^{-1} \cdot \beta \in \Phi^+$, since $l(w^{-1}) = l(w)$ and this is less than the depth of $\beta$. Furthermore, $w^{-1} \cdot \beta \neq \alpha_r$ since $\alpha$ and $\beta$ are distinct, and thus $rw^{-1} \cdot \beta \in \Phi^+$ by (1.4). Now

$$w^{-1} \cdot (-r_\alpha \cdot \beta) = w^{-1} \cdot (-wrw^{-1} \cdot \beta) = -(rw^{-1} \cdot \beta) \in \Phi^-,$$

and so $\text{dp}(-r_\alpha \cdot \beta) \leq l(w^{-1}) < \text{dp}(\alpha)$, as required. \qed

Proof of (2.9). Clearly $\Psi \subseteq \Phi_\Gamma$, and thus $\Phi_\Psi \subseteq \Phi_\Gamma$. Assume now for a contradiction that $\Phi_\Gamma \neq \Phi_\Psi$; then $(\Phi_\Gamma \setminus \Phi_\Psi) \cap \Phi^+ \neq \emptyset$, and thus there exists a positive root $\gamma$ in $\Phi_\Gamma \setminus \Phi_\Psi$ of minimal depth. In particular, $\gamma \notin \Psi$, and hence

$$(N(r_\gamma) \cap \Phi_\Gamma) \setminus \{\gamma\} \neq \emptyset.$$ 

Let $\beta$ be an element of the above set of minimal depth; then $-r_\gamma \cdot \beta$ is also an element of $N(r_\gamma) \setminus \{\gamma\}$ by (1.10), and since $\beta$ and $\gamma$ are in $\Phi_\Gamma$ by construction, it follows that $-r_\gamma \cdot \beta$ is in $(N(r_\gamma) \cap \Phi_\Gamma) \setminus \{\gamma\}$. If $\text{dp}(\beta) \geq \text{dp}(\gamma)$, then

$$\text{dp}(-r_\gamma \cdot \beta) < \text{dp}(\gamma) \leq \text{dp}(\beta)$$

by (2.22), contradicting the minimality of $\beta$. So $\text{dp}(\gamma) > \text{dp}(\beta)$, and minimality of $\gamma$ forces $\beta \in \Phi_\Psi$. 

If \( r_\beta \cdot \gamma \) is positive, then \( dp(\gamma) > dp(r_\beta \cdot \gamma) \) by (2.20), since \( \beta \in N(r_\gamma) \) implies that \( \langle \gamma, \beta \rangle > 0 \); thus \( r_\beta \cdot \gamma \in \Phi_\Psi \) by minimality of \( \gamma \). If \( r_\beta \cdot \gamma \) is negative, then \( -r_\beta \cdot \gamma \) is positive, and since \( dp(-r_\beta \cdot \gamma) < dp(\beta) < dp(\gamma) \) by (2.22), minimality of \( \gamma \) forces \( -r_\beta \cdot \gamma \in \Phi_\Psi \). So in any case \( r_\beta \cdot \gamma \in \Phi_\Psi \). But since \( r_\beta \in W_\Psi \), this yields \( \gamma \in \Phi_\Psi \), a contradiction. Hence \( \Phi_\Gamma = \Phi_\Psi \), and thus \( W_\Gamma = W_\Psi \).

Now let \( \alpha, \beta \in \Psi \) with \( \alpha \neq \beta \). It remains to show that \( \langle \alpha, \beta \rangle \leq -1 \) or \( \langle \alpha, \beta \rangle = -\cos(\pi/m) \) for some integer \( m \). If \( \langle \alpha, \beta \rangle \leq -1 \) or \( \langle \alpha, \beta \rangle = 0 \) this is certainly true, so assume without loss of generality that \( \langle \alpha, \beta \rangle > -1 \) and \( \langle \alpha, \beta \rangle \neq 0 \). Then \( r_\alpha \cdot \beta, r_\beta \cdot \alpha \in \Phi^+ \) since \( \alpha, \beta \in \Psi \); furthermore, \( r_\beta \cdot (r_\alpha \cdot \beta) \) is also positive, since \( r_\alpha \cdot \beta \) is an element of \( \Phi_\Psi \cap \Phi^+ \) but not equal to \( \beta \) (as \( \langle \alpha, \beta \rangle \neq 0 \)), and \( N(r_\beta) \cap \Phi_\Psi = \{ \beta \} \). Now \( \beta = r_\alpha \cdot \beta + 2\langle \alpha, \beta \rangle \alpha \), and thus

\[
-\beta = r_\beta \cdot \beta = r_\beta \cdot (r_\alpha \cdot \beta) + 2\langle \alpha, \beta \rangle r_\beta \cdot \alpha,
\]

which forces \( \langle \alpha, \beta \rangle \) to be less than 0; that is, \( \langle \alpha, \beta \rangle \in (-1, 0) \).

We show now that if \( \lambda \alpha + \mu \beta \) is a root in \( \Phi_\Gamma \), then \( \lambda \) and \( \mu \) are either both nonnegative or both nonpositive. Assume for a contradiction that there exist \( \lambda, \mu > 0 \) such that \( \lambda \alpha - \mu \beta \) is a root in \( \Phi_\Gamma \), and assume without loss of generality that this is a positive root. Then

\[
r_\alpha \cdot (\lambda \alpha - \mu \beta) = (-\lambda + 2\langle \alpha, \beta \rangle \mu) \alpha - \mu \beta
\]

is in \( \Phi^- \), since \( \lambda, \mu > 0 \) and \( \langle \alpha, \beta \rangle < 0 \). Thus \( \lambda \alpha - \mu \beta \in N(r_\alpha) \cap \Phi_\Gamma \), forcing \( \lambda \alpha - \mu \beta = \alpha \), and contradicting \( \mu > 0 \).

Now let \( \theta \in (0, \frac{\pi}{2}) \) such that \( \langle \alpha, \beta \rangle = -\cos(\theta) \), and let \( m \in \mathbb{N} \) be minimal such that \( (m+1)\theta > \pi \); then \( \sin(\theta(m+1)) < 0 \) and \( \sin(\theta m) \geq 0 \). If \( m \) is even,

\[
(r_\alpha r_\beta)^{m/2} \cdot \alpha = \frac{1}{\sin(\theta)}(\sin((m+1)\theta)\alpha + \sin(m\theta)\beta)
\]

by (1.1)(i), while if \( m \) is odd,

\[
(r_\alpha r_\beta)^{(m+1)/2} \cdot \beta = -\frac{1}{\sin(\theta)}(\sin((m+1)\theta)\alpha + \sin(m\theta)\beta).
\]

Since \( \sin((m+1)\theta) < 0 \), the previous paragraph forces \( \sin(m\theta) \leq 0 \) in both cases, and we deduce that \( \sin(m\theta) = 0 \); hence \( m\theta = \pi \), as required. \( \square \)
The following corollary is an easy consequence of the previous theorem together with (1.8).

(2.23) COROLLARY Suppose \( \Gamma \subseteq \Phi \). Then there exists a standard geometric realization of \( W_\Gamma \) with root system \( \tilde{\Phi} \) and bilinear form \( (\ ,\ ) \), and a bijection \( \psi : \tilde{\Phi} \to \Phi \) with

(i) \( \psi(\tilde{\Phi}^+) = \Phi_\Gamma \cap \Phi^+ \),

(ii) \( \psi(w \cdot \tilde{\alpha}) = w \cdot \psi(\tilde{\alpha}) \) for \( w \in W_\Gamma \) and \( \tilde{\alpha} \in \tilde{\Phi} \), and

(iii) \( \langle \psi(\tilde{\alpha}), \psi(\tilde{\beta}) \rangle = (\tilde{\alpha}, \tilde{\beta}) \) for all \( \tilde{\alpha}, \tilde{\beta} \in \tilde{\Phi} \).

(2.24) COROLLARY Let \( \alpha \) and \( \beta \) be roots such that \( \langle \alpha, \beta \rangle \in (-1, 1) \). Then \( W_{\{\alpha, \beta\}} \) is finite.

Proof. We show first that \( \langle \ ,\ \rangle \) restricted to \( V_{\{\alpha, \beta\}} \) is positive definite. So suppose that \( v \in V_{\{\alpha, \beta\}} \) with \( \langle v, v \rangle \leq 0 \), and let \( \lambda, \mu \in \mathbb{R} \) such that \( v = \lambda \alpha + \mu \beta \). Then

\[
\langle v, v \rangle = \lambda^2 + \mu^2 + 2\langle \alpha, \beta \rangle \lambda \mu = (\lambda + \langle \alpha, \beta \rangle \mu)^2 + (1 - \langle \alpha, \beta \rangle^2) \mu^2;
\]

since \( 1 - \langle \alpha, \beta \rangle^2 > 0 \), this forces \( \lambda = \mu = 0 \). Thus \( v \) equals the zero-vector, as required.

Now let \( \Gamma = \{\alpha, \beta\} \); as in (2.22), define \( \Psi \) to be the set of all \( \gamma \in \Phi_\Gamma \cap \Phi^+ \) with \( N(\gamma) \cap \Phi_\Gamma = \{\gamma\} \). Since \( \langle \ ,\ \rangle \) restricted to \( V_{\{\alpha, \beta\}} \) is positive definite, the remark following (1.8) yields that the elements of \( \Psi \) must be linearly independent. So \( \Psi = \{\alpha', \beta'\} \) for some roots \( \alpha' \) and \( \beta' \), as \( V_{\{\alpha, \beta\}} = V_\Psi \) is of dimension 2. Since \( \langle \ ,\ \rangle \) is positive definite on \( V_{\{\alpha', \beta'\}} \), we know further that

\[
0 < \langle \alpha' + \beta', \alpha' + \beta' \rangle = 2 + 2\langle \alpha', \beta' \rangle,
\]

and thus \( \langle \alpha', \beta' \rangle > -1 \). So \( \langle \alpha', \beta' \rangle = -\cos(\pi/m) \) for some integer \( m \) by (2.21), and it follows by (1.1)(i) and the faithfulness of the standard geometric realization of \( W \) that \( W_\Gamma = W_\Psi \) is finite.

(2.25) LEMMA The support of a root is always a connected subgraph of the Coxeter graph.

Proof. Let \( \alpha \) be a root. Since \( \text{supp}(\alpha) = \text{supp}(-\alpha) \), we may assume without loss of generality that \( \alpha \) is positive. If \( \alpha \) has depth 1, then \( |\text{supp}(\alpha)| = 1 \),
and a graph with one vertex is connected. So suppose next that \( \alpha \) is of depth greater than 1, and assume that the assertion is true for all positive roots of depth less than \( \text{dp}(\alpha) \). Let \( r \in R \) such that \( \text{dp}(r \cdot \alpha) = \text{dp}(\alpha) - 1 \); that is, \( \langle \alpha, \alpha_r \rangle > 0 \). Since \( \alpha \) is a positive root, and \( \langle \alpha_s, \alpha_r \rangle \leq 0 \) for all \( s \in R \) with \( s \neq r \), this yields that \( \alpha_r \) is an element of \( \text{supp}(\alpha) \). By induction, the support of \( r \cdot \alpha \) is connected, and the definition of the action of \( W \) on \( V \) yields that

\[
\text{supp}(\alpha) = \text{supp}(r \cdot \alpha) \cup \{\alpha_r\}.
\]

If \( \alpha_r \) is in the support of \( r \cdot \alpha \), then \( \text{supp}(\alpha) = \text{supp}(r \cdot \alpha) \), which is connected. If \( \alpha_r \notin \text{supp}(r \cdot \alpha) \), then \( \alpha_r \) must be adjoined to an element of \( \text{supp}(r \cdot \alpha) \), since \( \langle r \cdot \alpha, \alpha_r \rangle = -\langle \alpha, \alpha_r \rangle \neq 0 \); hence \( \text{supp}(\alpha) \) is again connected, and this finishes the proof.

The next proposition is a generalization of the well known fact that if the coefficient of a simple root in a root is greater than 0, it must be greater than or equal to 1.

\[ (2.26) \text{ Proposition } \]

Let \( \alpha \) be a positive root, and let \( r \in R \). Then the coefficient of \( \alpha_r \) in \( \alpha \) is either greater than or equal to 2, or equals 0, 1 or \( 2 \cos(\pi/m) \) for some \( s, t \in R \) with \( 4 \leq m_{st} < \infty \). In particular, since \( 2 \cos(\pi/m) \geq \sqrt{2} \) for \( m \geq 4 \), this yields that the coefficient of \( \alpha_r \) in \( \alpha \) equals 0, 1 or is greater than or equal to \( \sqrt{2} \).

\[ \text{Proof.} \] Let \( \alpha = \sum_{s \in R} \lambda_s \alpha_s \), and assume without loss of generality that \( \lambda_r \) is positive. If \( |R| = 1 \), then \( \alpha = \alpha_r \) and \( \lambda_r = 1 \), and there is nothing left to show; thus we may assume from now on that \( |R| \geq 2 \).

Suppose now that \( I(\alpha) \subseteq \{r, s\} \) for some \( s \in R \setminus \{r\} \); (we call this the rank 2 case). If \( \langle \alpha_r, \alpha_s \rangle \) is less than or equal to -1 (that is, \( rs \) has infinite order), then the result is an easy consequence of (1.1)(ii). So let \( m = m_{rs} < \infty \). We deduce from (1.1)(i) that there exists an \( l \in \{0, \ldots, 2m\} \) such that \( \lambda_r \) equals \( \sin(l\pi/m)/\sin(\pi/m) \). Since \( \lambda_r > 0 \), clearly \( 1 \leq l \leq m \), and by symmetry of sine on the interval \([0, \pi]\), we may assume without loss of generality that \( 1 \leq l \leq \frac{m}{2} \). Now \( \lambda_r = 1 \) if \( l = 1 \), and \( \lambda_r = 2 \cos(\pi/m) \) if \( l = 2 \); finally, if \( l \geq 3 \), in particular \( m \geq 2l \geq 6 \), and thus

\[
\frac{\sin(l\pi/m)}{\sin(\pi/m)} \geq \frac{\sin(3\pi/m)}{\sin(\pi/m)} = 2 \cos(2\pi/m) + 1 \geq 2 \cos(2\pi/6) + 1 = 2,
\]

as required.
The general case is shown by induction on the depth of \( \alpha \). If \( \alpha \) has depth 1, then \( \alpha = \alpha_r \), and the result follows trivially. So suppose \( \text{dp}(\alpha) > 1 \), and let \( s \in R \) and \( w \in W \) with \( \alpha = w \cdot \alpha_s \) and \( l(w) = \text{dp}(\alpha) - 1 \). Further, let \( t \in R \) such that \( l(wt) < l(w) \); then clearly \( t \neq s \), as \( w \cdot \alpha_s \in \Phi^+ \). Choose \( w' \in wW_{\{s,t\}} \) of minimal length. Then \( l(w') \leq l(wt) < l(w) \), and also \( l(w's) \geq l(w') \) and \( l(w't) \geq l(w') \) by minimality of \( w' \). So the roots \( w' \cdot \alpha_s \) and \( w' \cdot \alpha_t \) are positive by (1.3); moreover, each of these roots is of depth at most \( l(w') + 1 \), which is less than \( \text{dp}(\alpha) \). Hence the inductive hypothesis applies to \( w' \cdot \alpha_s \) and \( w' \cdot \alpha_t \).

There exists a \( u \in W_{\{s,t\}} \) with \( w = w'u \), and we have \( u \cdot \alpha_s = \mu \alpha_s + \nu \alpha_t \) for some \( \mu, \nu \in \mathbb{R} \). If \( u \cdot \alpha_s \in \Phi^- \), then \( \mu, \nu \leq 0 \), and so

\[
\alpha = (w'u) \cdot \alpha_s = \mu(w' \cdot \alpha_s) + \nu(w' \cdot \alpha_t)
\]

is negative, contrary to our hypothesis. Hence \( \mu, \nu \geq 0 \); in fact, \( \mu, \nu > 0 \), since otherwise \( \alpha = w'u \cdot \alpha_s \) would equal either \( w' \cdot \alpha_s \) or \( w' \cdot \alpha_t \), contradicting \( \text{dp}(\alpha) > l(w') + 1 \). Since the assertion is true for \( u \cdot \alpha_s \) by the rank 2 case, this implies that \( \mu, \nu \geq 1 \).

Let \( \mu_r, \nu_r \) be the coefficients of \( \alpha_r \) in \( w' \cdot \alpha_s, w' \cdot \alpha_t \) respectively, so that \( \lambda_r = \mu \mu_r + \nu \nu_r \geq \mu_r + \nu_r \). If \( \mu_r > 0 \), then by inductive hypothesis \( \mu_r \geq 1 \), and the same is true for \( \nu_r \). So if both \( \mu_r \) and \( \nu_r \) are nonzero, then \( \lambda_r \geq 1 + 1 = 2 \); since \( \lambda_r > 0 \) by assumption, it will suffice to consider the case \( \nu_r = 0 \) and \( \mu_r \geq 1 \), and hence \( \lambda_r = \mu_r \geq 1 \). If \( \lambda_r = 1 \), there is nothing left to show, so suppose \( \lambda_r > 1 \). We must have either \( \mu > 1 \) or \( \mu_r > 1 \). If \( \mu \) and \( \mu_r \) are both strictly greater than 1, then \( \mu \geq \sqrt{2} \) (by the rank 2 case) and \( \mu_r \geq \sqrt{2} \) (by induction), and thus \( \lambda_r \geq \sqrt{2} \sqrt{2} = 2 \), as required. This leaves us with the case that one of \( \mu \) and \( \mu_r \) is 1, and the other one is equal to \( \lambda_r \). Then by induction or the rank 2 case, \( \lambda_r \geq 2 \) or \( \lambda_r = 2 \cos(\pi/m_{xy}) \) for some \( x, y \in R \) with \( 4 \leq m_{xy} < \infty \).

The arguments in the above proof also yield the following scholium.

(2.27) Suppose \( \langle \alpha_s, \alpha_t \rangle \geq -1 \) for all \( s, t \in R \), and let \( \alpha \in \Phi^+ \) and \( r \in R \). Then the coefficient of \( \alpha_r \) in \( \alpha \) is a polynomial in \( C \) with coefficients in \( \mathbb{N}_0 \), where \( C \) is the set

\[
\left\{ \frac{\sin(l \pi/m)}{\sin(\pi/m)} \mid 4 \leq m = m_{st} < \infty \text{ for } s, t \in R \text{ and } l \in \mathbb{N} \text{ with } l \leq \frac{m}{2} \right\}.
\]
Chapter 3

Dominance and Elementary Roots

The main result of this chapter is the finiteness of the set of elementary roots defined below (given that $R$ is finite). It can be shown that this is equivalent to the Parallel Wall Theorem of the preprint [2], in which the proof given is incomplete.

For $\alpha, \beta \in \Phi^+$, we say that $\alpha$ dominates $\beta$ (we write $\alpha \text{ dom } \beta$) if and only if $w \cdot \beta$ is negative for all $w \in W$ with $w \cdot \alpha$ negative; equivalently, $\alpha$ dominates $\beta$ if and only if $w \cdot \alpha \in \Phi^+$ for all $w \in W$ with $w \cdot \beta \in \Phi^+$.

Observe that if $\alpha$ dominates $\beta$ and $w \cdot \beta$ is positive, it follows trivially that $(w \cdot \alpha) \text{ dom } (w \cdot \beta)$. Note also that it is not a priori clear that the notion of dominance does not depend on $W$. That is, if $\Gamma \subseteq \Phi$, and $\alpha$ and $\beta$ are roots in $\Phi_\Gamma$ such that $w \cdot \beta$ negative whenever $w \cdot \alpha \in \Phi^-$ for $w \in W_\Gamma$, it is not obvious that this means that $\alpha \text{ dom } \beta$. We will see shortly that it is true, however.

Define $\Delta$ to be the set of positive roots $\alpha$ such that $\alpha$ dominates some $\beta$ in $\Phi^+ \setminus \{\alpha\}$, and define the set of elementary roots $\mathcal{E}$ to be $\Phi^+ \setminus \Delta$.

Note that since $r \cdot \alpha_s \in \Phi^-$ and $r \cdot (\Phi \setminus \{\alpha_r\}) \subseteq \Phi^+$ for $r \in R$, every simple root is elementary. Observe also that if $\alpha$ dominates $\beta$ and $w \cdot \alpha$ is an elementary root for some $w \in W$, either $\alpha = \beta$ or $w \cdot \beta \in \Phi^-$. If $\alpha \in \Delta$, and $w^{-1} \cdot \alpha$ and $u \cdot \alpha$ are elementary for some $u, w \in W$, then $N(u) \cap N(w^{-1}) \neq \emptyset$; for if $\alpha$ dominates $\beta \in \Phi^+ \setminus \{\alpha\}$, then $w^{-1} \cdot \beta$ and $u \cdot \beta$ must be negative by the above. Thus $l(uw) \neq l(u) + l(w)$ by (1.11).

(3.28) Lemma Let $\alpha \in \Delta$ and $u, w \in W$ with $u \cdot \alpha, w^{-1} \cdot \alpha \in \Pi$. Then $l(uw) \neq l(u) + l(w)$.

By using $r_1 r_2 \cdots r_{j-1} r_j$ in place of $u$ and $r_{j+1} r_{j+2} \cdots r_{l-1} r_l$ in place of $w$ (with $\alpha = (r_{j+1} \cdots r_{l-1} r_l) \cdot \alpha_s$), we obtain

(3.29) Corollary Let $r_1, \ldots, r_l \in R$ such that $l(r_1 \cdots r_l) = l$, and suppose furthermore that $(r_1 \cdots r_l) \cdot \alpha_s = \alpha_t$ for some $s, t \in R$. Then

$$(r_{j} r_{j-1} \cdots r_2 r_1) \cdot \alpha_t = (r_{j+1} r_{j+2} \cdots r_{l-1} r_l) \cdot \alpha_s \in \mathcal{E} \text{ for all } j \in \{1, \ldots, l\}.$$
Chapter 3 Dominance and elementary roots

(3.30) Lemma Let $\alpha$ and $\beta$ be distinct positive roots. Then $\alpha \text{ dom } \beta$ if and only if $\text{dp}(w \cdot \alpha) > \text{dp}(w \cdot \beta)$ for all $w \in W$.

Proof. Suppose first that $w \cdot \alpha$ is of depth strictly greater than $\text{dp}(w \cdot \beta)$ for all $w \in W$. If $u \cdot \alpha$ is negative for some $u \in W$, then $\text{dp}(u \cdot \beta) < \text{dp}(u \cdot \alpha)$, and this is less than or equal to 0; whence $u \cdot \beta$ is negative, and we conclude that $\alpha \text{ dom } \beta$.

For the converse suppose that $\alpha \text{ dom } \beta$, and let $w \in W$. Suppose first that $w \cdot \alpha$ is positive, and let $u \in W$ and $r \in R$ such that $w \cdot \alpha = u \cdot \alpha_r$ and $l(u) = \text{dp}(w \cdot \alpha) - 1$. Since $r \cdot (u^{-1}w) \cdot \alpha = -\alpha_r \in \Phi^-$ and $\alpha \text{ dom } \beta$, we know that $(ru^{-1}w) \cdot \beta$ is negative. So by (1.4), either $(u^{-1}w) \cdot \beta = \alpha_r$ or $(u^{-1}w) \cdot \beta$ is negative. But since $\alpha$ and $\beta$ are distinct, the former is impossible, and thus $(u^{-1}w) \cdot \beta \in \Phi^-$. Hence $\text{dp}(w \cdot \beta) \leq l(u) < \text{dp}(w \cdot \alpha)$, as required.

Finally, suppose that $w \cdot \alpha$ is negative, and thus $w \cdot \beta \in \Phi^-$ since $\alpha \text{ dom } \beta$. Take $u \in W$ and $r \in R$ with $l(u) = -\text{dp}(w \cdot \beta)$ and $u \cdot (w \cdot \beta) = -\alpha_r$; then $(ruw) \cdot \beta = \alpha_r$ is positive, and thus $(ruw) \cdot \alpha$ must also be positive. Since $\alpha \neq \beta$ further $(ruw) \cdot \alpha \neq \alpha_r$, so $(uw) \cdot \alpha$ is still positive; that is, $u \cdot (-w \cdot \alpha) = -uw \cdot \alpha \in \Phi^-$. Hence

$$ -\text{dp}(w \cdot \beta) = l(u) \geq \text{dp}(-w \cdot \alpha) = 1 - \text{dp}(w \cdot \alpha), $$

and thus $\text{dp}(w \cdot \beta) < \text{dp}(w \cdot \alpha)$, as required.

Note that if $\alpha \text{ dom } \beta$, the previous lemma implies that $\text{dp}(\alpha) \geq \text{dp}(\beta)$, with equality only if $\alpha = \beta$, and so $\text{dom}$ is antisymmetric. It is clear that $\text{dom}$ is also transitive, and so $\text{dom}$ is a partial order on $\Phi^+$. The elementary roots are the minimal elements in this partial order, and for each $\alpha \in \Phi^+$ there exists a $\beta \in \mathcal{E}$ such that $\alpha \text{ dom } \beta$. So if $u, w \in W$, then $N(u) \cap N(w^{-1}) = \emptyset$ if and only if $N(u) \cap N(w^{-1}) \cap \mathcal{E} = \emptyset$; hence (1.11) yields the following result.

(3.31) Lemma Let $u, w \in W$. Then $l(uw) = l(u) + l(w)$ if and only if $N(u) \cap N(w^{-1}) \cap \mathcal{E} = \emptyset$.

Suppose next that $W$ is finite, and let $w_R$ denote an element of $W$ of maximal length. Then $l(w_Rr) \leq l(w_R)$ for all $r \in R$, and thus $w_R \cdot \alpha_r \in \Phi^-$. Hence $N(w_R) = \Phi^+$ (note that this also yields the uniqueness of $w_R$). We show now that $\text{dp}(w_R \cdot \alpha) = \text{dp}(-\alpha)$ for all $\alpha \in \Phi$. By (2.13) it suffices
Chapter 3 Dominance and elementary roots

28

to show this for \( \alpha \in \Phi^+ \). Then \( \langle \alpha, \beta \rangle > 0 \) for \( \beta \in N(r_\alpha) \); furthermore, \( r_\alpha \) permutes \( \Phi^+ \setminus N(r_\alpha) \) with \( \langle \alpha, \beta \rangle > 0 \) if and only if \( \langle \alpha, r_\alpha \cdot \beta \rangle < 0 \) for \( \beta \in \Phi^+ \setminus N(r_\alpha) \). Therefore

\[
|\Phi^+ \cap V_+(\alpha)| - |\Phi^+ \cap V_-(\alpha)| = |N(r_\alpha)|;
\]

that is, \( |N_+(w_R, \alpha)| - |N_-(w_R, \alpha)| = |N(r_\alpha)| = l(r_\alpha) \). Now (2.18) and (2.19) yield that

\[
\text{dp}(w_R \cdot \alpha) = \text{dp}(\alpha) - l(r_\alpha) = \text{dp}(\alpha) - (2 \text{dp}(\alpha) - 1) = 1 - \text{dp}(\alpha),
\]

as required. So if \( \text{dp}(\alpha) > \text{dp}(\beta) \) for some roots \( \alpha \) and \( \beta \), then

\[
\text{dp}(w_R \cdot \alpha) = 1 - \text{dp}(\alpha) < 1 - \text{dp}(\beta) = \text{dp}(w_R \cdot \beta).
\]

This together with (3.30) yield that there is no non-trivial dominance in finite root systems.

The next proposition provides us with an alternative characterization of dominance, which shows that dominance is independent of \( W \).

### (3.32) Proposition

Let \( \alpha \) and \( \beta \) be positive roots. Then \( \alpha \) dom \( \beta \) if and only if \( \langle \alpha, \beta \rangle \geq 1 \) and \( \text{dp}(\alpha) \geq \text{dp}(\beta) \).

**Proof.** Suppose first that \( \alpha \) dom \( \beta \). By (3.30) we need only show that \( \langle \alpha, \beta \rangle \) is greater than or equal to 1. Since \( r_\alpha \cdot \alpha \) is negative, \( r_\alpha \cdot \beta \) must be negative, and this forces \( \langle \alpha, \beta \rangle > 0 \).

Assume for a contradiction that \( \langle \alpha, \beta \rangle \in (0, 1) \); then (2.24) yields that \( W_{\langle \alpha, \beta \rangle} \) is a finite Coxeter group, and by (1.12) there exists a finite parabolic subgroup \( W_J \) of \( W \) and a \( w \in W \) such that \( wW_{\langle \alpha, \beta \rangle}w^{-1} \subseteq W_J \). Then \( w \cdot \alpha \) and \( w \cdot \beta \) are in \( \Phi_J \), and by (3.30) and (2.16),

\[
\text{dp}_J(w \cdot \alpha) = \text{dp}(w \cdot \alpha) > \text{dp}(w \cdot \beta) = \text{dp}_J(w \cdot \beta).
\]

But now the remark preceding this proposition together with (2.16) imply

\[
\text{dp}(w_Jw \cdot \alpha) = \text{dp}_J(w_J \cdot (w \cdot \alpha)) < \text{dp}_J(w_J \cdot (w \cdot \beta)) = \text{dp}(w_Jw \cdot \beta),
\]

where \( w_J \) denotes the element of \( W_J \) of maximal length. This contradicts (3.30) with \( w_Jw \) in place of \( w \), and hence \( \langle \alpha, \beta \rangle \geq 1 \) after all.
For the converse, assume that $\langle \alpha, \beta \rangle \geq 1$ and $\text{dp}(\alpha) \geq \text{dp}(\beta)$. First consider the case $\beta \in \Pi$, say $\beta = \alpha_r$. If $\alpha = \alpha_r$ there is nothing left to show, so suppose $r \cdot \alpha \in \Phi^+$. Then

$$\langle \alpha, r \cdot \alpha \rangle = \langle \alpha, \alpha \rangle - 2\langle \alpha, \alpha_r \rangle = 1 - 2\langle \alpha, \alpha_r \rangle \leq -1.$$  

By (1.1)(ii) there are infinitely many roots of the form $\lambda \alpha + \mu (r \cdot \alpha)$ with $\lambda, \mu > 0$. Assume for a contradiction that $\alpha$ does not dominate $\beta$, and choose $w \in W$ such that $w \cdot \alpha \in \Phi^-$ and $w \cdot \alpha_r \in \Phi^+$. Then

$$w \cdot (r \cdot \alpha) = w \cdot \alpha + 2\langle \alpha, \alpha_r \rangle (-w \cdot \alpha_r)$$

is a positive linear combination of negative roots, and must therefore be negative. So $N(w)$ contains $\alpha$ and $r \cdot \alpha$, and hence also contains all roots of the form $\lambda \alpha + \mu (r \cdot \alpha)$ with $\lambda, \mu > 0$. This contradicts the finiteness of $N(w)$ (see (1.9)).

Proceeding by induction on $\text{dp}(\beta)$, suppose now that $\text{dp}(\beta) > 1$, and choose $r \in R$ such that $\text{dp}(r \cdot \beta) = \text{dp}(\beta) - 1$. Since $\text{dp}(\alpha) \geq \text{dp}(\beta) > 1$, clearly $r \cdot \alpha \in \Phi^+$. Further $\langle r \cdot \alpha, r \cdot \beta \rangle \geq 1$, and

$$\text{dp}(r \cdot \alpha) \geq \text{dp}(\alpha) - 1 \geq \text{dp}(\beta) - 1 = \text{dp}(r \cdot \beta).$$

Now $(r \cdot \alpha) \text{ dom } (r \cdot \beta)$ by induction, and therefore $\alpha \text{ dom } \beta$.

Observe that if $\langle \alpha, \beta \rangle \geq 1$ for positive roots $\alpha$ and $\beta$, then by (3.32), $\alpha \text{ dom } \beta$ or $\beta \text{ dom } \alpha$.

Next, let $\Gamma \subseteq \Phi$ and let $\alpha$ and $\beta$ be positive roots in $\Phi_\Gamma$ such that $w \cdot \beta$ is negative for all $w \in W_\Gamma$ with $w \cdot \alpha$ negative. We show now that this yields that $\alpha \text{ dom } \beta$. If $\alpha = \beta$ this is certainly true, so suppose without loss of generality that $\alpha \neq \beta$. By (2.21) we know that $W_\Gamma$ is a Coxeter group, and by (2.23) there exists a standard geometric realization of $W_\Gamma$ with bilinear form $(,)$ and root system $\tilde{\Phi}$, and a bijection $\psi: \tilde{\Phi} \rightarrow \Phi_\Gamma$ such that

(i) $\psi(\tilde{\Phi}^+) = \Phi_\Gamma \cap \Phi^+$,

(ii) $\psi(w \cdot \tilde{\alpha}) = w \cdot \psi(\tilde{\alpha})$ for all $w \in W$ and $\tilde{\alpha} \in \tilde{\Phi}$, and

(iii) $\langle \psi(\tilde{\alpha}), \psi(\tilde{\beta}) \rangle = (\tilde{\alpha}, \tilde{\beta})$ for all $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Phi}$.

Now $\psi^{-1}(\alpha)$ and $\psi^{-1}(\beta)$ are in $\tilde{\Phi}^+$ by (i). If $w \cdot \psi^{-1}(\alpha) \in \tilde{\Phi}^-$ for some $w \in W_\Gamma$, then $\psi(w \cdot \psi^{-1}(\alpha)) \in \Phi^-$ by (i), and thus $w \cdot \alpha \in \Phi^-$ by (ii);
hence \( w \cdot \beta \in \Phi^- \) by hypothesis, and (again by (i) and (ii)), we find that \( w \cdot \psi^{-1}(\beta) = \psi^{-1}(w \cdot \beta) \in \widetilde{\Phi}^- \). So \( \psi^{-1}(\alpha) \) dominates \( \psi^{-1}(\beta) \) with respect to \( W_\Gamma \). Proposition (3.32) now yields that \( (\psi^{-1}(\alpha), \psi^{-1}(\beta)) \geq 1 \), and thus \( \langle \alpha, \beta \rangle \geq 1 \) by (iii). Hence \( \alpha \text{ dom } \beta \) or \( \beta \text{ dom } \alpha \). Since \( \alpha \neq \beta \), and thus \( \psi^{-1}(\alpha) \neq \psi^{-1}(\beta) \), Lemma (3.30) implies that the depth of \( \psi^{-1}(\alpha) \) with respect to the distinguished generating set of \( W_\Gamma \) is strictly greater than the depth of \( \psi^{-1}(\beta) \). Let \( w \in W_\Gamma \) such that \( w \cdot \psi^{-1}(\beta) \in \widetilde{\Phi}^- \) with the length of \( w \) (with respect to the distinguished generating set of \( W_\Gamma \)) equal to the depth of \( \psi^{-1}(\beta) \). Then \( w \cdot \psi^{-1}(\alpha) \in \widetilde{\Phi}^+ \), since \( \psi^{-1}(\alpha) \) is of depth greater than the length of \( w \). Therefore \( w \cdot \alpha \in \Phi^+ \) and \( w \cdot \beta \in \Phi^- \) by (i) and (ii), and thus \( \beta \) cannot dominate \( \alpha \); whence \( \alpha \text{ dom } \beta \).

We now make use of (3.32) to give an alternative derivation of the well known classification of finite Coxeter groups. Suppose that \( W_J \) is a parabolic subgroup of \( W \) with Coxeter diagram

![Coxeter diagram](image)

with \( m, n \geq 4 \). Denote the simple roots corresponding to \( r, s_j, t \) by \( x, y_j \) and \( z \) respectively, and define \( \gamma \) to be \( t(s_l \cdots s_1) \cdot x \). Then

\[
\gamma = x + c_m(y_1 + \cdots + y_l) + c_mc_nz,
\]

where \( c_m = 2\cos(\pi/m) \) and \( c_n = 2\cos(\pi/n) \), and an easy calculation yields that \( \langle \gamma, z \rangle \geq 1 \); since \( \gamma \) is certainly of depth greater than 1, we conclude that \( \gamma \text{ dom } z \) and \( \gamma \in \Delta \). So \( W_J \) must be infinite, and hence \( W \) must be infinite. The root \( \gamma \) appearing above can be conveniently described by means of the following diagram:

![Coxeter diagram](image)

Note that the vertex in the above diagram corresponding to \( z \) (which is dominated by \( \gamma \)) is denoted by a circuit rather than a dot. Similarly, roots described by the following diagrams are necessarily in \( \Delta \).
Chapter 3 Dominance and elementary roots

\[ m \geq 4 \]
\[ 2c_m^2 - 1 \quad 2c_m \quad 2c_m \quad 2c_m \quad 2c_m \]
\[ c_m \]

\[ m \geq 6 \]
\[ c_m \quad c_m^2 \quad 1 \]

\[ 5 \]
\[ 1 \quad c_5 + 1 \quad 2c_5 + 1 \quad c_5 \]
Consequently a finite Coxeter group cannot have a parabolic subgroup of type corresponding to any of the above diagrams. In (3.12) and (3.14) we show that $\Delta$ is also non-empty if $R$ contains a circuit or an infinite bond, and this yields the following well-known theorem:

(3.33) THEOREM Suppose $W$ is finite. Then the Coxeter graph of $W$ has finitely many connected components, and each of these is one of the following shapes:

- $I_2(m)$
- $A_n$
- $B_n$
- $D_n$
- $F_4$
It is a straightforward matter to show that the form $\langle \cdot, \cdot \rangle$ is positive
definite for each of the diagrams in this list. Consequently there can be no
nontrivial dominance in these root systems, since if $\alpha$ dominates $\beta$,

$$\langle \alpha - \beta, \alpha - \beta \rangle = \langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle - 2\langle \alpha, \beta \rangle = 2(1 - \langle \alpha, \beta \rangle) \leq 0.$$ 

From Theorem (3.17) below it follows that the root systems (and hence the
groups) are finite in these cases.

We now define a second partial order $\preceq$ on $\Phi$, which will enable us
to stop our search for elementary roots in an ascending chain with respect
to $\preceq$, as soon as we find a non-elementary root (see (3.9)). This fact is an
important tool in the proof of the finiteness of the set of elementary roots.

For roots $\alpha$ and $\beta$ we say that $\alpha$ precedes $\beta$ (and write $\alpha \preceq \beta$) if there
exists a $w \in W$ with $\beta = w \cdot \alpha$ and $N(w) \subseteq V_-(\alpha)$; that is, $N(w) = N_-(w, \alpha)$
and $N_+(w, \alpha) = N_0(w, \alpha) = \emptyset$. If $\alpha \preceq \beta$, we also write $\beta \succeq \alpha$ and say that $\beta$
is a successor of $\alpha$. We write $\alpha < \beta$ or $\beta > \alpha$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$. 
Note that if $\alpha$ is a root and $r \in \mathbb{R}$, then

$$r \cdot \alpha \begin{cases} < \alpha & \text{if } \langle \alpha, \alpha_r \rangle > 0; \text{ that is, } \text{dp}(r \cdot \alpha) = \text{dp}(\alpha) - 1, \\ = \alpha & \text{if } \langle \alpha, \alpha_r \rangle = 0, \\ > \alpha & \text{if } \langle \alpha, \alpha_r \rangle < 0; \text{ that is, } \text{dp}(r \cdot \alpha) = \text{dp}(\alpha) + 1. \end{cases}$$

In particular, if $\alpha$ is a positive root, then there exists an $r \in \mathbb{R}$ with $\alpha \succ r \cdot \alpha$. Since $\text{dp}(r \cdot \alpha) = \text{dp}(\alpha) - 1$ in this case, an iteration yields that each positive root is preceded by a simple one.

If $\beta = w \cdot \alpha$ for some $w \in W$, then

$$\text{dp}(\beta) - \text{dp}(\alpha) = \text{dp}(w \cdot \alpha) - \text{dp}(\alpha) = |N_-(w, \alpha)| - |N_+(w, \alpha)|$$

by (2.18), and it is clear that $\alpha \preceq \beta$ if and only if there exists a $w \in W$ of length equal to $\text{dp}(\beta) - \text{dp}(\alpha)$ such that $\beta = w \cdot \alpha$. Therefore $\preceq$ is antisymmetric, and we show now that $\preceq$ is also transitive, and thus a partial order. So let $\alpha, \beta$ and $\gamma$ be roots with $\alpha \preceq \beta$ and $\beta \preceq \gamma$. Then there exist $u, w \in W$ such that $\beta = w \cdot \alpha$ and $\gamma = u \cdot \beta$ with $N(w) = N_-(w, \alpha)$ and $N(u) = N_-(u, \beta)$. Thus $\gamma = uw \cdot \alpha$, and (2.17) gives

$$N_+(uw, \alpha) \subseteq N_+(w, \alpha) \cup w^{-1} \cdot N_+(u, w \cdot \alpha) = N_+(w, \alpha) \cup w^{-1} \cdot N_+(u, \beta) = \emptyset,$$

and similarly $N_0(uw, \alpha) = \emptyset$; hence $N(uw) = N_-(uw, \alpha)$ and $\alpha \preceq \gamma$, as required.

We can now state a slightly weaker criterion for precedence.

(3.34) Lemma \hspace{1em} Let $\alpha, \beta \in \Phi$ such that $\beta = w \cdot \alpha$ for some $w \in W$ with $N_+(w, \alpha) = \emptyset$. Then $\alpha \preceq \beta$.

Proof. If $w = 1$, this is certainly true; so suppose $l(w) > 0$, and proceed by induction. Let $r \in \mathbb{R}$ and $u \in W$ such that $w = ur$ and $l(w) = l(u) + 1$. Then $N_+(u, r \cdot \alpha) \subseteq r \cdot N_+(w, \alpha) = \emptyset$ by (2.17), and as $\beta = u \cdot (r \cdot \alpha)$, it follows by induction that $\beta \succeq r \cdot \alpha$. Lemma (2.17) implies furthermore that $N_+(r, \alpha) \subseteq N_+(w, \alpha) = \emptyset$, and thus $\langle \alpha, \alpha_r \rangle \leq 0$; we deduce that $r \cdot \alpha \succeq \alpha$, and hence $\beta \succeq \alpha$ by transitivity of $\succeq$, as required. \qed
We show now that if $\beta \succeq \alpha$, then there exist roots $\alpha_1, \ldots, \alpha_{d-1}$ with $dp(\alpha_i) = dp(\alpha) + i$ and $d = dp(\beta) - dp(\alpha)$ such that

$$\alpha \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{d-1} \prec \beta.$$ 

If $\alpha = \beta$ this is trivially true; so suppose $\alpha \prec \beta$, and let $w \in W$ with $l(w) = dp(\beta) - dp(\alpha)$ such that $\beta = w \cdot \alpha$. Then $w \neq 1$ as $\alpha \neq \beta$, and thus there exist $u \in W, r \in R$ such that $w = ru$ and $l(w) = l(u) + 1$. Now

$$dp(\beta) - 1 \leq dp(r \cdot \beta) = dp(u \cdot \alpha)$$

$$= dp(\alpha) + |N_-(u, \alpha)| - |N_+(u, \alpha)|$$

$$\leq dp(\alpha) + l(u) = dp(\alpha) + l(w) - 1 = dp(\beta) - 1,$$

and we must have equality everywhere; in particular, $dp(r \cdot \beta) = dp(\beta) - 1$ and $l(w) = |N_-(u, \alpha)|$. So $\beta \succeq r \cdot \beta = u \cdot \alpha$ and $N(u) \subseteq V_-(\alpha)$, and thus $r \cdot \beta \succeq \alpha$. Since $dp(r \cdot \beta) - dp(\alpha) = dp(\beta) - dp(\alpha) - 1$, induction yields that there exist roots $\alpha_1, \ldots, \alpha_{d-2}$ with $dp(\alpha_i) = dp(\alpha) + i$ such that

$$\alpha \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{d-2} \prec r \cdot \beta (= \alpha_{d-1} \prec \beta),$$

as required. In particular, if $\alpha \prec \beta$, then there exist $r, s \in R$ such that $\alpha \preceq r \cdot \beta \prec \beta$ and $\alpha \prec s \cdot \alpha \preceq \beta$.

(3.35) **Lemma** Let $\alpha = \sum_{r \in R} \lambda_r \alpha_r$ and $\beta = \sum_{r \in R} \mu_r \alpha_r$ be such that $\alpha \preceq \beta$. Then $\lambda_r \preceq \mu_r$ for all $r \in R$.

**Proof.** If $\alpha = \beta$, the assertion is trivially true, so suppose $dp(\beta) - dp(\alpha) > 0$, and let $s \in R$ with $\beta \succeq s \cdot \alpha \succ \alpha$; then $dp(s \cdot \alpha) = dp(\alpha) + 1$, and thus $dp(\beta) - dp(s \cdot \alpha) < dp(\beta) - dp(\alpha)$. Further, $\langle \alpha, \alpha_s \rangle < 0$ and

$$s \cdot \alpha = \sum_{r \in R \setminus \{s\}} \lambda_r \alpha_r + (\lambda_s - 2\langle \alpha, \alpha_s \rangle) \alpha_s;$$

so by induction, $\mu_r \geq \lambda_r$ for all $r \in R \setminus \{s\}$ and $\mu_s \geq \lambda_s - 2\langle \alpha, \alpha_s \rangle > \lambda_s$, as required. \qed
Note that this also yields that if $\beta = w \cdot \alpha$ and $N(w) \subseteq V_-(\alpha)$ for $w \in W$, then $w \in W_I$, where $I$ consists of all $r \in R$ such that the coefficient of $\alpha_r$ in $\alpha$ is strictly less than the coefficient of $\alpha_r$ in $\beta$.

\begin{equation}
3.36 \text{ Lemma} \quad \text{Let } \alpha, \beta \in \Phi^+ \text{ such that } \alpha \preceq \beta \text{ and } \alpha \in \Delta. \text{ Then } \beta \in \Delta.
\end{equation}

\textbf{Proof.} If $\alpha = \beta$, the assertion is again trivially true. Assume next that $\text{dp}(\beta) - \text{dp}(\alpha) > 0$, and let $s \in R$ with $\beta \succeq s \cdot \alpha \succ \alpha$. Then $s \cdot \alpha$ is of depth $\text{dp}(\alpha) + 1$, and thus $\text{dp}(\beta) - \text{dp}(s \cdot \alpha) < \text{dp}(\beta) - \text{dp}(\alpha)$; furthermore, $\langle \alpha, \alpha_s \rangle < 0$. Next, let $\gamma \in \Phi^+ \setminus \{\alpha\}$ such that $\alpha$ dominates $\gamma$; then $\langle \alpha, \gamma \rangle \geq 1$ by (3.32), and thus clearly $\gamma \neq \alpha_s$. So $s \cdot \gamma \in \Phi^+$ by (1.4), and it follows easily that $(s \cdot \alpha) \text{ dom } (s \cdot \gamma)$; since obviously $s \cdot \alpha \neq s \cdot \gamma$, we deduce that $s \cdot \alpha$ is in $\Delta$, and thus $\beta \in \Delta$ by induction. 

The next lemma gives an algorithm that has as its input the set of elementary roots of depth $n$, and computes the set of elementary roots of depth $n + 1$.

\begin{equation}
3.37 \text{ Lemma} \quad \text{For all } n \in \mathbb{N}, \text{ define } \mathcal{E}_n = \{ \alpha \in \mathcal{E} \mid \text{dp}(\alpha) = n \}. \text{ Then } \mathcal{E}_{n+1} = \{ r \cdot \alpha \mid \alpha \in \mathcal{E}_n \text{ and } r \in R \text{ with } \langle \alpha, \alpha_r \rangle \in (-1, 0) \}.
\end{equation}

\textbf{Proof.} First, let $\alpha \in \mathcal{E}_n$ and $r \in R$ with $\langle \alpha, \alpha_r \rangle \in (-1, 0)$, and suppose that $r \cdot \alpha$ dominates some $\beta \in \Phi^+$; then $\langle r \cdot \alpha, \beta \rangle \geq 1$, and thus $\beta \neq \alpha_r$, since $\langle r \cdot \alpha, \alpha_r \rangle = -\langle \alpha, \alpha_r \rangle \in (0, 1)$. So $r \cdot \beta \in \Phi^+$ by (1.4), and it follows that $\alpha \text{ dom } (r \cdot \beta)$. As $\alpha$ is elementary, this implies that $r \cdot \beta$ equals $\alpha$; that is, $\beta = r \cdot \alpha$. Therefore $r \cdot \alpha$ is elementary, and since $\text{dp}(r \cdot \alpha) = \text{dp}(\alpha) + 1$ by (2.14) we have $r \cdot \alpha \in \mathcal{E}_{n+1}$.

For the converse, suppose that $\alpha$ is an elementary root of depth $n + 1$ (which is greater than 1), and let $r \in R$ with $r \cdot \alpha \prec \alpha$. Then $r \cdot \alpha \in \mathcal{E}_n$, since $r \cdot \alpha$ is of depth $n$ by (2.14), and elementary by (3.36). Furthermore, $\langle \alpha, \alpha_r \rangle > 0$ since $r \cdot \alpha \prec \alpha$; on the other hand, $\langle \alpha, \alpha_r \rangle < 1$ by (3.32), as $\alpha$ is of depth greater than 1 and cannot dominate $\alpha_r$. Thus $\langle \alpha, \alpha_r \rangle \in (0, 1)$, and hence $\langle r \cdot \alpha, \alpha_r \rangle \in (-1, 0)$, as required.
Lemma (3.38) Let $\alpha, \beta \in \Phi^+$ with $\beta \leq \alpha$, and let $r \in R$ such that $\langle \beta, \alpha_r \rangle$ is less than or equal to $-1$. Then $\alpha \in \Delta$, or the coefficients of $\alpha_r$ in $\alpha$ and $\beta$ coincide.

Proof. Suppose that the coefficients of $\alpha_r$ in $\alpha$ and $\beta$ do not coincide, and let $\gamma$ be a root of maximal depth with $\beta \leq \gamma \leq \alpha$ such that the coefficients of $\alpha_r$ in $\beta$ and $\gamma$ coincide; then $\gamma < r \cdot \gamma \leq \alpha$ by maximality of $\gamma$. Now $\gamma - \beta = \sum_{s \in R \setminus \{r\}} \lambda_s \alpha_s$ for some $\lambda_s \geq 0$, and hence

$$\langle r \cdot \gamma, \alpha_r \rangle = -\langle \gamma, \alpha_r \rangle = -\langle \beta, \alpha_r \rangle - \sum_{s \in R \setminus \{r\}} \lambda_s \langle \alpha_s, \alpha_r \rangle \geq 1,$$

since $\langle \alpha_s, \alpha_r \rangle \leq 0$ for $s \neq r$. As $dp(r \cdot \gamma) > 1 = dp(\alpha_r)$, this implies that $r \cdot \gamma \in \Delta$, and thus $\alpha \in \Delta$ by (3.36) \(\square\).

Corollary (3.39) Let $\alpha \in \Phi^+$ such that supp($\alpha$) contains a circuit. Then $\alpha \in \Delta$.

Proof. Let $\beta$ be a positive root of minimal depth preceding $\alpha$ such that supp($\beta$) contains a circuit, and let $r \in R$ with $r \cdot \beta < \beta$. By minimality of $\beta$ we find that supp($r \cdot \beta$) does not contain a circuit. Now let $\beta = \sum_{x \in R} \lambda_x \alpha_x$ and $r \cdot \beta = \sum_{x \in R} \mu_x \alpha_x$. Since $\lambda_x = \mu_x$ for all $x \neq r$, and the support of $\beta$ contains a circuit, while the support of $r \cdot \beta$ does not, it follows that $\mu_r = 0$, and that $\alpha_r$ is part of a circuit in supp($\beta$). Hence there exist at least two elements $\alpha_s, \alpha_t$ of supp($\beta$) \(\setminus \{\alpha_r\}\) such that $\alpha_r$ is adjoined to $\alpha_s$ as well as $\alpha_t$. By definition of $\langle \cdot, \cdot \rangle$, it follows that $\langle \alpha_r, \alpha_s \rangle$ and $\langle \alpha_r, \alpha_t \rangle$ are both at most $-\cos(\pi/3) = -\frac{1}{2}$, while $\langle \alpha_r, \alpha_x \rangle \leq 0$ for all other $x \in R \setminus \{r\}$. Furthermore, $\mu_s, \mu_t \geq 1$ by (2.26), and thus

$$\langle r \cdot \beta, \alpha_r \rangle = \sum_{x \in R} \mu_x \langle \alpha_x, \alpha_r \rangle = \sum_{x \neq r} \mu_x \langle \alpha_x, \alpha_r \rangle \leq \mu_s \langle \alpha_s, \alpha_r \rangle + \mu_t \langle \alpha_t, \alpha_r \rangle \leq -1.$$

Since the coefficient of $\alpha_r$ in $\beta$ is not equal to the coefficient of $\alpha_r$ in $r \cdot \beta$, Lemma (3.38) implies that $\beta \in \Delta$; hence $\alpha \in \Delta$ by (3.36), as required. \(\square\)
Chapter 3  Dominance and elementary roots

(3.40) **Corollary**  Let \( r, s \in R \) be adjoined. Further, suppose that \( \alpha \) and \( \beta \) are positive roots with \( \alpha \succeq \beta \) such that \( r \in I(\alpha) \setminus I(\beta) \), and the coefficient of \( \alpha_s \) in \( \beta \) is greater than or equal to \( |\langle \alpha_r, \alpha_s \rangle|^{-1} \). Then \( \alpha \in \Delta \).

**Proof.** Let \( \beta = \sum_{t \in R \setminus \{r\}} \lambda_t \alpha_t \). Then

\[
\langle \beta, \alpha_r \rangle = \sum_{t \in R \setminus \{r\}} \lambda_t \langle \alpha_t, \alpha_r \rangle = \sum_{t \neq r, s} \lambda_t \langle \alpha_t, \alpha_r \rangle + \lambda_s \langle \alpha_s, \alpha_r \rangle \leq \lambda_s \langle \alpha_s, \alpha_r \rangle \leq -1.
\]

Since the coefficient of \( \alpha_r \) in \( \alpha \) is not equal to the coefficient of \( \alpha_r \) in \( \beta \), Lemma (3.38) implies that \( \alpha \in \Delta \). \( \square \)

(3.41) **Corollary**  Let \( r, s \in R \) such that \( r \) and \( s \) are adjoined by an infinite bond, and suppose that \( \alpha \) is a positive root with both \( \alpha_r \) and \( \alpha_s \) in its support. Then \( \alpha \in \Delta \).

**Proof.** Interchanging \( r \) and \( s \) if necessary, we may choose \( \beta \preceq \alpha \) such that \( r \in I(\beta) \) and \( s \not\in I(\beta) \). Since the coefficient of \( \alpha_r \) in \( \beta \) is greater than or equal to 1 by (2.26), and thus greater than or equal to \( |\langle \alpha_r, \alpha_s \rangle|^{-1} \), (3.40) implies that \( \alpha \in \Delta \). \( \square \)

Our proof of the fact that \( E \) is finite (if \( R \) is finite) depends on the finiteness of the set of real numbers \( \{ \langle \alpha, \alpha_r \rangle \mid \alpha \in \mathcal{E} \text{ and } r \in R \} \). The next definition facilitates the statement of the relevant facts.

Define \( C(R) \) to be the set of all real numbers of the form \( \cos(n \pi/m) \) with \( n \in \{1, \ldots, m-1\} \) and \( m = m_{rs} < \infty \) for some \( r, s \in R \). If \( R \) is finite, then \( |C(R)| \) is less than or equal to the sum of all \( m-1 \) with \( m = m_{rs} < \infty \) for some \( r, s \in R \).

The next proposition is a slight variation of (1.12). It yields that if \( \langle \alpha, \beta \rangle \in (-1, 1) \) for some roots \( \alpha \) and \( \beta \), it follows that \( \langle \alpha, \beta \rangle \in C(R) \). For (2.24) implies that \( W_{\{\alpha, \beta\}} \) is finite if \( \langle \alpha, \beta \rangle \in (-1, 1) \), and by the next assertion there exist \( r, s \in R \) and \( w \in W \) such that \( w \cdot \alpha, w \cdot \beta \in \Phi_{\{r, s\}} \); therefore we can deduce from (1.1)(i) that

\[
\langle \alpha, \beta \rangle = \langle w \cdot \alpha, w \cdot \beta \rangle = \cos(n \pi/m_{rs})
\]

for some \( n \in \mathbb{N} \), as required.
(3.42) **Proposition** Let $\Gamma \subseteq \Phi$ such that $W_{\Gamma}$ is finite. Then there exists a finite parabolic subgroup $W_I$ of $W$ with $|I| \leq |\Gamma|$ such that $W_{\Gamma}$ is conjugate to a subgroup of $W_I$.

**Proof.** Since $W_{\Gamma}$ is finite, we know by (1.12) that $W_{\Gamma}$ is conjugate to a subgroup of a finite parabolic subgroup $W_J$, and we may assume without loss of generality that $W_{\Gamma} \subseteq W_J$. If $|J| = |\Gamma|$ the assertion is true, so suppose now that $|J| > |\Gamma|$. We show that there exists an $I \subseteq J$ with $I \neq J$ such that $W_{\Gamma}$ is conjugate to a subgroup of $W_I$, and the assertion will follow by induction.

As in the proof of (1.12), let $V_J^*$ denote the dual space of $V_J$, acted upon from the right by $W_J$, and for $f \in V_J^*$ define $S(f) = \{ \gamma \in \Phi_J^+ \mid f(\gamma) < 0 \}$. Since the space spanned by $\Gamma$ is a subspace of $V_J$, and has dimension less than $|J|$ (the dimension of $V_J$), there exists a nonzero vector $v_0 \in V_J$ such that $\langle v_0, \gamma \rangle = 0$ for all $\gamma \in \Gamma$. Define $F \in V_J^*$ by $F : v \mapsto \langle v, v_0 \rangle$; then $S(F)$ is finite, since $\Phi_J^+$ is finite. As in the proof of (1.12) there exists an $x \in W$ such that $S(Fx) = \emptyset$ and $xW_{\Gamma}x^{-1} \subseteq W_I$, where $I = \{ r \in J \mid (Fx)(\alpha_r) = 0 \}$. Theorem (3.33) states that the Coxeter graph of $J$ consists of finitely many connected components, each of which is of one of the shapes described in (3.33), and thus it can be easily verified that $\langle \cdot, \cdot \rangle$ restricted to $V_J$ is positive definite; therefore $F \neq 0$, and thus $I \neq J$, as required.

The following technical lemma, though trivial, provides the key for our proof of the main theorem.

(3.43) **Lemma** Let $\alpha = \sum_{r \in R} \lambda_r \alpha_r$ and $\beta = \sum_{r \in R} \mu_r \alpha_r$ be positive roots. Furthermore, suppose that there exist $R_1, R_2 \subseteq R$ with $R = R_1 \cup R_2$ such that $\langle \alpha, \alpha_r \rangle = \langle \beta, \alpha_r \rangle$ for all $r \in R_1$, and $\lambda_r = \mu_r$ for all $r \in R_2$. Then $\langle \alpha, \beta \rangle = 1$.

**Proof.** Since $\alpha - \beta = \sum_{r \in R_1} (\lambda_r - \mu_r) \alpha_r$, we have

$$\langle \alpha, \alpha - \beta \rangle = \sum_{r \in R_1} (\lambda_r - \mu_r) \langle \alpha, \alpha_r \rangle = \sum_{r \in R_1} (\lambda_r - \mu_r) \langle \beta, \alpha_r \rangle = \langle \beta, \alpha - \beta \rangle,$$

and as $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$, this becomes $1 - \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle - 1$, and the result follows.
Chapter 3  
Dominance and elementary roots

(3.44) **Theorem**  
\( \mathcal{E} \) is finite, provided that \( R \) is finite.

**Proof.** If \( R \) is finite, it is clear that \( \mathcal{C}(R) \) is finite, and we set \( c = |\mathcal{C}(R)| \). Since every root of depth \( d \) can be expressed as \( (r_d r_{d-1} \cdots r_2) \cdot \alpha_{r_1} \) with each \( r_i \in R \), there are no more than \( |R|^d \) roots of depth \( d \). If we can show that no root in \( \mathcal{E} \) can have depth exceeding \( c|R|(|R|+1) + 1 \), the proof will be complete. So let \( \beta \in \mathcal{E} \) have depth \( d \), and let \( \beta_1 \prec \cdots \prec \beta_d = \beta \) be a sequence of positive roots such that \( \text{dp}(\beta_i) = i \). Note that \( \beta_i \in \mathcal{E} \) for each \( i \in \{1, \ldots, d\} \). For \( i \in \{1, \ldots, d\} \), define \( J_i = \{ r \in R \mid \langle \beta_i, \alpha_r \rangle > -1 \} \). If \( r \notin J_i \), the coefficient of \( \alpha_r \) in \( \beta_j \) is constant for all \( j \geq i \) by (3.38); since \( \langle \alpha_{\gamma}, \alpha_r \rangle \leq 0 \) for all \( s \in R \setminus \{r\} \), it follows from (3.35) that \( \langle \beta_j, \alpha_r \rangle \leq \langle \beta_i, \alpha_r \rangle \) for \( j \geq i \), and hence \( r \notin J_j \) for all \( j \geq i \). Thus the sets \( J_i \) form a decreasing chain.

Suppose \( J_i = \cdots = J_j = J \) for some \( 2 \leq i \leq j \). If \( k \in \{i, \ldots, j\} \) and \( r \in R \), then \( \langle \beta_k, \alpha_r \rangle < 1 \) by (3.32); (since \( \beta_k \) is of depth greater than 1 and cannot dominate \( \alpha_r \)). Hence \( \langle \beta_k, \alpha_r \rangle \in (-1,1) \) for \( r \in J \), and thus \( \langle \beta_k, \alpha_r \rangle \in \mathcal{C}(R) \) by the remark preceding (3.42). So if \( j - i \geq c|R| \), then there exist \( m, n \in \{i, i+1, \ldots, j\} \) with \( n > m \) and \( \langle \beta_n, \alpha_r \rangle = \langle \beta_m, \alpha_r \rangle \) for all \( r \in J \). But if \( r \notin J \), then \( \alpha_r \) has the same coefficient in \( \beta_m \) as in \( \beta_n \), and it follows by (3.43) that \( \langle \beta_n, \beta_m \rangle = 1 \). This contradicts (3.32), since \( \beta_m \notin \Delta \). We conclude that if \( j - i \geq c|R| \), then \( J_j \) is strictly smaller than \( J_i \). Since \( J_2 \subseteq R \), it follows that the chain \( J_2 \supseteq J_3 \supseteq \cdots \supseteq J_d \) can have length at most \( c|R|(|R|+1) \), and this finishes the proof.

(3.45) **Lemma**  
Let \( \alpha \in \Phi^+ \). Then \( \alpha \in \mathcal{E} \) or there exists \( \beta \in \Phi^+ \) such that \( \alpha \dom \beta \) and \( r_\beta \cdot \alpha \in \Phi^+ \).

**Proof.** If \( \text{dp}(\alpha) = 1 \) then \( \alpha \in \mathcal{E} \) and there is nothing left to show. So suppose \( \text{dp}(\alpha) > 1 \) and \( \alpha \notin \mathcal{E} \). Now let \( \gamma \) be a positive root different from \( \alpha \) which is dominated by \( \alpha \). and let \( r \in R \) such that \( r \cdot \alpha \prec \alpha \). If \( \gamma = \alpha_r \) choose \( \beta = \alpha_r \). Then \( r_\beta \cdot \alpha = r \cdot \alpha \) has depth greater or equal to 1 and hence is in \( \Phi^+ \).

Suppose next \( \gamma \neq \alpha_r \). Then \( r \cdot \gamma \in \Phi^+ \) by (1.6) and hence \( r \cdot \alpha \) dominates \( r \cdot \gamma \), whence \( r \cdot \alpha \in \Delta_W \). By induction there exists a root \( \beta' \in \Phi^+ \) which is dominated by \( r \cdot \alpha \) such that \( r_\beta' \cdot (r \cdot \alpha) \in \Phi^+ \). Since \( \langle r \cdot \alpha, \alpha_r \rangle < 0 \) as \( r \cdot \alpha \prec \alpha \) while on the other hand \( \langle r \cdot \alpha, \beta' \rangle \geq 1 \) by (3.22), we know \( \beta' \neq \alpha_r \). Hence \( r \cdot \beta' \in \Phi^+ \) by (1.6) and thus \( \alpha \dom r \cdot \beta' \) as \( \beta' \in \Phi^+ \).

If \( r_{r \cdot \beta'} \cdot \alpha \in \Phi^+ \) choose \( \beta = r \cdot \beta' \). Then clearly \( \alpha \dom \beta \) and \( r_\beta \cdot \alpha \in \Phi^+ \). This leaves us with the case \( r_{r \cdot \beta'} \cdot \alpha \in \Phi^- \). Since \( r_{r \cdot \beta'} = rr_{\beta'} r \) by (1.9) we
find
\[ r_{r' \cdot \alpha} \cdot \alpha = r r_{r' \cdot \alpha} = r \cdot (r_{r' \cdot \alpha} \cdot (r \cdot \alpha)). \]

As \( r_{r' \cdot (r \cdot \alpha)} \in \Phi^+ \) by choice of \( r' \), this forces \( r_{r' \cdot (r \cdot \alpha)} = \alpha \).

That is,
\[ r \cdot \alpha = r_{r' \cdot \alpha} = \alpha - 2\langle \alpha, r' \rangle \beta'. \]

Now \( 1 \leq \langle r \cdot \alpha, \beta' \rangle \) by (3.32) and this equals
\[ \langle \alpha, \beta' \rangle - 2\langle \alpha, \beta' \rangle \langle \beta', \beta' \rangle = -\langle \alpha, \beta' \rangle. \]

Thus \( \langle r \cdot \beta', \alpha \rangle = -\langle \alpha, \beta' \rangle \geq 1 \) and clearly \( dp(r \cdot \beta') \geq 1 = dp(\alpha) \); by (3.32) this yields \( r \cdot \beta' \text{ dom } \alpha \), and by transitivity of dominance we find \( \alpha \text{ dom } \alpha \).

So if we choose \( \beta = \alpha \), then certainly \( \alpha \text{ dom } \beta \) and \( r_{r' \cdot \alpha} = r \cdot \alpha \in \Phi^+ \).

\[ \square \]

**Chapter 4**

**The Stabilizer of a Root**

We now show that the stabilizer of a root is the semidirect product of a Coxeter group and a free group.

For a root \( \alpha \), we denote the stabilizer of \( \alpha \) in \( W \) by \( W(\alpha) \). Any root can be written as \( w \cdot \alpha_r \) for some \( w \in W \) and \( r \in R \), and an easy calculation yields that \( W(w \cdot \alpha_r) = wW(\alpha_r)w^{-1} \); therefore we can restrict our attention to \( W(\alpha_r) \) for \( r \in R \).

Let \( \Gamma(r) \) be the set of roots \( \gamma \) with \( \langle \alpha_r, \gamma \rangle = 0 \); that is, \( \Gamma(r) \) equals \( V_0(\alpha_r) \cap \Phi \). The group \( W_{\Gamma(r)} \) generated by the reflections corresponding to the roots in \( \Gamma(r) \) is a normal subgroup of \( W(\alpha_r) \); moreover, Theorem (2.21) states that \( W_{\Gamma(r)} \) is a Coxeter group. We will show that \( W_{\Gamma(r)} \) has a complement \( Y_r \) in \( W(\alpha_r) \), and that \( Y_r \) is isomorphic to the fundamental group of a certain graph. Well known arguments then show that \( Y_r \) is a free group.

Next, let \( X_r \) be a set of coset representatives of \( W_{\Gamma(r)} \) in \( W(\alpha_r) \) of minimal length. For \( w \in X_r \), the minimality of \( l(w) \) yields that \( l(wr_\gamma) \geq l(w) \) for all roots \( \gamma \in \Gamma(r) \), and it follows by the Strong Exchange Condition that
$N(w) \cap \Gamma(r) = \emptyset$; that is, $N_0(w, \alpha_r) = \emptyset$. So $X_r$ is a subset of $Y_r$, the set of all $w \in W(\alpha_r)$ with $N_0(w, \alpha_r) = \emptyset$. We will see shortly that $X_r = Y_r$, and then use the concepts developed in the preceding chapter to prove that $Y_r$ has the properties we have described.

(4.46) Lemma Let $\alpha \in \Phi$ and $w \in W$ such that $N_0(w, \alpha) = \emptyset$. Then $N_0(w^{-1}, w \cdot \alpha) = \emptyset$.

Proof. Since $V_0(w \cdot \alpha) = w \cdot V_0(\alpha)$, we know that

$$N_0(w^{-1}, w \cdot \alpha) = N(w^{-1}) \cap V_0(w \cdot \alpha) = w \cdot (w^{-1} \cdot N(w^{-1}) \cap V_0(\alpha)).$$

Further, $w^{-1} \cdot N(w^{-1}) = -N(w)$ by (1.10), while $-V_0(\alpha) = V_0(\alpha)$; therefore

$$N_0(w^{-1}, w \cdot \alpha) = -w \cdot (N(w) \cap V_0(\alpha)) = -w \cdot N_0(w, \alpha) = \emptyset.$$

Now let $w \in Y_r$. Then $N_0(w^{-1}, \alpha_r) = N_0(w^{-1}, w \cdot \alpha_r) = \emptyset$ by (4.46), and thus $w^{-1} \in Y_r$. If $u, w \in Y_r$, (2.17) yields that

$$N_\circ (uw, \alpha_r) \subseteq N_0(w, \alpha_r) \cup w^{-1} \cdot N_0(u, w \cdot \alpha_r) = N_0(w, \alpha_r) \cup w^{-1} \cdot N_0(u, \alpha_r) = \emptyset,$$

and therefore also $uw \in Y_r$. Since clearly $1 \in Y_r$, this proves that $Y_r$ is a group.

We show next that $Y_r = X_r$. Let $w \in Y_r$, and let $u \in X_r$ be the representative of the coset $wW_{\Gamma(r)}$; then $u \in Y_r$, and thus $w^{-1}u \in Y_r$ and

$$N(w^{-1}u) \cap \Gamma(r) = N_0(w^{-1}u, \alpha_r) = \emptyset.$$ 

It is clear that $\Phi_{\Gamma(r)} = \Gamma(r)$, and so the above becomes:

$$(w^{-1}u) \cdot (\Phi_{\Gamma(r)} \cap \Phi^+) \subseteq \Phi_{\Gamma(r)} \cap \Phi^+.$$
As $w^{-1}u \in W_{\Gamma(r)}$, faithfulness of the standard geometric realization together with (2.23) now imply that $w^{-1}u$ equals 1; that is, $w = u \in X_r$. Whence $X_r = Y_r$, and it remains to show that $Y_r$ is a free group.

We obtain the odd Coxeter graph of $W$ by deleting all edges of even weight as well as all edges of infinite weight from the Coxeter graph. Let the set $\mathcal{F}$ consist of all $(l + 1)$-tuples $(r_0, \ldots, r_l)$ with $r_0, \ldots, r_l \in R$ such that $r_{i-1}$ and $r_i$ are adjoined in the odd Coxeter graph for all $i \in \{1, \ldots, l\}$. If $s, t \in R$ are adjoined by a bond of weight $2n + 1$, we define $\pi(s, t) = (ts)^n$; then $s\pi(s, t)$ is the uniquely determined word in $W_{\{s, t\}}$ of maximal length, and thus

$$N(\pi(s, t)) = \Phi^+_{\{s, t\}} \setminus \{\alpha_i\}.$$ 

Define $\pi: \mathcal{F} \to W$ by $\pi(r_0) = 1$ and for $l \geq 1$,

$$\pi(r_0, \ldots, r_l) = \pi(r_0, r_1)\pi(r_1, r_2)\pi(r_2, r_3) \cdots \pi(r_{l-2}, r_{l-1})\pi(r_{l-1}, r_l).$$

Now let $(r_0, \ldots, r_l) \in \mathcal{F}$, and set $w = \pi(r_0, \ldots, r_l)$. It follows from (1.1)(i) that $\pi(r_{i-1}, r_i) \cdot \alpha_{r_i} = \alpha_{r_{i-1}}$ for all $i$, and thus $\pi(r_1, \ldots, r_l) \cdot \alpha_{r_i} = \alpha_{r_i}$; in particular, $w \cdot \alpha_{r_i} = \alpha_{r_0}$. An iteration of (2.17) yields that

$$N_0(w, \alpha_{r_i}) \subseteq \bigcup_{i=1}^{l} \pi(r_i, \cdots r_i) \cdot N_0(\pi(r_{i-1}, r_i), \pi(r_i, \ldots, r_l) \cdot \alpha_{r_i})$$

$$= \bigcup_{i=1}^{l} \pi(r_i, \cdots r_i) \cdot N_0(\pi(r_{i-1}, r_i), \alpha_{r_i}).$$

Since the order of $r_{i-1}r_i$ is odd, we deduce from (1.1)(i) that $\Phi_{\{r_{i-1}, r_i\}}$ contains no roots perpendicular to $\alpha_{r_i}$, and thus $N_0(\pi(r_{i-1}, r_i), \alpha_{r_i})$ is empty for all $i$; hence $N_0(w, \alpha_{r_i}) = \emptyset$. In particular, if $r_0 = r_l = r$ then $\pi(r_0, \ldots, r_l)$ is an element of $Y_r$.

Next, define $L: \mathcal{F} \to \mathbb{N}_0$ by $L(r_0, \ldots, r_l) = \sum_{i=1}^{l} l(\pi(r_{i-1}, r_i))$. It is clear that $L(s) \geq l(\pi(s))$ for all $s \in \mathcal{F}$, and we define $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ to be the set of all $s \in \mathcal{F}$ with $L(s) = l(\pi(s))$. Note that $(s, t) \in \tilde{\mathcal{F}}$ for all $(s, t) \in \mathcal{F}$. If $(r_0, \ldots, r_l) \in \tilde{\mathcal{F}}$, it is certainly necessary that $r_{i-1} \neq r_{i+1}$ for all $i$, since

$$\pi(s, t, s) = \pi(s, t)\pi(t, s) = 1.$$ 

We will soon see that this condition is also sufficient.
(4.47) Proposition. Let \( w \in W \) and \( r, s \in R \). Then \( w \cdot \alpha_r = \alpha_s \) with \( N_0(w, \alpha_r) = \emptyset \) if and only if there exists some \((r_0, \ldots, r_l) \in \mathcal{F} \) with \( r_0 = s \) and \( r_l = r \) such that \( w = \pi(r_0, \ldots, r_l) \).

Proof. We have seen above that \( w \cdot \alpha_r = \alpha_s \) and that \( N_0(w, \alpha_r) \) is empty for \( w = \pi(r_0, \ldots, r_l) \) with \( r_l = r \), \( r_0 = s \) and \((r_0, \ldots, r_l) \in \mathcal{F} \subseteq \mathcal{F} \); so it suffices to show the converse.

Suppose that \( w \cdot \alpha_r = \alpha_s \) and \( N_0(w, \alpha_r) = \emptyset \). If \( w = 1 \), the assertion is true with \( l = 0 \). So assume that \( w \neq 1 \), and proceed by induction. Let \( t \in R \) with \( l(wt) < l(w) \); then \( t \neq r \) by (1.5), as \( w \cdot \alpha_r = \alpha_s \) is positive. Set \( I = \{r, t\} \), and let \( u \in W_I \) be of maximal length such that \( w = du \) for some \( d \in W \) with \( l(w) = l(d) + l(u) \). Then by (2.17),

\[
N_0(u, \alpha_r) = \emptyset \text{ and } N_0(d, u \cdot \alpha_r) = \emptyset. \tag{*} \]

Now let \( \lambda, \mu \in \mathbb{R} \) such that \( u \cdot \alpha_r = \lambda \alpha_r + \mu \alpha_t \); then

\[
\alpha_s = w \cdot \alpha_r = d \cdot (u \cdot \alpha_r) = \lambda(d \cdot \alpha_r) + \mu(d \cdot \alpha_t) \]

and \( \lambda, \mu \geq 0 \) or \( \lambda, \mu \leq 0 \). Maximality of \( u \) together with (1.5) force \( d \cdot \alpha_r \) and \( d \cdot \alpha_t \) to be positive, and as \( \alpha_t \) is positive, we deduce that \( \lambda, \mu \geq 0 \); moreover, \( \lambda = 0 \) or \( \mu = 0 \) since \( \alpha_s \) is simple. That is, \( u \cdot \alpha_r = \alpha_r \) or \( u \cdot \alpha_r = \alpha_t \), and we denote \( u \cdot \alpha_r \) by \( \alpha_z \).

Since \( u \cdot \alpha_r \) is positive, we know by (1.5) that \( l_I(wr) = l_I(u) + 1 \), and as \( u \) is an element of \( W_I \) and \( u \neq 1 \), this forces \( l_I(u) < l_I(u) \); that is, \( u \cdot \alpha_t \in \Phi^- \).

Furthermore, \( u \cdot \alpha_r \) and \( u \cdot \alpha_t \) are linearly independent since \( \alpha_r \) and \( \alpha_t \) are linearly independent, and thus in particular \( u \cdot \alpha_t \neq -u \cdot \alpha_r = -\alpha_z \). So \((zu) \cdot \alpha_t \in \Phi^- \) by (1.4) and clearly \((zu) \cdot \alpha_r = -\alpha_z \in \Phi^- \). Now \( N(zu) \) includes all positive roots which are linear combinations of \( \alpha_r \) and \( \alpha_t \), and thus \( rt \) must have finite order, and \( zu \) is the uniquely determined word in \( W_I \) of maximal length.

Assume for a contradiction that the order of \( rt \) is even and equals \( 2m \). Then \( u = t(rt)^{m-1} \), and we can deduce from (1.1) (i) that \((tr)^{m-1} \cdot \alpha_t \) is in \( N_0(u, \alpha_r) \) if \( m \) is odd, and \( t(rt)^{m-1} \cdot \alpha_r \) is in \( N_0(u, \alpha_r) \) if \( m \) is even. Both of these contradict (**), so \( rt \) must have odd order. Then \( u = \pi(t, r) \); moreover, \( d \cdot \alpha_t = d \cdot (u \cdot \alpha_r) = w \cdot \alpha_r = \alpha_s \) with \( N_0(d, \alpha_r) = N_0(d, u \cdot \alpha_r) = \emptyset \) by (**). Since \( l(u) \geq l(t) = 1 \), we know that \( l(d) < l(w) \), and by induction there
exists an \((r_0, \ldots, r_l) \in \tilde{F}\) with \(r_0 = s\) and \(r_l = t\) such that \(d = \pi(r_0, \ldots, r_l)\). Now \((r_0, \ldots, r_l, r) \in F, and
\[ w = du = \pi(r_0, \ldots, r_l)\pi(r_l, r) = \pi(r_0, \ldots, r_l, r); \]

furthermore, \(l(w) = l(d) + l(u) = L(r_0, \ldots, r_l) + L(r_l, r) = L(r_0, \ldots, r_l, r)\), and thus \((r_0, \ldots, r_l, r)\) is in \(F\), as required.

It is clear that the elements of the fundamental group of a connected graph \(\Xi\) can be identified with paths in \(\Xi\) which start and end at a fixed vertex \(v\), and which never back-track upon themselves; that is, at no stage does the path traverse an edge and then immediately traverse it again in the opposite direction. It is well known (see, for example, [8], Chapter 6, Theorem (5.2), p.198) that this fundamental group is free of rank \(\epsilon - \nu + 1\), where \(\epsilon\) is the number of edges, and \(\nu\) is the number of vertices of \(\Xi\). (This follows from the fact that \(\Xi\) is homotopy equivalent to a graph on one vertex with \(\epsilon - \nu + 1\) edges - topologically, a bouquet of circles - as can be seen by shrinking a spanning tree of \(\Gamma\) to a single vertex.)

Specifically, if the graph \(\Xi\) is the connected component of the odd Coxeter graph containing the vertex \(r \in R\), then the elements of the fundamental group of \(\Xi\) can be identified with the set
\[ \mathcal{F}_r = \{(r_0, \ldots, r_l) \in F \mid r_0 = r_l = r \text{ and } r_{i-1} \neq r_{i+1} \text{ for all } i \in \{1, \ldots, l\}\}, \]
multiplication being defined by the rule
\[ (r_0, \ldots, r_l) \ast (s_0, \ldots, s_m) = (r_0, \ldots, r_{l-i-1}, s_i, \ldots, s_m), \]
where \(i\) is the maximal integer such that \(s_j = r_{l-j}\) for all \(j \in \{0, \ldots, i\}\). Note that the identity of this group is \((r)\). At the beginning of this chapter we have seen that \(\pi\) maps \(\mathcal{F}_r\) into \(Y_r\), and since \(\pi(s, t, s) = 1\) for \((s, t, s) \in F\), we conclude that \(\pi\) induces a homomorphism \(\pi_r\) from \(\mathcal{F}_r\) to \(Y_r\). If \((r_0, \ldots, r_l)\) is in \(\tilde{F}\) with \(r_0 = r_l = r\), clearly \((r_0, \ldots, r_l) \in F_r\), and so (4.47) implies that \(\pi_r\) is surjective. The next proposition yields that \(\mathcal{F}_r \subseteq \tilde{F}\). So if \(s \in \mathcal{F}_r\) is in the kernel of \(\pi_r\), then
\[ L(s) = l(\pi(s)) = l(\pi_r(s)) = l(1) = 0, \]
and thus \(s = (r)\). Hence \(\pi_r\) is also injective, and therefore a group isomorphism.
(4.48) Proposition. Let \((r_0, \ldots, r_l) \in \mathcal{F} \setminus \tilde{\mathcal{F}}\). Then \(r_{i-1} = r_{i+1}\) for some \(i \in \{1, \ldots, l-1\}\).

Proof. In order to avoid double indices, we denote the simple root corresponding to \(r_i\) by \(\alpha_i\) for all \(i\). Now let \(m \in \{0, \ldots, l\}\) be minimal such that \((r_0, \ldots, r_m) \notin \mathcal{F}\), and let \(n \in \{0, \ldots, m\}\) be maximal such that \((r_n, \ldots, r_m)\) is not in \(\mathcal{F}\). Since \((r_n, \ldots, r_m)\) is also in \(\mathcal{F} \setminus \tilde{\mathcal{F}}\), and as it suffices to find one \(i\) such that \(r_{i-1} = r_{i+1}\), we may assume without loss of generality that \(n = 0\) and \(m = l\). Then \((r_0, \ldots, r_{l-1}), (r_1, \ldots, r_l) \in \tilde{\mathcal{F}}\) by minimality of \(m\) and maximality of \(n\) respectively. Further \(l \geq 2\), since \((s, t) \in \tilde{\mathcal{F}}\) for all \((s, t) \in \mathcal{F}\).

Now define \(u_1 = \pi(r_0, r_1), w = \pi(r_1, \ldots, r_{l-1})\) and \(u_2 = \pi(r_{l-1}, r_l)\). The above yields that \(l(u_1) = l(u_1) + l(w)\) as well as \(l(wu_2) = l(w) + l(u_2)\), and thus by (1.11),

\[
N(u_1)w = w^{-1}N(u_1) \cup N(w) \text{ and } N(w) \cap N(u_2^{-1}) = \emptyset.
\]

On the other hand, \((r_0, \ldots, r_l) \notin \tilde{\mathcal{F}}\) implies that \(l(u_1wu_2) < l(u_1) + l(u_2)\), and therefore \(N(u_1w) \cap N(u_2^{-1}) \neq \emptyset\) by (1.11); hence \(w^{-1}N(u_1) \cap N(u_2^{-1}) \neq \emptyset\) by the above. Now

\[
N(u_1) = \Phi_{\{\alpha_0, \alpha_1\}}^+ \setminus \{\alpha_1\} \text{ and } N(u_2^{-1}) = \Phi_{\{\alpha_{l-1}, \alpha_l\}}^+ \setminus \{\alpha_{l-1}\};
\]

thus there exist \(\lambda > 0, \mu \geq 0\) and \(x \geq 0, y > 0\) such that

\[
w^{-1} \cdot (\lambda \alpha_0 + \mu \alpha_1) = x \alpha_{l-1} + y \alpha_l.
\]

Since \(w^{-1} \cdot \alpha_1 = \alpha_{l-1}\), this yields that \(\lambda (w^{-1} \cdot \alpha_0) = (x - \mu) \alpha_{l-1} + y \alpha_l\); as \(y > 0\) and \(w^{-1} \cdot \alpha\) is either positive or negative, this forces \(x \geq \mu\). Symmetrically, \(y(w \cdot \alpha_l) = (\mu - x) \alpha_1 + \lambda \alpha_0\) and \(\lambda > 0\), and thus \(\mu = x\) and \(w \cdot \alpha_l = \alpha_0\). If \(l = 2\), then \(w = 1\) and \(\alpha_0 = 1 \cdot \alpha_2 = \alpha_2\), and thus \(r_0 = r_2\), as required.

Assume for a contradiction that \(l > 2\); then \((r_0, r_1, r_2)\) is in \(\tilde{\mathcal{F}}\) by minimality of \(m = l\), and this yields that \(r_0 \neq r_2\), and thus \(\alpha_0 \neq \alpha_2\). Let \(u = \pi(r_2, \ldots, r_{l-1})\); then \(w = \pi(r_1, r_2)u\) with

\[
l(w) = l(\pi(r_1, r_2)u) = l(\pi(r_1, r_2)) + l(u). \quad (*)
\]

Next, define \(\gamma\) to be \(\pi(r_1, r_2)^{-1} \cdot \alpha_0\); since \(w^{-1} \cdot \gamma = w^{-1} \cdot \alpha_0 = \alpha_l\) and \(\pi(r_1, r_2) \cdot \gamma = \alpha_0\) are both in \(\Pi\), Lemma (3.28) and (*) force \(\gamma\) to be an
elementary root. We show that this cannot be true, and this will be the desired contradiction.

We show first that $\gamma$ is preceded by $r_1 r_2 \cdot \alpha_0$. By definition of $\pi$, it is clear that $\pi^{-1}(r_1, r_2)$ equals $w' r_1 r_2$ for some $w' \in W_{\{r_1, r_2\}}$ of length $l(\pi^{-1}(r_1, r_2)) - 2$. Now

$$N_+(w', r_1 r_2 \cdot \alpha_0) \subseteq r_1 r_2 \cdot N_+(\pi^{-1}(r_1, r_2), \alpha_0)$$

by (2.17), and since $N(\pi^{-1}(r_1, r_2)) \subseteq \Phi^+_{\{r_1, r_2\}}$ and $\langle \alpha_0, \alpha_1 \rangle, \langle \alpha_0, \alpha_2 \rangle \leq 0$, it follows that $N_+(\pi^{-1}(r_1, r_2), \alpha_0)$ is empty. Therefore $N_+(w', r_1 r_2 \cdot \alpha_0)$ must be empty, and (3.34) yields that $\gamma \geq r_1 r_2 \cdot \alpha_0$. Now

$$r_1 r_2 \cdot \alpha_0 = \alpha_0 + (4\langle \alpha_0, \alpha_2 \rangle \langle \alpha_1, \alpha_2 \rangle - 2\langle \alpha_0, \alpha_1 \rangle) \alpha_1 + (-2\langle \alpha_0, \alpha_2 \rangle) \alpha_2,$$

with $\langle \alpha_0, \alpha_1 \rangle, \langle \alpha_1, \alpha_2 \rangle < 0$ (by construction), and $\langle \alpha_0, \alpha_2 \rangle \leq 0$ (as $r_0 \neq r_2$); so the coefficient of $\alpha_1$ in $r_1 r_2 \cdot \alpha_0$ is positive. Lemma (3.35) implies that there exist $a, b \in \mathbb{R}$ with $a > 0$ and $b \geq 0$ such that $\gamma = \alpha_0 + a \alpha_1 + b \alpha_2$. Further,

$$\alpha_1 = u^{-1} \cdot \gamma = (u^{-1} \cdot \alpha_0) + a(u^{-1} \cdot \alpha_1) + b(u^{-1} \cdot \alpha_2)$$

$$= (u^{-1} \cdot \alpha_0) + a(u^{-1} \cdot \alpha_1) + b \alpha_{l-1},$$

with $u^{-1} \cdot \alpha_1 \in \Phi^+$ (since $l(r_1 u) > l(u)$ by (*)). As $a > 0$ and $b \geq 0$, this forces $u^{-1} \cdot \alpha_0$ to be negative; that is, $\alpha_0 \in N(u^{-1})$. Now $N_0(u, \alpha_{l-1})$ is empty by (4.47), and thus (4.46) yields that

$$N(u^{-1}) \cap V_0(\alpha_2) = N_0(u^{-1}, \alpha_2) = N_0(u^{-1}, u \cdot \alpha_{l-1}) = \emptyset.$$ 

So $\alpha_0 \notin V_0(\alpha_2)$; that is, $\langle \alpha_0, \alpha_2 \rangle \neq 0$. Hence $r_0$ and $r_2$ are adjoined and $\langle \alpha_0, \alpha_2 \rangle < 0$. In particular, the coefficient of $\alpha_2$ in $r_1 r_2 \cdot \alpha_0$ is positive, and so the coefficient of $\alpha_2$ in $\gamma$ must also be positive by (3.35); that is, $b > 0$. Now $\alpha_0, \alpha_1, \alpha_2 \in \text{supp}(\gamma)$, and since these form a circuit, (3.39) forces $\gamma \in \Delta$. This contradicts our earlier conclusion that $\gamma$ is elementary, and thus $l = 2$ and $r_0 = r_2$ after all, as required.
Theorem 4.49 The stabilizer of $\alpha_r$ in $W$ is the semidirect product of $W_{\Gamma(r)}$ and $Y_r$. Moreover, $W_{\Gamma(r)}$ is a Coxeter group, and $Y_r$ is a free group of rank $e(r) - n(r) + 1$, where $e(r)$ denotes the number of edges and $n(r)$ the number of vertices of the connected component of the odd Coxeter graph of $W$ containing $r$.

Chapter 5

An Automatic Structure

The principal result of this chapter is that Coxeter groups with finite distinguished generating sets are automatic. This is proved in [2] under the assumption that the Parallel Wall Theorem is valid. In our proof, the concept of dominance introduced in Chapter 3 replaces the parallel wall property.

For a finite set $A$, let $A^*$ be the free monoid on $A$ with multiplication $\ast$. Any subset $L$ of $A^*$ is a language over the alphabet $A$, the elements of $A$ being the letters, and the elements of $L$ the words of the language. A language is regular if and only if there exists a deterministic finite state automaton which accepts the words of the language and rejects words which are not in the language. A deterministic finite state automaton is a quintuple $(S, A, \mu, Y, S_0)$, where $S$ is a finite set of states, $Y \subseteq S$ is the set of accept states, $S_0 \in S$ is the starting state and $\mu : S \times A \to S$ is the transition function. The automaton reads the letters of a word one at a time, starting from the left and in state $S_0$, and if it was in state $S$ before reading the letter $a$, its state after reading $a$ is $\mu(S, a)$. The automaton accepts the word if it is in an accept state after reading the final letter, and rejects it otherwise. We say that $(S, A, \mu, Y, S_0)$ recognizes $L$.

In order to define automaticity for groups, we shall also have to consider languages over the alphabet $\mathfrak{A} = ((A \cup \{\$\}) \times (A \cup \{\$\})) \setminus \{($, $\}$, where $\$ denotes a symbol which is not in $A$. For $(a_1, b_1), \ldots, (a_n, b_n)$ in $\mathfrak{A}$ we identify $(a_1, b_1) \ast \ldots \ast (a_n, b_n)$ with $(a_1 \ast \ldots \ast a_n, b_1 \ast \ldots \ast b_n)$. For $(a, b) \in A^* \times A^*$ we define $(a, b)^\$ \in \mathfrak{A}^*$ as follows: if $a$ and $b$ are of the same length, then $(a, b)^\$ equals $(a, b)$, while if $a$ and $b$ are of unequal length, then as many $\$’s are appended to the shorter of $a$ and $b$ as are necessary to make the lengths equal.
Chapter 5  An automatic structure

A group $G$ is said to be automatic if there exists a finite set $A$ of semigroup generators for $G$, and a language $L$ over $A$ such that the following are satisfied:

(i) the natural homomorphism $\pi: L \to G$ is surjective, and

(ii) $L_a = \{(x_1, x_2)^{\mathbf{A}} \in \mathbf{A}^* \mid x_1, x_2 \in L$ and $\pi(x_1) = \pi(x_2)a\}$ is a regular language over $\mathbf{A}$ for all $a \in A \cup \{1_G\}$.

We then say that $L$ yields an automatic structure for $G$.

Let $W$ be a Coxeter group with finite distinguished generating set $R$. We are going to construct a language $L$ over $R$ that yields an automatic structure for $W$.

If $x$ and $y$ are in $R^*$, we say that $y$ is a segment of $x$ if there exist $y' \in R^*$ such that $x = y' * y * y''$. For $r_1, r_2, \ldots, r_l \in R$ we define the length of $r_1 * r_2 * \cdots * r_l$ to be $\ell(r_1 * r_2 * \cdots * r_l) = l$. Recall that if $w \in W$, then $l(w) = \min\{\ell(x) \mid x \in \pi^{-1}(w)\}$, where $\pi: R^* \to W$ denotes the natural homomorphism. An element $x \in R^*$ is called a reduced word if $\ell(x) = l(\pi(x))$. Define $L'$ to be the language of all reduced words. Note that if $x$ is in $L'$, then any segment of $x$ must also be in $L'$.

Now let $\preceq$ be the lexicographical order on $R^*$ for some (arbitrary) ordering of $R$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. For $r \in R$ we define $R_r$ to be the set of all simple reflections $s$ with $s \prec r$, and $\Pi_r$ to be the set of $\alpha_s$ with $s \in R_r$. It is clear that for each $w \in W$ there exists a unique $\nu(w) \in \pi^{-1}(w)$ such that $\nu(w) \in L'$ and $\nu(w) \preceq x$ for all $x \in \pi^{-1}(w) \cap L'$. We define the language $L$ to consist of all these lexicographically minimal reduced words for the various elements of $W$:

$$L = \{\nu(w) \mid w \in W\} = \{y \in L' \mid y \preceq x \text{ for all } x \in L' \text{ with } \pi(x) = \pi(y)\}.$$  

Observe that $L$ coincides with ShortLex as defined in [6]. As above for $L'$, it is clear that if $x$ is in $L$, each segment of $x$ has to be in $L$.

(5.50) Proposition  Suppose $w \in W$ and $r \in R$ with $l(wr) = l(w) + 1$, and let $r_1, r_2, \ldots, r_l \in R$ with $\nu(w) = r_1 * r_2 * \cdots * r_l$. For $j \in \{1, 2, \ldots, l\}$ define $R_j = R_{r_j}$, and set $R_{l+1} = \{r\}$. Then $\nu(wr) = r_1 * \cdots * r_{l-1} * s * r_l * \cdots * r_l$, where $s \in R_{r_j}$ for some $j$.
where \(i \in \{1, 2, \ldots, l + 1\}\) is minimal such that there exists an \(s \in R_i\) with 
\((r_ir_{l-1} \cdots r_i) \cdot \alpha_s = \alpha_r.\)

Proof. Since \(l(wr) = l(w) + 1\) we know that \(\nu(w) * r\) is in \(L'\), and furthermore that there exist \(s_1, s_2, \ldots, s_{l+1} \in R\) with \(\nu(wr) = s_1 * s_2 * \cdots * s_{l+1}\). Then

\[
\nu(wr) = s_1 * s_2 * \cdots * s_{l+1} \preceq r_1 * \cdots * r_l * r
\]

by minimality of \(\nu(wr)\) in \(L' \cap \pi^{-1}(wr)\). Now \((s_1s_2 \cdots s_{l+1})r = w\), and by the Exchange Condition there exists an \(i \in \{1, \ldots, l + 1\}\) such that \(w\) equals \(s_1 \cdots s_{i-1}s_{i+1} \cdots s_{l+1}\). Thus

\[
\nu(w) = r_1 * \cdots * r_l \preceq s_1 * \cdots * s_{i-1} * s_{i+1} * \cdots * s_{l+1},
\]

and it is immediate from (1) and (2) that \(r_j = s_j\) for all \(j \in \{1, 2, \ldots, i - 1\}\).

We deduce that \(s_{i+1} \cdots s_{l+1} = r_i \cdots r_l\), and since both \(s_{i+1} * \cdots * s_{l+1}\) and \(r_i * \cdots * r_l\) are in \(L\) (as these are segments of elements of \(L\)), this yields that

\[r_i * \cdots * r_l = s_{i+1} * \cdots * s_{l+1},\]

and thus \(r_j = s_{j+1}\) for all \(j \in \{i, \ldots, l\}\). Now define \(s = s_i\); then \(\nu(wr)\) equals \(r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l\). If \(i = l + 1\) it is clear that \(s = r\). If \(i \leq l\), equation (1) yields that \(s \preceq r_i\), and since \(s * r_i = s * s_{i+1}\) is reduced, it follows that \(s < r_i\); that is, \(s \in R_i\).

Furthermore, \((r_1r_2 \cdots r_{l-1}r_l)r = wr = r_1r_2 \cdots r_{l-1}sr_i \cdots r_{l-1}r_l\), and thus

\[(r_ir_{i+1} \cdots r_l)(r_l \cdots r_{i+1}r_i) = s.\]

Therefore \((r_ir_{i+1} \cdots r_l) \cdot \alpha_r = \pm \alpha_s\), and since \(\alpha_s\) and \((r_ir_{i+1} \cdots r_l) \cdot \alpha_r\) are both positive (the latter one because \(r_i * r_{i+1} * \cdots * r_l * r\) is reduced), it follows that \((r_ir_{i+1} \cdots r_l) \cdot \alpha_r = \alpha_s\); that is, \((r_l \cdots r_{i+1}r_i) \cdot \alpha_s = \alpha_r\).

Assume for a contradiction that \((r_ir_{i-1} \cdots r_j) \cdot \alpha_t = \alpha_r\) for some \(j < i\) and \(t \in R_j\). Then \(\pi(r_1 * \cdots * r_{j-1} * t * r_j * \cdots * r_l) = wr\), and further

\[r_1 * \cdots * r_{j-1} * t * r_j * \cdots * r_l \preceq r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l = \nu(wr),\]

contradicting the minimality of \(\nu(wr)\). Hence \(i\) is minimal with the above property. \[\square\]
We now describe a finite state automaton $W$ which recognizes $L$. The accept states of the automaton $W$ will be the subsets $E$ of $\mathcal{E}$ such that $E = PLC(E) \cap \mathcal{E}$, where $PLC(E)$ denotes the set of nonnegative linear combinations of roots in $E$. We denote the set of accept states by $\mathcal{P}(\mathcal{E})$. There will be one reject state $F$, and the starting state is the empty set in $\mathcal{P}(E)$. The transition function $\mu: \mathcal{S} \times R \rightarrow \mathcal{S}$ is given by

$$
\mu(X, r) = \begin{cases} 
F & \text{if } X = F, \\
F & \text{if } X \in \mathcal{P}(\mathcal{E}) \text{ and } \alpha_r \in X, \\
PLC(r \cdot X \cup r \cdot \Pi_r \cup \{\alpha_r\}) \cap \mathcal{E} & \text{if } X \in \mathcal{P}(\mathcal{E}) \text{ and } \alpha_r \not\in X.
\end{cases}
$$

(5.51) Proposition The automaton $W$ recognizes the language $L$.

Proof. Let $r_1 \ast r_2 \ast \cdots \ast r_n \in R^*$, and denote the simple root corresponding to $r_i$ by $\alpha_i$. Set $X_0 = \emptyset$, and for $i \in \{1, \ldots, n\}$ define

$$
X_i = PLC(r_i \cdot X_{i-1} \cup r_i \cdot \Pi_r \cup \{\alpha_i\}) \cap \mathcal{E}.
$$

A straightforward induction yields that $W$ is either in state $X_i$ or $F$ after reading $r_1 \ast r_2 \ast \cdots \ast r_i$; moreover, if $W$ is in $F$ after reading $r_1 \ast r_2 \ast \cdots \ast r_i$, then there exists an $l \in \{1, \ldots, i\}$ such that $\alpha_l \in X_{l-1}$. We show now that $r_1 \ast r_2 \ast \cdots \ast r_n \in L$ if and only if $\alpha_l \not\in X_{l-1}$ for all $l \in \{1, \ldots, n\}$.

Suppose first that $\alpha_l \in X_{l-1}$ for some $l \in \{0, \ldots, n-1\}$. An easy induction yields that $X_{l-1}$ is a subset of

$$
PLC\left(\bigcup_{i=1}^{l-1} (r_{i-1} \cdots r_i) \cdot \Pi_r \cup \{(r_{l-1} \cdots r_{i+1}) \cdot \alpha_i \mid i \in \{1, \ldots, l-1\}\}\right);
$$

thus there exist nonnegative coefficients $\lambda^i_s$ and $\mu_i$ such that

$$
\alpha_l = \sum_{i=1}^{l-1} \sum_{\alpha_s \in \Pi_{r_i}} \lambda^i_s (r_{i-1} \cdots r_i) \cdot \alpha_s + \sum_{i=1}^{l-1} \mu_i (r_{l-1} \cdots r_{i+1}) \cdot \alpha_i,
$$

and this yields

$$
-\alpha_l = r_l \cdot \alpha_l = \sum_{i=1}^{l-1} \sum_{\alpha_s \in \Pi_{r_i}} \lambda^i_s (r_l r_{i-1} \cdots r_i) \cdot \alpha_s + \sum_{i=1}^{l-1} \mu_i (r_l r_{i-1} \cdots r_{i+1}) \cdot \alpha_i.
$$
that \( (r) \) and maximality of required. (3.29) yields that \( (r) \)

\[ \text{\textit{\textbf{\it{\alpha}}}} \]

it follows by a straightforward induction that \( (r) \) is not reduced. Then \( (r) \)

\[ \text{\textit{\textbf{\it{j}}} \in \text{\textit{\textbf{\it{L}}}} \} \]

Suppose next that \( (r) \) is not reduced. Then \( (r) \) is not reduced. (3.29) yields that there exists an \( (r) \).

Suppose next that \( (r) \) is not in \( L \); hence \( (r) \) is reduced, (3.29) yields that there exists an \( (r) \).

Suppose now that \( (r) \) is not reduced. Then \( (r) \) is not reduced, (3.29) yields that there exists an \( (r) \).

Assume first that \( (r) \) is not reduced. Then \( l(\pi(r)) = m - 1 \) by minimality of \( l \), and (1.5) yields that \( (r) \) is negative. Let \( i \in \{1, \ldots, m - 1\} \) be maximal such that \( (r) \) is negative; then \( (r) \) is negative by (1.4) and maximality of \( i \). Since \( r \) is reduced (as it is a segment of a reduced word), (3.29) yields that \( (r) \) is in \( E \) for all \( j \in \{i + 1, \ldots, m\} \). As \( \alpha_i \) is \( \alpha_i \) in \( \alpha_i \), an easy induction now yields that \( (r) \) is in \( \alpha_i \) for all \( j \in \{i + 1, \ldots, m\} \); in particular, \( \alpha_i \) is \( \alpha_i \) as required.

Assume now that \( (r) \) is reduced, but \( (r) \) is not reduced, \( (r) \) is reduced, but not in \( L \), Proposition (5.50) yields that there exists an \( i \in \{1, 2, \ldots, m - 1\} \) such that \( (r) \) is \( \alpha_i \) for some \( s \in R \). As \( (r) \) is reduced, (3.29) implies that \( (r) \) is \( E \) for all \( j \in \{i, \ldots, m - 1\} \), and since \( \alpha_s \) is \( \alpha_i \), it follows by a straightforward induction that \( \alpha_i = (r) \cdot \alpha_s \in X_i \), as required.

(5.52) Lemma Suppose \( w \in W \) and \( r \in R \) with \( l(wr) = l(w) - 1 \), and let \( r_1, \ldots, r_l \in R \) with \( (r) = r_1 \cdot \ldots \cdot r_l \). For \( i \in \{1, \ldots, l\} \) define \( R_i = R_{r_i} \), and set \( R_{l+1} = \{r\} \). Then there exists exactly one \( i \in \{1, 2, \ldots, l\} \) such that \( (r) \) is \( (r) \cdot \alpha_i = \alpha_r \); moreover, \( r_i \in R_{l+1} \) and

\[ \nu(wr) = r_1 \cdot \ldots \cdot r_{l-1} \cdot r_{l+1} \cdot \ldots \cdot r_l. \]
Chapter 5  An automatic structure  

Proof. Let \( s_1, s_2, \ldots, s_{l-1} \in R \) such that \( \nu(wr) = s_1 \ast s_2 \ast \cdots \ast s_{l-1} \). It is clear that \( \nu(wr) \ast r \) is reduced, and (5.50) yields that there exists some \( i \in \{1, \ldots, l\} \) such that

\[
\nu((wr)r) = s_1 \ast \cdots \ast s_{i-1} \ast s \ast s_i \ast \cdots \ast s_{l-1},
\]

with \( (s_{l-1} \cdots s_i) \cdot \alpha_s = \alpha_r \), and \( s = r \) if \( i = l \), while \( s < s_i \) if \( i \leq l - 1 \). But \( \nu((wr)r) = \nu(w) \) also equals \( r_1 \ast r_2 \ast \cdots \ast r_l \), and hence \( s_k = r_k \) for \( k \) in \( \{1, \ldots, i-1\} \), \( r_i = s \) and \( r_{j+1} = s_j \) for all \( j \in \{i, \ldots, l-1\} \). Thus

\[
\nu(wr) = r_1 \ast \cdots \ast r_{i-1} \ast r_{i+1} \ast \cdots \ast r_l
\]

with \( r_i < r_{i+1} \) if \( i < l \), and \( r_i = r \) if \( i = l \); that is, \( r_i \in R_{i+1} \).

Assume for a contradiction that there exists a \( j \in \{1, \ldots, l\} \setminus \{i\} \) with \( (r_i \cdots r_{j+1}) \cdot \alpha_{r_j} = \alpha_r \), and suppose without loss of generality that \( i < j \). Then \( (r_i \cdots r_{j+1}) \cdot \alpha_{r_i} = \alpha_{r_j} \) and thus \( r_i r_{i+1} \cdots r_j r_{j+1} = r_{i+1} \cdots r_{j+1} \); hence

\[
r_1 r_2 \cdots r_l = r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_j \cdots r_{j+1} \cdots r_l,
\]

and thus \( r_1 r_2 \cdots r_n \) is of length less than \( l \), contradicting our assumption that \( r_1 \ast r_2 \ast \cdots \ast r_l \) is reduced.

Observe that this yields an algorithm which determines \( \nu(wr) \) if \( \nu(w) \) is given: Suppose that \( \nu(w) = r_1 \ast r_2 \ast \cdots \ast r_l \), and set \( \beta_i = r_i \cdot \alpha_r \). If \( i \geq 2 \) and \( \beta_i \not= \alpha_{r_{i-1}} \), define \( \beta_{i-1} = r_{i-1} \cdot \beta_i \); otherwise

\[
\nu(wr) = r_1 \ast \ldots \ast r_{i-1} \ast r_{i+1} \ldots \ast r_l
\]

by the previous lemma, and the algorithm can terminate. If \( \beta_i \not= \alpha_{r_{i-1}} \) for all \( i \geq 2 \), two cases arise. Firstly, if \( \beta_i \in \Pi_{r_i} \) for some \( i \), let \( i \) be minimal with this property, and let \( s \in R_{r_i} \) with \( \beta_i = \alpha_s \); then

\[
\nu(wr) = r_1 \ast \ldots \ast r_{i-1} \ast s \ast r_i \ast \ldots \ast r_l
\]

by (5.50). Secondly, if \( \beta_i \not\in \Pi_{r_i} \) for all \( i \), then \( \nu(wr) = \nu(w) \ast r \) by (5.50).

Note that if \( s \in R \) with \( (r_i \cdots r_j) \cdot \alpha_r = \alpha_s \) for some \( i \), then by (3.29), \( (r_j \cdots r_l) \cdot \alpha_r \) must be elementary for all \( j \in \{i, \ldots, l\} \) since \( r_1 \cdots r_l \) is reduced. Hence we can stop our search for \( i \) according to the above description as soon as \( \beta_j \) is in \( \Delta \).
Now let $\mathcal{R} = ((R \cup \{\$\}) \times (R \cup \{\$\})) \setminus \{\{\$, \$\}\}$, and for each $r \in R \cup \{1\}$ define

$$L_r = \{ (x_1, x_2)^\$ \in \mathcal{R}^* \mid x_1, x_2 \in L \text{ and } \pi(x_1)r = \pi(x_2) \}.$$ 

Since the words of the language $L$ correspond bijectively to the elements of $W$, we find that $L_1 = \{ (\nu(w), \nu(w)) \mid w \in W \}$; therefore a trivial modification of $W$ will yield a finite state automaton that recognizes $L_1$. So we only need to show that $L_r$ is regular for $r \in R$. Then

$$L_r = \{ (\nu(w), \nu(wr))^\$ \mid w \in W \}$$
$$= \{ (\nu(w) \ast \$, \nu(wr)) \mid w \in W \text{ with } l(wr) = l(w) + 1 \}$$
$$\cup \{ (\nu(w), \nu(wr) \ast \$) \mid w \in W \text{ with } l(wr) = l(w) - 1 \}$$
$$= \{ \nu(wr) \ast \$, \nu(w) \mid w \in W \text{ with } l(wr) = l(w) - 1 \}$$
$$\cup \{ (\nu(w), \nu(wr) \ast \$) \mid w \in W \text{ with } l(wr) = l(w) - 1 \}.$$ 

In particular, $(x_1, x_2) \in L_r$ if and only if $(x_2, x_1) \in L_r$; moreover, (5.52) yields:

(5.53) **Corollary** Let $r \in R$ and $(a, b) \in \mathcal{R}^*$ with $a = s_1 \ast s_2 \ast \cdots \ast s_n$, and $b = t_1 \ast t_2 \ast \cdots \ast t_n$ for some $(s_1, t_1), \ldots, (s_n, t_n) \in \mathcal{R}$. Furthermore, let $l$ be maximal in $\{1, \ldots, n + 1\}$ such that $s_{l-1} = t_{l-1} \in R$. Then $(a, b) \in L_r$ if and only if

(i) $l = n$, $\{s_n, t_n\} = \{r, \$\}$ and $s_1 \ast \cdots \ast s_{n-1} \ast r \in L$, or

(ii) $l < n$, $s_l, t_l \in R$ and $s_l < t_l$, $s_i = t_{i-1} \in R$ for all $i \in \{l + 1, \ldots, n\}$, $t_n = \$ and $a \in L$ with $(s_n \cdot s_{l+1}) \cdot \alpha_{s_l} = \alpha_r$, or

(iii) $l < n$, $s_l, t_l \in R$ and $s_l > t_l$, $s_{i-1} = r_i \in R$ for all $i \in \{l + 1, \ldots, n\}$, $s_n = \$ and $b \in L$ with $(t_n \cdot t_{l+1}) \cdot \alpha_{t_l} = \alpha_r$.

We now describe a finite state automaton $W_r$ which recognizes $L_r$. The automaton $W_r$ has one accept state $A$, and there is one **failure state**, $F$ (from which there are no transitions to other states). All elements of $\mathfrak{P}(\mathcal{E})$ are states, and the remaining states are the elements of the Cartesian products $\mathfrak{P}(\mathcal{E}) \times \mathcal{E}_r \times R$ and $\mathfrak{P}(\mathcal{E}) \times R \times \mathcal{E}_r$, where $\mathcal{E}_r$ denotes the set of elementary roots that can be written as $w \cdot \alpha_r$ for some $w \in W$. Let $\mathcal{G}_r$ be the set of all these states, and let $\emptyset \in \mathfrak{P}(\mathcal{E})$ be the starting state. Note that the subset $\mathfrak{P}(\mathcal{E}) \cup \{F\}$ of $\mathcal{G}_r$ can be identified with the set of states of $W$. Next, let $\mu$
be the transition function, described above, for \( W \). The transition function 
\( \mu_r: S_r \times R \rightarrow S_r \) for the automaton \( W_r \) is defined by the rules listed below. 
Let \( X \in S_r \) and \((s, t) \in R\), and for brevity let \( Y = \mu_r(X, (s, t)) \).

**Case 1:** \( X \in \mathcal{P}(E) \).

(i) If \( s = t \in R \), then \( Y = \mu(X, s) \).

(ii) If either \( s \) or \( t \) is \$, then \( Y = \{ A \} \) if \( \mu(X, s) \neq \mathcal{F} \) and \( \alpha_s \in E_r \), 
and \( Y = \{ \mathcal{F} \} \) if \( \mu(X, s) = \mathcal{F} \) or \( \alpha_s \notin E_r \).

(iii) If \( s < t \in R \), then \( Y = \{ (\mu(X, s), \alpha_s, t) \} \) if \( \mu(X, s) \neq \mathcal{F} \) and \( \alpha_s \in E_r \), 
and \( Y = \{ \mathcal{F} \} \) if \( \mu(X, s) = \mathcal{F} \) or \( \alpha_s \notin E_r \).

(iv) If \( t < s \in R \), then \( Y = \{ (\mu(X, t), s, \alpha_t) \} \) if \( \mu(X, t) \neq \mathcal{F} \) and \( \alpha_t \in E_r \), 
and \( Y = \{ \mathcal{F} \} \) if \( \mu(X, t) = \mathcal{F} \) or \( \alpha_t \notin E_r \).

**Case 2:** \( X = (X, \beta, u) \in \mathcal{P}(E) \times E_r \times R \).

Let \( Y = \mu(X, s) \) and \( \gamma = s \cdot \beta \).

(i) If \( Y = \mathcal{F} \), then \( Y = \mathcal{F} \).

(ii) If \( s \neq u \), then \( Y = \mathcal{F} \).

(iii) If \( s = u \) and \( t \in R \) while \( Y \neq \mathcal{F} \), then \( Y = \{ (Y, \gamma, t) \} \) if \( \gamma \in E_r \), 
and \( Y = \{ \mathcal{F} \} \) if \( \gamma \notin E_r \).

(iv) If \( (s, t) = (u, \$) \) while \( Y \neq \mathcal{F} \), then \( Y = \{ A \} \) if \( \gamma = \alpha_r \), 
and \( Y = \{ \mathcal{F} \} \) if \( \gamma \neq \alpha_r \).

**Case 3:** \( X = (X, u, \beta) \in \mathcal{P}(E) \times R \times E_r \).

Let \( Y = \mu(X, t) \) and \( \gamma = t \cdot \beta \).

(i) If \( Y = \mathcal{F} \), then \( Y = \mathcal{F} \).

(ii) If \( t \neq u \), then \( Y = \mathcal{F} \).

(iii) If \( t = u \) and \( s \in R \) while \( Y \neq \mathcal{F} \), then \( Y = \{ (Y, s, \gamma) \} \) if \( \gamma \in E_r \), 
and \( Y = \{ \mathcal{F} \} \) if \( \gamma \notin E_r \).

(iv) If \( (s, t) = (\$, u) \) while \( Y \neq \mathcal{F} \), then \( Y = \{ A \} \) if \( \gamma = \alpha_r \), 
and \( Y = \{ \mathcal{F} \} \) if \( \gamma \neq \alpha_r \).

**Case 4:** \( X = A \) or \( \mathcal{F} \).

\( Y = \mathcal{F} \) in all cases.

It can be easily seen that \( W_r \) accepts \((x_1, x_2)\) if and only \( W_r \) accepts \((x_2, x_1)\).
Chapter 5 An automatic structure

(5.54) PROPOSITION The automaton $W_r$ defined above recognizes the language $L_r$.

Proof. Let $(s_i, t_i) \in \mathcal{R}$ and set $a = s_1 \ast s_2 \cdots \ast s_n$ and $b = t_1 \ast t_2 \cdots \ast t_n$. For $i \geq 0$, denote the state of $W_r$ after reading $(s_1 \cdots \ast s_i, t_1 \cdots \ast t_i)$ by $X_i$, and let $l \in \{1, \ldots, n + 1\}$ be maximal such that $X_{l-1} = X_{l-1} \in \mathcal{P}(\mathcal{E})$. It is clear that $s_i = t_i \in R$ for all $i \in \{1, \ldots, l - 1\}$.

We show that $X_n$ equals $A$ if and only if $(a, b) \in L_r$.

Suppose first that $(a, b) \in L_r$. Then an easy induction shows that $W$ is in state $X_i$ after reading $s_1 \ast \cdots \ast s_i = t_1 \ast \cdots \ast t_i$ for $i \in \{0, \ldots, l - 1\}$. If $l = n$, then $\{s_n, t_n\} = \{r, s\}$ by (5.53)(i); moreover, $W$ is in $\mu(X_{n-1}, r)$ after reading $s_1 \ast \cdots \ast s_{n-1} \ast r$, and since $s_1 \ast \cdots \ast s_{n-1} \ast r$ is in $L$ by (5.53)(i), we deduce that $\mu(X_{n-1}, r) \in \mathcal{P}(\mathcal{E})$. It follows by rule (ii) of Case 1 that $X_n = A$.

Next, suppose that $l < n$. Since $(a, b)$ is in $L_r$, it follows easily that $s_l, t_l \in R$, and by symmetry of both $W_r$ and $L_r$ we may assume without loss of generality that $s_l < t_l$. Then by (5.53)(ii), $s_i = t_{i+1} \in R$ for all $l < i \leq n - 1$ and $(s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r$. It follows by (3.29) that $(s_i \cdots s_{l+1}) \cdot \alpha_{s_l}$ is in $\mathcal{E}_r$ for all $i \in \{l, \ldots, n - 1\}$, and a straightforward induction yields that

$$X_i = (X_i, (s_l \cdots s_{l+1}) \cdot \alpha_{s_l}, s_{l+1})$$

for $i \in \{l, \ldots, n - 1\}$, where $X_i$ denotes the state of $W$ after reading $s_1 \ast \cdots \ast s_i$. In particular,

$$X_{n-1} = (X_{n-1}, (s_{n-1} \cdots s_{l+1}) \cdot \alpha_r, s_n).$$

Furthermore, $\mu(X_{n-1}, s_n) \in \mathcal{P}(\mathcal{E})$ since $s_1 \ast \cdots \ast s_n$ is an element of $L$; as $(s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r$ and $t_n = s$ by (5.53)(ii), rule (iv) of Case 2 yields that $X_n = A$.

It remains to show that $X_n = A$ implies $(a, b) \in L_r$. So let $X_n = A$; then in particular $X_i \neq F$ and $X_i \neq A$ for all $i < n$, since there are no transitions from $F$ or $A$ into states other than $F$. Further, $l - 1 < n$ since $A \notin \mathcal{P}(\mathcal{E})$.

If $l = n$, then $W$ is in $X_{n-1}$ after reading $s_1 \ast s_2 \cdots \ast s_{n-1}$, and since $X_n = A$, rule (ii) of Case 1 yields $\{s_n, t_n\} = \{r, s\}$ and $\mu(X_{n-1}, r) \in \mathcal{P}(\mathcal{E})$. So $W$ accepts $s_1 \ast \cdots \ast s_{n-1} \ast r$, and it follows by (5.53)(i) that $(a, b)$ is in $L_r$.

Suppose now that $l < n$. Then $X_i \neq A$ and $X_i \neq F$, and thus $s_l, t_l \in R$ by rule (ii) of Case 1; by symmetry of both $W_r$ and $L_r$ we may assume
without loss of generality that \( s_t < t_t \). We show now that \( r_i = s_{i-1} \) and

\[
X_i = (X_i, (s_i \cdots s_{l+1}) \cdot \alpha_{s_l}, t_i)
\]

for \( i \in \{l, \ldots, n - 1\} \), where \( X_i \in \mathcal{P}(\mathcal{E}) \) denotes the state of \( W \) after reading \( s_1 \ast s_2 \ast \cdots \ast s_i \). By rule (iii) of Case 1, \( X_i \) equals \( (X_i, \alpha_{s_l}, t_i) \). Suppose next that \( i \in \{l + 1, \ldots, n - 1\} \), and assume furthermore that

\[
X_{i-1} = (X_{i-1}, (s_{i-1} \cdots s_{l+1}) \cdot \alpha_{s_l}, t_{i-1})
\]

Since \( X_i \neq \mathcal{F} \) and \( X_i \neq A \), rule (iii) of Case 2 yields \( s_i = t_{i-1}, X_i \in \mathcal{P}(\mathcal{E}) \) and further \( (s_i \cdots s_{l+1}) \cdot \alpha_{s_l} \in \mathcal{E}_r \); hence \( X_i = (X_i, (s_i \cdots s_{l+1}) \cdot \alpha_{s_l}, t_i) \), and this finishes the induction. In particular,

\[
X_{n-1} = (X_{n-1}, (s_{n-1} \cdots s_{l+1}) \cdot \alpha_{s_l}, t_{n-1}),
\]

with \( X_{n-1} \in \mathcal{P}(\mathcal{E}) \) and \( s_i = t_{i-1} \) for all \( i \in \{l, \ldots, n - 1\} \). Since \( X_n = A \), rule (iv) of Case 2 implies further that

\[
s_n = t_{n-1}, t_n = $, (s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r \text{ and } \mu(X_{n-1}, s_n) \in \mathcal{P}(\mathcal{E}).
\]

Now \( a = s_1 \ast s_2 \ast \cdots \ast s_n \in L \), since \( \mu(X_{n-1}, s_n) \in \mathcal{P}(\mathcal{E}) \), while

\[
b = s_1 \ast s_2 \ast \cdots \ast s_{l-1} \ast s_{l+1} \ast \cdots \ast s_n \ast $
\]

with \( (s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r \); thus \( (a, b) \in L_r \) by (5.53)(ii), as required.

(5.55) **Theorem**  \( W \) is automatic, provided that \( R \) is finite.

Observe that the automaton \( W_r \) described above is by no means minimal. For example, the state \( (X, \beta, u) \in \mathcal{P}(\mathcal{E}) \times \mathcal{E}_r \times R \) is inaccessible if \( \beta \notin X \); that is, \( (X, \beta, u) \) cannot be reached from the starting state. Moreover, \( (X, \beta, u) \) is dead if \( u \cdot \beta \notin \mathcal{E} \); that is, the accept state cannot be reached from \( (X, \beta, u) \). Without changing the language recognized by \( W_r \) we may delete all inaccessible states and amalgamate all dead states with the failure state \( \mathcal{F} \), and obtain a normalized automaton with fewer states that also recognizes \( L_r \).

A group \( G \) is said to be bi-automatic, if there exists a set \( A \) of semigroup generators of \( G \), and a language \( L \) over \( A \) which yields an automatic structure for \( G \) such that, additionally,
(iii) \( L^a = \{(x_1, x_2) \in A^* \mid x_1, x_2 \in L \text{ and } \pi(x_1) = a\pi(x_2)\} \) is a regular language over \( A \) for all \( a \in A \).

We then say that \( L \) yields a biautomatic structure for \( W \).

The above constructed language for finitely generated Coxeter groups does not in general yield a biautomatic structure for \( W \). For example, suppose that \( W \) has the following Coxeter graph

\[ 
\begin{array}{ccc}
  e & \infty & a \\
  &    & b \\
  &    & \infty \\
  c & \infty & d \\
\end{array}
\]

and let \( R \) be ordered alphabetically. Assume for a contradiction that \( \mathcal{W}_e \) is a finite state automaton recognizing \( L^e \), and let \( n \) be the number of states of \( \mathcal{W}_e \). For \( i \in \{0, \ldots, n\} \) define

\[ w(i) = ((a \ast b)^i, (c \ast d)^i). \]

Since the number of states of \( \mathcal{W}_e \) is less than \( n+1 \), there exist \( i, j \in \{0, \ldots, n\} \) with \( i < j \) such that \( \mathcal{W}_e \) is in the same state after reading \( w(i) \) as after reading \( w(j) \). Now \( \mathcal{W}_e \) will certainly accept

\[ ((a \ast b)^n \ast (c \ast d)^n \ast \$, (c \ast d)^n \ast e \ast (a \ast b)^n) \]

\[ = ((a \ast b)^j, (c \ast d)^j) \ast ((a \ast b)^{n-j} \ast (c \ast d)^n \ast \$, (c \ast d)^{n-j} \ast e \ast (a \ast b)^n), \]

and thus \( \mathcal{W}_e \) is forced to accept

\[ ((a \ast b)^i, (c \ast d)^i) \ast ((a \ast b)^{n-j} \ast (c \ast d)^n \ast \$, (c \ast d)^{n-j} \ast e \ast (a \ast b)^n) \]

\[ = ((a \ast b)^{n-j+i} \ast (c \ast d)^n \ast \$, (c \ast d)^{n-j+i} \ast e \ast (a \ast b)^n). \]

But

\[ e\pi((a \ast b)^{n-j+i} \ast (c \ast d)^n) = e(ab)^{n-j+i}(cd)^n \]

and

\[ \pi((c \ast d)^{n-j} \ast e \ast (a \ast b)^n) = e(ab)^n(cd)^{n-j+i}, \]

and these are not equal. Hence \((a \ast b)^{n-j+i} \ast (c \ast d)^n, (c \ast d)^{n-j+i} \ast e \ast (a \ast b)^n)\) is not in \( L^e \), and \( \mathcal{W}_e \) does not recognize \( L^e \), contradicting our assumption.

**Chapter 6**
The Set of Elementary Roots

We have seen in Chapter 3 that the set of elementary roots is finite, provided that \( R \) is finite. Moreover, the proof of Theorem (3.44) yields that \( |E| \) is bounded by

\[
|E| \leq c |R|^{c |R| (|R|+1)+1} - 1),
\]

where \( c \) equals the cardinality of the set

\[
\{ \cos(n\pi/m_{rs}) \mid r, s \in R, m_{rs} < \infty \text{ and } n \in \{1, \ldots, m_{rs} - 1\} \}.
\]

This bound, however, is rather large. For example, if

\[
W = \langle r, s \mid r^2 = s^2 = (rs)^3 = 1 \rangle,
\]

then \( |E| = 3 \), but \( |R| = 2 \) and \( c = 3 \), and thus

\[
\frac{|R|}{|R|-1} (|R|^{c |R| (|R|+1)+1} - 1) = 2^{28} - 1.
\]

In this chapter we will explicitly determine the set of elementary roots and thus find \( |E| \) precisely.

For \( I \subseteq R \), let \( E_I \) denote the set of all elementary roots \( \alpha \) with \( I(\alpha) = I \); then \( E \) is the disjoint union of all \( E_I \) with \( \emptyset \neq I \subseteq R \) finite. Connectedness of the support of a root yields that \( E_I \) is empty if \( I \) is not connected; moreover, \( E_I \) is also empty by (3.39), (3.41) if \( I \) contains a circuit or an infinite bond.

For \( J \subseteq I \), define \( E_I^J \) to be the set of roots \( \alpha \) in \( E_I \) such that for \( r \in J \) the coefficient of \( \alpha_r \) in \( \alpha \) equals 1, and for \( s \in I \setminus J \) the coefficient of \( \alpha_s \) in \( \alpha \) is greater than 1; then \( E_I \) is the disjoint union of all \( E_I^J \) with \( J \subseteq I \), and thus

\[
E = \bigcup_{I \in \mathcal{I}(R)} \bigcup_{J \subseteq I} E_I^J = \bigcup_{I \in \mathcal{I}(R)} \bigcup_{J \subset I} E_I^J \cup \bigcup_{I \in \mathcal{I}(R)} E_I.
\]
Chapter 6  The set of elementary roots

where $\mathcal{I}(R)$ consists of the finite non-empty connected subsets of $R$ that do not contain any circuits or infinite bonds.

Note that $\sum_{r \in I} \alpha_r$ is an elementary root if $I \in \mathcal{I}(R)$ contains only simple bonds. For if $|I| = 1$, this is trivially true, and we proceed by induction. Suppose that $|I| > 1$. Since $I$ does not contain any circuits, we can choose an $s \in I$ such that $s$ is adjacent to exactly one element of $I \setminus \{s\}$, say $t$. Then $\sum_{r \in I \setminus \{s\}} \alpha_r$ is an elementary root by induction. Since $s$ is only adjoined to $t$ in $I \setminus \{s\}$, and $s$ and $t$ are adjoined by a simple bond, we find that

$$\langle \sum_{r \in I \setminus \{s\}} \alpha_r, \alpha_s \rangle = \langle \alpha_s, \alpha_t \rangle = -\frac{1}{2};$$

hence $\sum_{r \in I} \alpha_s = s \cdot \sum_{r \in I \setminus \{s\}} \alpha_r$ is an elementary root by (3.37).

On the other hand, the next lemma together with (2.26) yield that if $r$ and $s$ are adjoined by a non-simple bond, then no root can have coefficient 1 for both $\alpha_r$ and $\alpha_s$. So if $I$ does contain non-simple bonds, $\sum_{r \in I} \alpha_r$ cannot be a root, and thus

$$\mathcal{E}_I = \begin{cases} \{ \sum_{r \in I} \alpha_r \} & \text{if } I \text{ contains only simple bonds,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore

$$|\mathcal{E}| = \sum_{I \in \mathcal{I}(R)} \sum_{J \subset I} |\mathcal{E}_I| + n(R),$$

if $R$ is finite, where $n(R)$ denotes the the number of non-empty connected subsets of $R$ that contain only simple bonds and no circuits.

(6.56) Lemma  Let $x_1, x_2, y \in \Pi$ with $x_1 \neq x_2$ such that $\langle x_i, y \rangle$ equals $-\cos(\pi/m_i)$ for $i = 1, 2$. Furthermore, let $\alpha$ and $\beta$ be positive roots such that $\beta$ precedes $\alpha$, and $y$ is not in the support of $\beta$. Denote the coefficient of $x_i$ in $\beta$ by $\lambda_i$ for $i = 1, 2$. Then the coefficient of $y$ in $\alpha$ equals 0, or is greater than or equal to $2 \cos(\pi/m_1)\lambda_1 + 2 \cos(\pi/m_2)\lambda_2$.

Proof. Let $\gamma$ be of maximal depth with $\beta \preceq \gamma \preceq \alpha$ such that $y \notin \text{supp}(\gamma)$. If $\gamma = \alpha$, then $y \notin \text{supp}(\alpha)$, and the assertion is true. So suppose that $\gamma < \alpha$, and denote the coefficient of $x_i$ in $\gamma$ by $\mu_i$; then $\lambda_i \leq \mu_i$. Maximality of $\gamma$ now
yields that $\gamma \prec r_y \cdot \gamma \preceq \alpha$, and the coefficient of $y$ in $r_y \cdot \gamma$ equals $0 - 2\langle \gamma, y \rangle$. This is greater than or equal to

$$-2(\langle x_1, y \rangle \mu_1 + \langle x_2, y \rangle \mu_2) = 2 \cos(\pi/m_1)\mu_1 + 2 \cos(\pi/m_2)\mu_2,$$

which in turn is greater than or equal to $2 \cos(\pi/m_1)\lambda_1 + 2 \cos(\pi/m_2)\lambda_2$. Since $r_y \cdot \gamma \preceq \alpha$, the coefficient of $y$ in $\alpha$ is greater than or equal to the coefficient of $y$ in $r_y \cdot \gamma$ by (3.35), and this finishes the proof.

By the above we only need to determine $E_J^I$ for $I \in \mathcal{I}(R)$ and $J \subset I$. We will now further reduce the number of subsets $J$ of $I$ for which we need to calculate $E_J^I$ (see Theorem (6.5)). Once this is done, we show that we only need to consider $E_J^I$ for $I \in \mathcal{I}(R)$ containing at most one non-simple bond (see Lemma (6.7)). We then continue by determining $E_J^I$ in case $I$ contains only simple bonds, and finish this chapter by dealing with the case that $I$ contains exactly one non-simple bond of finite weight.

(6.57) **Proposition** Let $I \subseteq R$, $r \in I$ and $K_1, \ldots, K_n \subseteq I \setminus \{r\}$ such that $I \setminus \{r\}$ is the disjoint union of $K_1, \ldots, K_n$. Suppose further that no element of $K_i$ is adjoined to any element of $K_j$ if $i \neq j$, and set $I_i = K_i \cup \{r\}$ for all $i \in \{1, \ldots, n\}$. Then

$$\phi: (\beta_1, \ldots, \beta_n) \mapsto \beta_1 + \cdots + \beta_n - (n-1)\alpha_r$$

defines a one-one correspondence between the set of $n$-tuples in $\Phi_{I_1}^+ \times \cdots \times \Phi_{I_n}^+$ such that the coefficient of $\alpha_r$ in each component equals 1, and the set of roots in $\Phi_I^+$ with coefficient 1 for $\alpha_r$. Moreover, this map restricts to a one-one correspondence between the set of $n$-tuples in $E_{I_1} \times \cdots \times E_{I_n}$ such that $\alpha_r$ has coefficient 1 in each component, and the set of roots in $E_I$ with coefficient 1 for $\alpha_r$.

Before we can show (6.57) we need to prove the next two technical results.

(6.58) **Lemma** Let $\alpha$ and $\beta$ be positive roots with $\alpha \succeq \beta$. Further let $r \in R$, and define $I$ to be the set of simple reflections $s \in R$ which are adjoined to $r$. Suppose that the coefficient of $\alpha_r$ in $\alpha$ is strictly greater than the coefficient of $\alpha_r$ in $\beta$, while for $s \in I$ the coefficients of $\alpha_s$ in $\alpha$ and $\beta$ coincide. Then $\alpha \succeq r \cdot \beta$.

**Proof.** The assertion is trivially true if $r \cdot \beta \preceq \beta$, so suppose that $r \cdot \beta \succeq \beta$; that is, $dp(r \cdot \beta) = dp(\beta) + 1$. Let $\gamma$ be of maximal depth with $\alpha \succeq \gamma \succeq \beta$
such that the coefficients of $\alpha_r$ in $\gamma$ and $\beta$ coincide, and let $w \in W$ such that $\gamma = w \cdot \beta$ and $l(w) = \text{dp}(\gamma) - \text{dp}(\beta)$. Since the coefficients of $\alpha_s$ in $\gamma$ and $\beta$ coincide for all $s \in I \cup \{r\}$, it is clear that $w \in W_{R \setminus (I \cup \{r\})}$. So $rw = wr$ and

$$r \cdot \gamma = rw \cdot \beta = w \cdot (r \cdot \beta).$$

Now $\alpha \preceq r \cdot \gamma \succ \gamma$ by maximality of $\gamma$, and thus $\text{dp}(r \cdot \gamma) = \text{dp}(\gamma) + 1$. Hence $l(w) = (r \cdot \gamma) - \text{dp}(r \cdot \beta)$ and $r \cdot \gamma \succeq r \cdot \beta$; transitivity of $\succeq$ yields that $\alpha$ is preceded by $r \cdot \beta$, as required.

(6.59) **Lemma** Let $\alpha \in \Phi^+$ and $r \in R$ such that the coefficient of $\alpha_r$ in $\alpha$ equals 1. Then $\alpha \succeq \alpha_r$, and thus there exists a $w \in W_{I(\alpha) \setminus \{r\}}$ such that $\alpha = w \cdot \alpha_r$ and $l(w) = \text{dp}(\alpha) - 1$.

**Proof.** If $\alpha$ is simple, the assertion is trivially true. Suppose now that $\alpha$ is of depth greater than 1, and assume that all positive roots $\beta < \alpha$ are preceded by $\alpha_s$, whenever $\alpha_s \in \Pi$ has coefficient 1 in $\beta$. Let $t \in R$ such that $t \cdot \alpha < \alpha$. If $t \neq r$, then $t \cdot \alpha \geq \alpha_r$ by induction, and thus $\alpha \succ t \cdot \alpha \geq \alpha_r$, as required. Suppose next that $t = r$. The coefficient of $\alpha_r$ in $r \cdot \alpha$ is less than 1, and thus must be equal to 0 by (2.26). That is, $\alpha_r \notin \text{supp}(r \cdot \alpha)$, and by the connectedness of the support of $\alpha$ there exists an $x \in \text{supp}(r \cdot \alpha)$ such that $\alpha_r$ and $x$ are adjoined by a bond of weight $m \geq 3$. Since $\alpha \succeq r \cdot \alpha$ and the coefficient of $\alpha_r$ in $\alpha$ equals 1, Lemma (6.56) yields that $m = 3$ and that the coefficient of $x$ in $r \cdot \alpha$ equals 1; moreover, (6.56) also yields that $\alpha_r$ cannot be adjacent to any element of $\text{supp}(\alpha) \setminus \{x\}$. Now $r \cdot \alpha \succeq x$ by inductive hypothesis, and since the coefficients of $x$ in $\alpha$ and $x$ coincide, while the coefficient of $\alpha_r$ in $\alpha$ is greater than the coefficient of $\alpha_r$ in $x$, the previous lemma yields that $\alpha$ is preceded by $r_1 \cdot x_1 = x_1 + \alpha_r$. It is clear that $x_1 + \alpha_r$ is a successor of $\alpha_r$, hence $\alpha \succeq \alpha_r$ by transitivity of $\succeq$, as required.

**Proof of (6.2).** We show first that $\phi$ is well defined. So let $(\beta_1, \ldots, \beta_n)$ be an $n$-tuple in $\Phi^+_I \times \cdots \times \Phi^+_I$ such that the coefficient of $\alpha_r$ in $\beta_i$ equals 1 for all $i$. By the previous lemma there exist $w_i \in W_{K_i}$ with $l(w_i) = \text{dp}(\beta_i) - 1$ such that $\beta_i$ equals $w_i \cdot \alpha_r$. Observe that the groups $W_{K_i}$ centralize each other by construction. Define $w = w_1 \cdots w_n$ and $\alpha = w \cdot \alpha_r$; then the coefficient of $\alpha_r$ in $\alpha$ equals 1, and a straightforward calculation yields that

$$\alpha = \beta_1 + \cdots + \beta_n - (n - 1)\alpha_r.$$
Thus $\phi(\beta_1, \ldots, \beta_n)$ is in $\Phi^+_I$, and since the coefficient of $\alpha_r$ in $\alpha$ clearly equals 1, we know that $\phi$ is well defined.

On the other hand, if $\alpha \in \Phi^+_I$ has coefficient 1 for $\alpha_r$, Lemma (6.59) yields that there exists a $w \in W_{I \setminus \{r\}}$ with $l(w) = dp(\alpha) - 1$ such that $\alpha = w \cdot \alpha_r$. Since $W_{I \setminus \{r\}}$ is the direct product of $W_{K_i}, \ldots, W_{K_n}$, there exist $w_i \in W_{K_i}$ for all $i \in \{1, \ldots, n\}$ such that $w = w_1 \cdots w_n$ with length adding. So if we define $\beta_i = w_i \cdot \alpha_r$, then $I(\beta_i) \subseteq I_i$ and $\alpha_r$ has coefficient 1 in $\beta_i$. By the above, $\alpha = \beta_1 + \cdots + \beta_n - (n - 1)\alpha_r = \phi(\beta_1, \ldots, \beta_n)$; whence $\phi$ is onto. Since $I(\beta_i) \cap I(\beta_j) = \{r\}$ for $i \neq j$, and the coefficient of $\alpha_r$ in all $\beta_i$ is 1, it is clear that $\phi$ is one-one, and thus $\phi$ is a one-one correspondence.

In order to show that $\phi$ induces a one-one correspondence between the set of $n$-tuples in $E_{I_1} \times \cdots \times E_{I_n}$ such that $\alpha_r$ has coefficient 1 in each component, and the set of roots in $E_I$ with coefficient 1 for $\alpha_r$, it suffices to show for $w_1 \in W_{K_1}, \ldots, w_n \in W_{K_n}$ and $w = w_0 \cdots w_n$ that $w \cdot \alpha_r \in E$ if and only if $w_i \cdot \alpha_r \in E$ for all $i \in \{1, \ldots, n\}$.

Suppose first that $w_i \cdot \alpha_r \in \Delta$ for some $i$; by symmetry of the $I_i$ we may assume without loss of generality that $i = 1$. Then $N_+(w_2 \cdots w_n, w_1 \cdot \alpha_r) = 0$; for $N(w_2 \cdots w_n) \subseteq \Phi^+_{I \setminus I_1}$ and $w_1 \cdot \alpha_r \in \Phi^+_{I_1}$, and clearly $\langle \gamma, w_1 \cdot \alpha_r \rangle \leq 0$ for all $\gamma \in \Phi^+_{I \setminus I_1}$. Thus $w \cdot \alpha_r \geq w_1 \cdot \alpha_r$ by (3.34), and (3.36) implies that $w \cdot \alpha_r \in \Delta$, as required.

For the converse, suppose that $w \cdot \alpha_r$ dominates some $\gamma \in \Phi^+ \setminus \{w \cdot \alpha_r\}$; then $w^{-1} \cdot \gamma \in \Phi^-$ since $\alpha_r \not\in \Delta$, and thus $\gamma \in N(w^{-1})$. Now

$$N((w_1 \cdots w_n)^{-1}) = N(w_1^{-1}) \cup \ldots \cup N(w_n^{-1}),$$

since the $W_{K_i}$ centralize each other, and this union is disjoint; by symmetry we may assume without loss of generality that $\gamma$ is in $N(w_1^{-1})$ and not in

$$N(w_2^{-1}) \cup \ldots \cup N(w_n^{-1}) = N((w_2 \cdots w_n)^{-1}).$$

So $(w_2 \cdots w_n)^{-1} \cdot \gamma$ is positive, and hence $(w_1 \cdot \alpha_r) \text{dom} (w_2 \cdots w_n)^{-1} \cdot \gamma$; since $w \cdot \alpha_r \neq \gamma$ it follows that $w_1 \cdot \alpha_r \neq (w_2 \cdots w_n)^{-1} \cdot \gamma$, and thus $w_1 \cdot \alpha_r \in \Delta$, as required. $\square$
Chapter 6  \hspace{1cm} \textit{The set of elementary roots}  \hspace{1cm} 64

Now let $I \in \mathcal{I}(R)$ and $J \subset I$. If $r \in J$, denote the connected components of $I \setminus \{r\}$ by $K_1, \ldots, K_n$. By (6.57) each element of $\mathcal{E}^J_I$ can be written as

$$\alpha_1 + \cdots + \alpha_n - (n-1)\alpha_r,$$

with $\alpha_i \in \mathcal{E}_I^{\{r\} \cup K_i}$, with coefficient 1 for $\alpha_r$ for all $i$. Furthermore, if $s \in K_i$, the coefficient of $\alpha_s$ in $\alpha_i$ equals 1 if $s \in J$, and is greater than 1 if $s \notin J$; that is, $\alpha_i \in \mathcal{E}_I^{\{r\} \cup (J \cap K_i)}$. It is clear, that

$$\beta_1 + \cdots + \beta_n - (n-1)\alpha_r$$

is in $\mathcal{E}^J_I$ if $\beta_i \in \mathcal{E}_I^{\{r\} \cup (K_i \setminus J)}$ for all $i$, and an iteration of this procedure yields the following theorem.

(6.60) \textbf{Theorem}  \hspace{1cm} Let $I \in \mathcal{I}$ and $J \subset I$, and denote the set of connected components of $I \setminus J$ by $\mathcal{K}(I \setminus J)$. Furthermore, for each connected component $K$ of $I \setminus J$ let $X(K, J)$ denote the set of $r \in J$ that are adjoined to some element of $K$, and define $Y(K, J) = K \cup X(K, J)$. Finally, for $r \in J$ let $n_r(I, J)$ denote the number of $K \in \mathcal{K}(I \setminus J)$ with $r \in X(K, J)$. Then $\mathcal{E}^J_I$ is the set of

$$\sum_{K \in \mathcal{K}(I \setminus J)} \alpha_K - \sum_{r \in J} (n_r(I, J) - 1)\alpha_r,$$

where $\alpha_K \in \mathcal{E}_Y^X(K, J)$ for all $K \in \mathcal{K}(I \setminus J)$. Hence

$$|\mathcal{E}^J_I| = \prod_{K \in \mathcal{K}(I \setminus J)} |\mathcal{E}_Y^X(K, J)|.$$

(6.61)  \hspace{1cm} From now on we only need to determine $\mathcal{E}_Y^X$ for $X, Y \subseteq R$ such that

(i) $Y$ does not contain any circuits or infinite bonds,

(ii) $X \subseteq Y$ and $X \neq Y$,

(iii) $Y \setminus X$ is connected,

(iv) every element of $X$ is adjacent to some element of $Y \setminus X$.

Note that since $Y$ does not contain any circuits, and $Y \setminus X$ is connected, every element of $X$ is adjoined to exactly one element of $Y \setminus X$, and no two elements of $X$ are adjoined.
The following lemma implies that $\mathcal{E}_Y^X$ is empty if $Y$ contains more than one non-simple bond, while $X, Y$ satisfy (6.61); therefore we only need to determine $\mathcal{E}_Y^X$ for $X, Y$ satisfying (6.61) such that $Y$ contains at most one non-simple bond.

(6.62) Lemma Let $r_0, \ldots, r_{l+1} \in R$ such that the subgraph of the Coxeter graph corresponding to $\{r_0, \ldots, r_{l+1}\}$ is of the following shape

```
\begin{center}
\begin{tikzpicture}
  \node (r0) at (0,0) {$r_0$};
  \node (r1) at (1,0) {$r_1$};
  \node (r2) at (2,0) {$r_2$};
  \node (rl-1) at (3,0) {$r_{l-1}$};
  \node (rl) at (4,0) {$r_l$};
  \node (rl+1) at (5,0) {$r_{l+1}$};
  \draw (r0) -- (r1);
  \draw (r1) -- (r2);
  \draw (rl-1) -- (rl);
  \draw (rl) -- (rl+1);
\end{tikzpicture}
\end{center}
```

with $m_1, m_2 \geq 4$. Denote the simple root corresponding to $r_i$ by $x_i$, and let $\alpha$ be a root with $x_0, x_{l+1} \in \text{supp}(\alpha)$ such that the coefficients of $x_1, \ldots, x_l$ in $\alpha$ are greater than 1. Then $\alpha \in \Delta$.

Proof. Let $\beta \preceq \alpha$ be a positive root of minimal depth such that $x_0$ and $x_{l+1}$ are in the support of $\beta$, and the coefficients of $x_1, \ldots, x_l$ in $\beta$ are greater than 1. By (3.36) it suffices to show that $\beta$ is in $\Delta$, and since this is certainly the case if the support of $\beta$ contains a circuit (by (3.39)), we can assume without loss of generality that $\text{supp}(\beta)$ does not contain any circuits.

For $i \in \{0, \ldots, l+1\}$, let $\lambda_i$ denote the coefficient of $x_i$ in $\beta$; then $\lambda_1, \ldots, \lambda_l \geq \sqrt{2}$ by (2.26). Next, let $r \in R$ such that $r \cdot \beta < \beta$. We show that $\langle r \cdot \beta, \alpha_r \rangle \leq -1$, which then implies $\langle \beta, \alpha_r \rangle \geq 1$; hence $\beta \in \Delta$ by (3.32), since $\beta$ is clearly of depth greater than $\text{dp}(\alpha_r) = 1$.

Minimality of $\beta$ forces $r = r_i$ for some $i \in \{0, \ldots, l+1\}$. If $i = 0$ or $l+1$, we may assume without loss of generality that $i = 0$. Then minimality of $\beta$ yields further that $x_0 \notin \text{supp}(r_0 \cdot \beta)$, and thus

$$\langle r_0 \cdot \beta, x_0 \rangle \leq 0 + \lambda_1 \langle x_1, x_0 \rangle;$$

since $\lambda_1 \geq \sqrt{2}$ and $\langle x_1, x_0 \rangle = \cos(\pi/m_1) \leq -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$, this is less than or equal to $-1$, as required.

Suppose next that $i \in \{1, \ldots, l\}$. Connectedness of the support of $r_i \cdot \beta$ together with the assumption that $\text{supp}(\beta)$ does not contain any circuits force $x_i \in \text{supp}(r_i \cdot \beta)$, and minimality of $\beta$ yields that the coefficient of $x_i$ in $r_i \cdot \beta$ is less than or equal to 1; therefore the coefficient of $x_i$ in $r_i \cdot \beta$ equals 1 by
Let $K_1, \ldots, K_n$ denote the connected components of $I(\beta) \setminus \{r_i\}$, and for $j \in \{0, \ldots, n\}$ let $\beta_j \in \Phi_{K_j \cup \{r_i\}}^+$ with coefficient 1 for $x_i$ such that
\[ r_i \cdot \beta = \beta_1 + \cdots + \beta_n - (n-1)x_i \]
according to (6.57). We may assume without loss of generality that $r_{i-1} \in K_1$.

If $i = 1$, then $\beta_1 \geq x_1$ by (6.59), and thus $\lambda_0 \geq 2 \cos(\pi/m)$ by (6.56); hence
\[ \lambda_{i-1} \langle x_{i-1}, x_i \rangle = \lambda_0 \langle x_0, x_1 \rangle \leq 2 \cos(\pi/m_1)(-\cos(\pi/m_1)) \leq -1, \]
as $\cos(\pi/m_1) \geq \frac{1}{\sqrt{2}}$. If $i > 1$, then $r_i$ and $r_{i-1}$ are adjoined by a simple bond, and since $I(\beta)$ does not contain any circuits, $r_i$ is adjoined only to $r_{i-1}$ in $I(\beta_1)$. The coefficient of $x_i$ in $r_i \cdot \beta_0$ equals $-1 + \lambda_{i-1}$, which is greater than 0 since $\lambda_{i-1} > 1$, and thus must be greater than or equal to 1 by (2.26). Therefore $\lambda_{i-1} \geq 2$, and thus again $\lambda_{i-1} \langle x_{i-1}, x_i \rangle \leq -1$. Symmetrical arguments also yield $\lambda_{i+1} \langle x_{i+1}, x_i \rangle \leq -1$ and thus
\[ \langle r_i \cdot \beta, x_i \rangle \leq 1 + \lambda_{i-1} \langle x_{i-1}, x_i \rangle + \lambda_{i+1} \langle x_{i+1}, x_i \rangle \leq -1, \]
as required.

(6.63) **Proposition** Suppose $X, Y$ satisfy (6.61), and assume furthermore that $Y$ contains two or more non-simple bonds. Then $\mathcal{E}_X^Y = \emptyset$.

§6a Simple bonds only*

For the duration of this section we assume that $X, Y \subseteq R$ satisfy (6.61) and, furthermore, that $Y$ contains only simple bonds.

It is clear that all coefficients of roots in $\Phi_Y$ are integers, and thus $\langle \alpha, \beta \rangle$ is an integer multiple of $\frac{1}{2}$ for $\alpha, \beta \in \Phi_Y$. Moreover, $\mathcal{E}_X^Y$ consists of the roots in $\mathcal{E}_Y$ with coefficient 1 for $\alpha_r$ with $r \in X$, and coefficient greater than or equal to 2 for $\alpha_s$ with $s \in Y \setminus X$.

The following is immediate.

*I learned recently that Professor J.-Y. Hée also has a description of the set of elementary roots for the case that the Coxeter graph contains only simple bonds.*
(6.64) Lemma  Let $\alpha \in \mathcal{E}$ be of depth greater than 1 with $I(\alpha) \subseteq Y$ and $r \in R$ such that $\alpha > r \cdot \alpha$. Then $\langle \alpha, \alpha_r \rangle = \frac{1}{2}$, and the coefficient of $\alpha_r$ in $r \cdot \alpha$ equals the coefficient of $\alpha_r$ in $\alpha$ minus 1.

The next lemma yields that $\mathcal{E}_Y^X$ is empty if $Y$ contains more than one vertex of valency greater than or equal to 3 (if $l > 1$, namely $r_1$ and $r_l$), or one or more vertices of valency greater than or equal to 4 (if $l = 1$, namely $r_1$).

(6.65) Lemma  Let $r_1, \ldots, r_l$ and $s_1, s_2, s_3, s_4$ be in $Y$ such that the subgraph of the Coxeter graph corresponding to $\{r_1, \ldots, r_l\} \cup \{s_1, s_2, s_3, s_4\}$ is of the following shape:

Denote the simple roots corresponding to $r_i$ and $s_j$ by $x_i$ and $y_j$ respectively, and let $\alpha \in \Phi_+^Y$ such that $y_1, y_2, y_3, y_4 \in \text{supp}(\alpha)$, and the coefficients of $x_1, \ldots, x_l$ in $\alpha$ are greater than or equal to 2. Then $\alpha \in \Delta$.

Proof. Let $\beta$ be a positive root of minimal depth preceding $\alpha$ such that $y_1, y_2, y_3$ and $y_4$ are in the support of $\beta$, and the coefficients of $x_i$ in $\beta$ are greater than or equal to 2 for all $i \in \{1, \ldots, l\}$. By (3.36) it suffices to show that $\beta$ is in $\Delta$. Denote the coefficients of $x_i$ and $y_j$ in $\beta$ by $\lambda_i$ and $\mu_j$ respectively, and let $r \in Y$ such that $r \cdot \beta < \beta$. We show that $\langle r \cdot \beta, \alpha_r \rangle \leq -1$, which then implies $\langle \beta, \alpha_r \rangle \geq 1$, and hence $\beta \in \Delta$ by (3.32); (since $\beta$ is clearly of depth greater than $\text{dp}(\alpha_r) = 1$).

By minimality of $\beta$ we know that $r = r_i$ or $r = s_j$ for some $i, j$. If $r = s_j$, we may assume without loss of generality that $r = s_1$; then $y_1$ cannot be in the support of $s_1 \cdot \beta$ by minimality of $\beta$, and since $\lambda_1 \geq 2$, we find that $\langle s_1 \cdot \beta, y_1 \rangle \leq 0 + (-\frac{1}{2})\lambda_1 \leq -1$, as required.

Suppose now that $r = r_i$ for some $i \in \{1, \ldots, l\}$; then the coefficient of $x_i$ in $r_i \cdot \alpha$ has to be less than or equal to 1 by minimality of $\beta$. Since $\lambda_{i-1} \geq 2$ if $i \geq 2$, and $\mu_1, \mu_2 \geq 1$ if $i = 1$, while $\lambda_{i+1} \geq 2$ if $i \leq l - 1$, and
\( \mu_3, \mu_4 \geq 1 \) if \( i = l \), this yields that
\[
\langle r_i \cdot \beta, x_i \rangle \leq \begin{cases} 
1 + (-\frac{1}{2})1 + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } i = 1 = l, \\
1 + (-\frac{1}{2})1 + (-\frac{1}{2})1 + (-\frac{1}{2})2 & \text{if } i = 1 < l, \\
1 + (-\frac{1}{2})2 + (-\frac{1}{2})2 & \text{if } 1 < i < l, \\
1 + (-\frac{1}{2})2 + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } 1 < i = l;
\end{cases}
\]
so \( \langle r_i \cdot \beta, x_i \rangle \leq -1 \) in all cases, as required. \( \square \)

An easy calculation yields the following result.

(6.66) **Lemma** Suppose that \( Y \) equals

\[
\begin{array}{ccc}
\bullet & \circ & \circ & \cdots & \circ & \bullet
\end{array}
\]

and denote the simple root corresponding to \( u_i \) by \( \alpha_i \). Then
\[
\Phi^+ = \{ \alpha_i + \cdots + \alpha_j \mid 1 \leq i \leq j \leq n \},
\]
and thus \( \mathcal{E}_Y^X = \emptyset \) if \( X \neq Y \).

(6.67) **Proposition** Suppose that \( Y \subseteq R \) contains only simple bonds and \( |Y| > 1 \). Further, let \( X \subseteq Y \) such that \( X, Y \) satisfy (6.61). Then \( \mathcal{E}_Y^X \) is empty unless \( Y \) equals

\[
\begin{array}{cccccccc}
\bullet & \circ & \cdots & \circ & \bullet & \\
& \circ & \circ & \cdots & \circ & \\
& & \cdots & \\
& & \cdots & \\
\bullet & \circ & \cdots & \circ & \bullet
\end{array}
\]

with \( l, m, n \geq 1 \).
If $Y$ is of the shape described above, and $X, Y$ satisfy (6.61), then $X$ must be contained in $\{r_l, s_m, t_n\}$, since each element of $X$ is adjoined to exactly one element of $Y \setminus X$, and $Y \setminus X$ is connected. It is convenient for us to define $s_0$ and $t_0$ to be equal to $r_0$. In order to avoid double indices, we denote the simple reflections corresponding to $r_i, s_j$ and $t_k$ respectively for $i = 0, \ldots, l$, $j = 0, \ldots, m$ and $k = 0, \ldots, n$.

Define $\rho_{l,m,n}$ to be the root

$$(t_{n-1} \cdots t_1)(s_{m-1} \cdots s_1)(r_{l-1} \cdots r_1)r_0(t_n \cdots t_1)(s_m \cdots s_1)(r_l \cdots r_1) \cdot x_0$$

$$= x_l + y_m + z_n$$

$$+ 2(x_{l-1} + \cdots + x_1 + y_{m-1} + \cdots + y_1 + x_0 + z_1 + \cdots + z_{n-1}) .$$

A straightforward calculation yields that

$$(r_i \cdots r_1) \cdot x_0 = x_i + \cdots + x_1 + x_0$$

and $\langle (r_i \cdots r_1) \cdot x_0, x_{i+1} \rangle = -\frac{1}{2}$ for all $i \in \{1, \ldots, l\}$. So by (3.37) we can deduce that $(r_{i+1} \cdots r_1) \cdot x_0$ is elementary if $(r_i \cdots r_1) \cdot x_0$ is elementary. Since $x_0 \in \mathcal{E}$, induction yields that $(r_1 \cdots r_1) \cdot x_0$ is elementary. A string of similar arguments yields that $\rho_{l,m,n}$ is elementary, and it follows that $\rho_{l,m,n}$ is an element of $\mathcal{E}_{Y}^{Y}$. 

The next two assertions will enable us to show that each root in $\mathcal{E}_{Y}^{X}$ is preceded by $\rho_{l,m,n}$ if $X, Y$ are of the shape described in (6.67). Note that the next lemma also yields that the depth function coincides with the height function defined by $\sum_{r \in Y} \lambda_r \alpha_r \mapsto \sum_{r \in Y} \lambda_r$ on the set of elementary roots with only simple bonds in their support.

(6.68) Lemma Suppose $Y$ contains only simple bonds, and let $\alpha \in \Phi^+$ such that $\alpha = \sum_{r \in Y} \lambda_r \alpha_r$ for some $(\lambda_r)_{r \in Y}$. Then $\text{dp}(\alpha) \leq \sum_{r \in Y} \lambda_r$, with equality if and only if $\alpha \in \mathcal{E}$.

Proof. If $\alpha$ is of depth 1, then $\alpha = \alpha_r$ for some $r \in Y$, and the assertion is immediate. So suppose now that $\text{dp}(\alpha) > 1$, and assume that for each positive root $\beta \prec \alpha$, the depth of $\beta$ is less than or equal to the sum of the coefficients of the simple roots in $\beta$, with equality if and only if $\beta \in \mathcal{E}$. Further, let $s \in R$ such that $s \cdot \alpha \prec \alpha$. 

If \( \alpha \) is elementary, (6.64) implies that the coefficient of \( \alpha_s \) in \( s \cdot \alpha \) equals \( \lambda_s - 1 \), and since \( r \cdot \alpha \in \mathcal{E} \) by (3.36), induction yields that

\[
\text{dp}(s \cdot \alpha) = \sum_{r \neq s} \lambda_r + (\lambda_s - 1) = \sum_{r \in Y} \lambda_r - 1;
\]

hence \( \text{dp}(\alpha) = \text{dp}(s \cdot \alpha) + 1 = \sum_{r \in Y} \lambda_r \).

Suppose next that \( \alpha \in \Delta \) and \( s \cdot \alpha \in \Delta \). Then the coefficient of \( \alpha_s \) in \( s \cdot \alpha \) is less than or equal to \( \lambda_s - 1 \); hence

\[
\text{dp}(s \cdot \alpha) < \sum_{r \neq s} \lambda_r + (\lambda_s - 1) = \sum_{r \in Y} \lambda_r - 1
\]

by induction, and thus \( \text{dp}(\alpha) = \text{dp}(s \cdot \alpha) + 1 < \sum_{r \in Y} \lambda_r \).

Finally, suppose that \( \alpha \in \Delta \) while \( s \cdot \alpha \) is elementary. This is possible only if \( \alpha \) dom \( \alpha_s \), and thus \( \langle \alpha, \alpha_s \rangle \geq 1 \) by (3.32). The coefficient of \( \alpha_s \) in \( s \cdot \alpha \) equals \( \lambda_s - 2\langle \alpha, \alpha_s \rangle \), and this is less than or equal to \( \lambda_s - 2 \). Thus

\[
\text{dp}(s \cdot \alpha) \leq \sum_{r \neq s} \lambda_r + (\lambda_s - 2) = \sum_{r \in Y} \lambda_r - 2
\]

by inductive hypothesis, and therefore

\[
\text{dp}(\alpha) = \text{dp}(s \cdot \alpha) + 1 \leq \sum_{r \in Y} \lambda_r - 1 < \sum_{r \in Y} \lambda_r,
\]

as required.

(6.69) Proposition Suppose \( Y \) contains only simple bonds. Let \( \beta \in \Phi^+ \) and \( \alpha \in \mathcal{E} \) with \( I(\alpha) \subseteq Y \). Then \( \alpha \geq \beta \) if and only if the coefficients of simple roots in \( \beta \) are less than or equal to the corresponding coefficients in \( \alpha \). (Note that if this is the case, (3.36) yields that \( \beta \in \mathcal{E} \).)

Proof. If \( \alpha \geq \beta \), Lemma (3.35) yields that the coefficients of simple roots in \( \beta \) are less than or equal to the corresponding coefficients in \( \alpha \), and it suffices to show the converse.

Suppose first that \( \text{dp}(\beta) = 1 \), and let \( \gamma \leq \alpha \) be a positive root of minimal depth such that \( \beta \in \text{supp}(\gamma) \). Assume for a contradiction that \( \gamma \neq \beta \). Then
dp(γ) > 1, since β is in the support of γ, but γ does not equal β. Minimality of γ yields that \( r_β \cdot γ < γ \) and \( β \notin \text{supp}(r_β \cdot γ) \). Since \( α \) is elementary, it follows that γ is elementary, and by (6.64), the coefficient of β in γ equals the coefficient of β in \( r_β \cdot γ \) plus 1, that is 1. But then \( γ \succeq β \) by (6.59), and since \( γ \neq β \) we find that \( α \succeq γ \succ β \), contradicting the minimality of γ. So \( γ = β \), as required.

Suppose next that dp(β) > 1, and let \( (μ_r)_{r \in Y} \) with \( β = \sum_{r \in Y} μ_r α_r \). Furthermore, assume that the assertion is true for all positive roots of depth less than the depth of β. Let \( α \) be an elementary root with \( (λ_r)_{r \in Y} \) such that \( α = \sum_{r \in Y} λ_r α_r \) and \( λ_r \geq μ_r \) for all \( r \in Y \). Then dp(α) ≥ dp(β) by (6.68).

If dp(α) = dp(β), Lemma (6.68) together with the hypothesis yield that \( α = β \). Suppose now that dp(α) > dp(β) and let \( s \in Y \) such that \( s \cdot α < α \); by (6.64) we know that the coefficient of \( α_s \) in \( s \cdot α \) equals \( λ_s - 1 \) and that \( ⟨α, α_s⟩ = \frac{1}{2} \). If \( λ_s - 1 \geq μ_s \), induction on dp(α) − dp(β) yields that \( s \cdot α \) is a successor of β, and thus \( α \succeq β \) by transitivity of \( \succeq \). Assume next that \( λ_s - 1 < μ_s \); then \( λ_s = μ_s \), since \( λ_s \geq μ_s \) and \( λ_s \) and \( μ_s \) are integers. Further, \( λ_r \geq μ_r \) and \( ⟨α_r, α_s⟩ \leq 0 \) for \( r \neq s \); therefore

\[
⟨α, α_s⟩ = λ_s + \sum_{r \neq s} λ_r ⟨α_r, α_s⟩ \leq μ_s + \sum_{r \neq s} μ_r ⟨α_r, α_s⟩ = ⟨β, α_s⟩. \quad (∗)
\]

So \( ⟨β, α_s⟩ > 0 \), and thus \( s \cdot β < β \). Denote the coefficient of \( α_s \) in \( s \cdot β \) by \( μ'_s \); then \( μ'_s \) is less than or equal to \( μ_s - 1 = λ_s - 1 \), and thus \( s \cdot α \succeq s \cdot β \) by induction on dp(β).

Assume for a contradiction that \( μ'_s < μ_s - 1 \), and thus \( ⟨s \cdot β, α_s⟩ \leq -1 \). Since \( s \cdot α \) is an elementary root preceded by \( s \cdot β \), Lemma (3.38) implies that the coefficients of \( α_s \) in \( s \cdot α \) and \( s \cdot β \) coincide; that is, \( μ'_s = λ_s - 1 \), and thus \( μ'_s = μ_s - 1 \), contradicting our assumption.

So \( μ'_s = μ_s - 1 \) and \( ⟨β, α_s⟩ = \frac{1}{2} = ⟨α, α_s⟩ \). Now (∗) together with the hypothesis force \( λ_r = μ_r \) for all \( r \in Y \) such that \( ⟨α_r, α_s⟩ \neq 0 \). So for \( r \in R \) adjoined to \( s \), the coefficients of \( α_r \) in \( α \) and \( s \cdot β \) coincide, while the coefficient of \( α_s \) in \( α \) is greater than the coefficient of \( α_s \) in \( s \cdot β \); thus \( α \succ s \cdot (s \cdot β) = β \) by (6.58), as required.

Note that (6.69) does not hold in general for arbitrary roots in \( Φ_Y \). For example, if \( Y \) equals...
Chapter 6  The set of elementary roots

then the coefficients in \( rs \cdot \alpha_t = 2\alpha_r + \alpha_s + \alpha_t \) are greater than or equal to the corresponding coefficients on \( \alpha_r \), but clearly \( \alpha_r \not\leq rs \cdot \alpha_t \). Note furthermore that (6.69) also does not hold in general for elementary roots for arbitrary \( Y \). For example, suppose that \( Y = \{r, s\} \) with \( m_{rs} = 4 \); then the coefficients in \( r \cdot \alpha_s = \sqrt{2}\alpha_r + \alpha_s \) are greater than or equal to the corresponding coefficients in \( \alpha_r \), but clearly \( \alpha_r \not\leq r \cdot \alpha_s \).

If \( Y \) is of the shape described in (6.67) and \( X \subseteq \{r, s, t\} \), Proposition (6.69) yields that every root in \( E_X^Y \) is a successor of \( \rho_{l,m,n} \). It can be easily seen that \( r_0 \) is the only element of \( Y \) such that \( r_0 \cdot \rho_{l,m,n} \succ \rho_{l,m,n} \); that is, \( \langle \rho_{l,m,n}, x_0 \rangle < 0 \).

If \( l, m, n \geq 2 \), then

\[
\langle r_0 \cdot \rho_{l,m,n}, x_0 \rangle = -\langle \rho_{l,m,n}, x_0 \rangle = -(2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2) = 1;
\]

thus \( r_0 \cdot \rho_{l,m,n} \in \Delta \), and no elementary root can be a successor of \( r_0 \cdot \rho_{l,m,n} \).

(6.70) Proposition  Suppose \( X, Y \) are of the shape described in (6.67) with \( l, m, n \geq 2 \). Then

\[
E_X^Y = \begin{cases} 
\{\rho_{l,m,n}\} & \text{if } X = \{r, s, t\}, \\
\emptyset & \text{if } X \neq \{r, s, t\}.
\end{cases}
\]

(6.71)  From now on suppose that \( Y \) equals

with \( m, n \geq 1 \) and \( X \subseteq \{r_1, s_m, t_n\} \).
Chapter 6  

The set of elementary roots

73

(6.72) Proposition  Suppose $Y$ is of the shape described in (6.71). Then $\mathcal{E}_Y^{\{r_1,s_m,t_n\}}$ is the set of

$$
x_1 + y_m + 2(y_{m-1} + \cdots + y_{j(3)}) + 3(y_{j(3)-1} + \cdots + y_{j(4)}) + \cdots
\cdots + (M-1)(y_{j(M-1)-1} + \cdots + y_{j(M)}) + M(y_{j(M)-1} + \cdots + y_1)
+ Mx_0 + M(z_1 + \cdots + z_{k(M)-1}) + \cdots + 2(z_{k(3)} + \cdots + z_{n-1}) + z_n
$$

with $M \in \{2, \ldots, \min(m,n) + 1\}$,

$$m > j(3) > j(4) > \ldots > j(M-1) > j(M) > 0$$

and

$$0 < k(M) < k(M-1) < \ldots < k(4) < k(3) < n.$$ 

Whence

$$|\mathcal{E}_Y^{\{r_1,s_m,t_n\}}| = \sum_{M=2}^{\min(m,n)+1} \binom{m-1}{M-2} \binom{n-1}{M-2} = \binom{m+n-1}{m-1}.$$ 

Before we show (6.72), we prove the next lemma, which yields a more interesting proof for (6.72).

(6.73) Lemma  Suppose $Y$ is of the shape described in (6.71), and let $\alpha$ be in $\Phi_Y^+$. Denote the coefficients of $x_i, y_j, z_k$ in $\alpha$ by $\lambda_i, \mu_j$ and $\nu_k$ respectively, and suppose that $\lambda_0 \geq 1$.

(i) Then $\lambda_1 \leq \lambda_0$, with equality only if $\lambda_0 = \lambda_1 = 1$; furthermore,

$$\mu_m \leq \mu_{m-1} \leq \ldots \leq \mu_1 \leq \lambda_0 \text{ and } \lambda_0 \geq \nu_1 \geq \ldots \geq \nu_{n-1} \geq \nu_n.$$ 

(ii) There exists a $\beta \preceq \alpha$ such that the coefficient of $x_0$ in $\beta$ equals $\lambda_0$, while the coefficient of $y_1$ in $\beta$ is less than or equal to $\lambda_0 - 1$.

(iii) If $\mu_j \geq \mu_{j+1} + 2$ for some $j \in \{0, \ldots, m-1\}$ (where $\mu_0 = \lambda_0$), there exists a $\beta \preceq \alpha$ such that the coefficient of $x_0$ in $\beta$ equals $\lambda_0$, while the coefficient of $y_1$ is less than or equal to $\lambda_0 - 2$.

Proof. The coefficient of $x_1$ in $r_1 \cdot \alpha$ equals $\lambda_0 - \lambda_1$, and since the coefficient of $x_0$ in $r_1 \cdot \alpha$ (namely $\lambda_0$) is greater than 0, this has to be nonnegative; that
is \( \lambda_0 \geq \lambda_1 \). If \( \lambda_0 = \lambda_1 \), then \( I(r_1 \cdot \alpha) \) is a subset of

![Diagram](image)

and (6.66) yields that \( \lambda_0 = 1 \). Now let \( j \in \{1, \ldots, m\} \). An easy induction yields for \( i \in \{j, \ldots, m-1\} \) that \( y_i \) has coefficient \( \mu_{i+1} + \mu_{j-1} - \mu_j \) in \((s_i \cdots s_j) \cdot \alpha\), and we deduce that the coefficient of \( y_m \) in \( \beta = (s_i \cdots s_j) \cdot \alpha \) equals \( \mu_{j-1} - \mu_j \). Since the coefficient of \( x_0 \) in \( \beta \) equals \( \lambda_0 \), and this is positive, \( \beta \) is a positive root, and thus \( \mu_{j-1} \geq \mu_j \). Symmetrical arguments yield the inequalities for the \( \nu_k \), and this proves (i).

If \( \lambda_0 = 1 \), then \( \alpha \geq x_0 \) by (6.69), and (ii) is certainly true. So suppose that \( \lambda_0 \geq 2 \), and let \( \beta \preceq \alpha \) be of minimal depth such that the coefficient of \( x_0 \) in \( \beta \) equals \( \lambda_0 \). Minimality of \( \beta \) implies that \( r_0 \cdot \beta \prec \beta \), and the coefficient of \( x_0 \) in \( r_0 \cdot \beta \) is less than or equal to \( \lambda_0 - 1 \). Now (i) yields that the coefficient of \( y_1 \) in \( r_0 \cdot \beta \) is less than or equal to \( \lambda_0 - 1 \), and so the coefficient of \( y_1 \) in \( \beta \) is less than or equal to \( \lambda_0 - 2 \), as required.

It remains to show (iii). So suppose that \( \mu_j \geq \mu_{j+1} + 2 \) for some \( j \in \{0, \ldots, m-1\} \), and let \( \beta \) be minimal with this property. If \( j = 0 \) we can choose \( \beta = \alpha \), so assume next that \( j > 0 \), and proceed by induction. By minimality of \( j \) we know that \( \mu_{j-1} \leq \mu_j + 1 \); whence \( \mu_{j-1} = \mu_j \) or \( \mu_{j-1} = \mu_j + 1 \) by (i). If \( \mu_{j-1} = \mu_j \), the coefficient of \( y_j \) in \( s_j \cdot \alpha \) equals \( \mu_{j+1} \), and this is less than or equal to \( \mu_j - 2 = \mu_{j-1} - 2 \); if \( \mu_{j-1} = \mu_j + 1 \), the coefficient of \( y_j \) in \( s_j \cdot \alpha \) equals \( \mu_{j+1} + 1 \), and this is less than or equal to \( \mu_j - 2 + 1 = \mu_{j-1} - 2 \). So in any case, induction yields that there exists a \( \beta \preceq s_j \cdot \alpha \) (\( \preceq \alpha \)) such that the coefficient of \( x_0 \) in \( \beta \) equals \( \lambda_0 \), while the coefficient of \( y_1 \) is less than or equal to \( \lambda_0 - 2 \), and this finishes the proof.

**Proof of (6.17).** Suppose first that \( \alpha \) is of the form described above; we show that \( \alpha \) is an elementary root, and since the coefficients of \( \alpha \) certainly satisfy the required conditions, it will follow that \( \alpha \in E_Y^{(r_1, s_m, t_n)} \). If \( M = 2 \), then \( \alpha = \rho_{1,m,n} \) is elementary. Suppose next that \( M \geq 3 \), and proceed by induction. If \( j(M) = k(M) = 1 \), then \( \langle \alpha, x_0 \rangle = \frac{1}{2} \) and \( r_0 \cdot \alpha \) is of the form described above with \( M - 1 \) in place of \( M \). Thus \( r_0 \cdot \alpha \in E \) by induction on \( M \), and since \( \langle r_0 \cdot \alpha, x_0 \rangle = -\frac{1}{2} \), Lemma (3.37) yields that \( \alpha \) is elementary.

Suppose next that \( j(M) > 1 \), and proceed by induction on \( j(M) \). Define
Chapter 6

The set of elementary roots

75

\[ l(i) = j(i) \] for \( i \in \{3, \ldots, N - 1\} \) and \( l(M) = j(M) - 1 \); then

\[ m > l(3) > l(4) > \ldots > l(M - 1) > l(M) > 0. \]

Further, \( \langle \alpha, y_{j(M) - 1} \rangle = \frac{1}{2} \) and \( s_{j(M) - 1} \cdot \alpha \) is of the form described above with \( l(i) \) in place of \( j(i) \) for all \( i \). By induction, \( s_{j(M) - 1} \cdot \alpha \) is elementary, and as \( \langle s_{j(M) - 1} \cdot \alpha, y_{j(M) - 1} \rangle = -\langle \alpha, y_{j(M) - 1} \rangle = -\frac{1}{2} \), it follows by (3.37) that \( \alpha \) is also elementary.

Symmetrical arguments apply if \( k(M) > 1 \), and it remains to show that we have listed all the elements of \( E_{Y}^{\{r_{1}, s_{m}, l_{n}\}} \). This could be done by an inductive proof similar to the previous one, but for the sake of variety we choose the following approach.

Let \( \mu_{m-1}, \ldots, \mu_{1}, \lambda_{0}, \nu_{1}, \ldots, \nu_{n-1} \geq 2 \) such that

\[ \alpha = x_{1} + y_{m} + \mu_{m-1}y_{m-1} + \cdots + \mu_{1}y_{1} + \lambda_{0}x_{0} + \nu_{1}z_{1} + \cdots + \nu_{n-1}z_{n-1} + z_{n} \]

is an elementary root, and assume for a contradiction that there exists a \( j \in \{0, \ldots, m-1\} \) with \( \mu_{j} \geq \mu_{j+1} + 2 \) (where \( \mu_{0} = \lambda_{0} \) and \( \mu_{m} = 1 \)). Let \( \beta \preceq \alpha \) be according to (6.73)(iii) such that the coefficient of \( x_{0} \) in \( \beta \) equals \( \lambda_{0} \), and the coefficient of \( y_{1} \) in \( \beta \) is less than or equal to \( \lambda_{0} - 2 \). Furthermore, let \( \gamma \preceq \beta \) be according to (6.73)(ii) such that the coefficient of \( x_{1} \) in \( \gamma \) equals \( \lambda_{0} \), while the coefficient of \( z_{1} \) in \( \gamma \) is less than or equal to \( \lambda_{0} - 1 \). Since \( \gamma \) precedes \( \alpha \) and \( \beta \), the coefficient of \( x_{1} \) in \( \gamma \) is less than or equal to 1, and the coefficient of \( y_{1} \) in \( \gamma \) is less than or equal to \( \lambda_{0} - 2 \). Hence

\[ \langle \gamma, x_{0} \rangle \geq \lambda_{0} + \left( -\frac{1}{2} \right)(\lambda_{0} - 2) + \left( -\frac{1}{2} \right)1 + \left( -\frac{1}{2} \right)(\lambda_{0} - 1) \geq 1; \]

since the coefficient of \( x_{0} \) in \( \gamma \) is greater than 1 we find that \( \gamma \) is of depth strictly greater than 1, and thus (3.32) implies that \( \gamma \) dom \( x_{0} \) and \( \gamma \in \Delta \). Now (3.36) forces \( \alpha \in \Delta \), contrary to our choice of \( \alpha \). So \( \mu_{j} \leq \mu_{j+1} + 1 \) for all \( j \in \{0, \ldots, m-1\} \), and thus \( \mu_{j} = \mu_{j+1} \) or \( \mu_{j+1} + 1 \) by (6.73)(i); symmetrically \( \nu_{k} \in \{\nu_{k+1}, \nu_{k+1} + 1\} \) for all \( k \in \{0, \ldots, n-1\} \), and \( \alpha \) is of the desired form.

The binomial identity employed in the latter part of the assertion is known as the Vandermonde-identity.
(6.74) **Proposition** Suppose $Y$ is of the shape described in (6.71). Then $\mathcal{E}^X_Y = \emptyset$ if $X = \{r_1\}, \{r_1, s_m\}, \{r_1, t_n\}$.

The next two results are part of the proof of (6.74), but are stated separately since they will be used again later.

(6.75) **Lemma** Suppose $X$ and $Y$ are of the shape described in (6.71) and $X \neq \{r_1, s_m, t_n\}$. Furthermore, let $\alpha$ be a minimal element of $\mathcal{E}^X_Y$ with respect to $\preceq$. Then there exists an $r \in \{r_1, s_m, t_n\} \setminus X$ with $r \cdot \alpha \in \mathcal{E}^{X \cup \{r\}}_Y$.

**Proof.** Since $X \subseteq \{r_1, s_m, t_n\}$, we know that $\alpha$ is a successor of $\rho_{1,m,n}$, and since $X \neq \{r_1, s_m, t_n\}$ clearly $\alpha \neq \rho_{1,m,n}$. So $\alpha \succ \rho_{1,m,n}$, and there exists an $r \in Y$ with $\alpha \succ r \cdot \alpha \succeq \rho_{1,m,n}$. For $s \in X$, the coefficients of $\alpha_s$ in $\alpha$ and $\rho_{1,m,n}$ coincide, and thus $r \notin X$. Denote the coefficient of $\alpha_r$ in $r \cdot \alpha$ by $\lambda$. If $\lambda \geq 2$, then $r \cdot \alpha$ is in $\mathcal{E}^X_Y$, contradicting the minimality of $\alpha$; therefore $\lambda \leq 1$. The coefficient of $\alpha_r$ in $\alpha$ equals $\lambda + 1$ by (6.64), and this has to be greater than or equal to 2 as $r \in Y \setminus X$; whence $\lambda = 1$ and $r \cdot \alpha \in \mathcal{E}^{X \cup \{r\}}_Y$. Since $r \cdot \alpha$ is a successor of $\rho_{1,m,n}$, the coefficients of $y_{m-1}, \ldots, y_1, x, z_1, \ldots, z_{n-1}$ in $r \cdot \alpha$ are greater than or equal to 2, and this forces $r \in \{r_1, s_m, t_n\}$, as required.

(6.76) **Corollary** Suppose $X$ and $Y$ are of the shape described in (6.71) with $X \neq \{r_1, s_m, t_n\}$ such that $\mathcal{E}^{X \cup \{r\}}_Y = \emptyset$ for all $r \in \{r_1, s_m, t_n\} \setminus X$. Then $\mathcal{E}^X_Y = \emptyset$.

**Proof of (6.19).** Assume for a contradiction that $\mathcal{E}^{\{r_1, s_m\}}_Y \neq \emptyset$, and let $\alpha$ be an element of minimal depth. As $t_n$ is the only element of $\{r_1, s_m, t_n\} \setminus \{r_1, s_m\}$, the previous lemma yields that $t_n \cdot \alpha \in \mathcal{E}^{\{r_1, s_m, t_n\}}_Y$. By (6.72), the coefficient of $z_{n-1}$ in $t_n \cdot \alpha$ equals 2, while the coefficient of $z_n$ in $t_n \cdot \alpha$ equals 1, and thus

$$\langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -(1 + (-1/2)2) = 0,$$

contradicting $t_n \cdot \alpha \prec \alpha$. So $\mathcal{E}^{\{r_1, s_m\}}_Y = \emptyset$ and symmetrically also $\mathcal{E}^{\{r_1, t_n\}}_Y = \emptyset$. Moreover, $\mathcal{E}^{\{r_1\}}_Y = \emptyset$ by (6.76).
If $m$ also equals 1, symmetrical arguments yield that $E_{Y}^{\{s_{1}, t_{n}\}}$ and $E_{Y}^{\{s_{1}\}}$ are empty; so $E_{Y}^{\{t_{n}\}}$ is empty by (6.76), and a repeated application of (6.76) yields that $E_{Y}^{\emptyset}$ is also empty. (Alternatively, it can be easily verified that $\rho_{1,1,n}$ is the only element of $\Phi_{Y}$ preceded by $\rho_{1,1,n}$.) This yields the next result.

(6.77) Lemma Suppose $Y$ equals

$$
\begin{array}{c}
\text{s}_1 \rightarrow r_0 \rightarrow t_1 \rightarrow t_{n-1} \rightarrow t_n \\
\end{array}
$$

with $n \geq 1$ and $X \subseteq \{r_1, s_1, t_n\}$. Then $E_{Y}^{X} = \emptyset$ if $X \neq \{r_1, s_1, t_n\}$ and $E_{Y}^{\{r_1, s_1, t_n\}} = \{\rho_{1,1,n}\}$.

(6.78) From now on suppose that $Y$ equals

$$
\begin{array}{c}
\text{s}_m \rightarrow s_{m-1} \rightarrow \text{s}_1 \rightarrow r_0 \rightarrow t_1 \rightarrow t_{n-1} \rightarrow t_n \\
\end{array}
$$

with $m, n \geq 2$ and $X \subseteq \{s_m, t_n\}$.

Define $\sigma_{m,n}$ to equal

$$(r_1 r_0) \cdot \rho_{1,m,n} = 2x_1 + y_m + 2(y_{m-1} + \cdots + y_1) + 3x_0 + 2(z_1 + \cdots + z_{n-1}) + z_n.$$ 

Since $\rho_{1,m,n}$ is elementary and $\langle \rho_{1,m,n}, x_0 \rangle = \langle r_0 \cdot \rho_{1,m,n}, x_1 \rangle = -\frac{1}{2}$, Lemma (3.37) yields that $\sigma_{m,n}$ is in $E_{Y}$, and hence in $E_{Y}^{\{s_m, t_n\}}$.

Now let $\alpha \in E_{Y}^{X}$. By (6.73)(i), the coefficient of $x_0$ in $\alpha$ is strictly greater than the coefficient of $x_1$ in $\alpha$, which in turn is greater than or equal to 2. So the coefficient of $x_0$ in $\alpha$ is greater than or equal to 3, and since for $r \in Y \setminus \{s_m, t_n\}$ the coefficient of $\alpha_r$ in $\alpha$ is greater than or equal to 2, Proposition (6.69) yields that $\alpha \succeq \sigma_{m,n}$.
Proof. We show first that if \( \alpha \) is of type (1), (2) or (3), then \( \alpha \in \mathcal{E} \); since the coefficients of \( \alpha \) satisfy the required conditions, it follows that \( \alpha \in \mathcal{E} \).

Suppose first that \( \alpha \) is of type (1). If \( j = 1 \), then \( \alpha = \sigma_{m,n} \) is certainly an elementary root. So suppose now that \( j + k > 2 \), and proceed by induction. By symmetry, we may assume without loss of generality that \( j > 1 \). It follows that \( \langle \alpha, y_{j-1} \rangle = \frac{1}{\lambda} \) and \( s_{j-1} \cdot \alpha \) is of type (1) with \( j - 1 \) in place of \( j \); induction yields that \( s_{j-1} \cdot \alpha \in \mathcal{E} \) and so \( \alpha \) is elementary by (3.37) since \( \langle s_{j-1} \cdot \alpha, y_{j-1} \rangle = \langle \alpha, y_{j-1} \rangle = -\frac{1}{\lambda} \).

Assume next that \( \alpha \) is of type (2). If \( k(2) = 1 \), then \( r_0 \cdot \alpha \) is of type (1) (since \( k(1) > 1 \)), and thus an elementary root by the previous paragraph. Now \( \langle r_0 \cdot \alpha, x_0 \rangle = \frac{1}{\lambda} \), so (3.37) yields that \( \alpha \) is elementary. Suppose next that \( k(2) > 1 \), and proceed by induction. Then \( \langle \alpha, z_{k(2)} \rangle = \frac{1}{\lambda} \) and \( t_{k(2)} \cdot \alpha \) is of type (2) with \( k(2) - 1 \) in place of \( k(2) \). So \( t_{k(2)} \cdot \alpha \) is elementary by induction, and (3.37) implies once again that \( \alpha \in \mathcal{E} \).

Symmetrical arguments apply if \( \alpha \) is of type (3), therefore it remains to show that all the elements of \( \mathcal{E}_{Y}^{\{s_m,t_n\}} \) have been accounted for. Again, this could be done by an inductive proof, but we choose the following one.
Let $\alpha \in \mathcal{E}_{\Delta}^{(s_{m},t_{n})}$. Then $\alpha \succeq \sigma_{m,n}$ by the above, whence the coefficient of $x_{0}$ in $\alpha$ is greater than or equal to 3. We show first that $\alpha$ is of type (1), (2) or (3) if $\alpha$ has coefficient 3 or 4 for $x_{0}$. Suppose now that

$$\alpha = \lambda_{1}x_{1} + y_{m} + \mu_{m-1}y_{m-1} + \cdots + \mu_{1}y_{1} + 3x_{0} + \nu_{1}z_{1} + \cdots + \nu_{n-1}z_{n-1} + z_{n}.$$ 

Then $\lambda_{1} < 3$ by (6.73)(i), and thus $\lambda_{1} = 2$ by hypothesis. If $\mu_{m-1} = 3$, then (6.73)(ii) and (iii) yield that there exists a $\beta \preceq r_{1} \cdot \alpha$ with coefficient of $x_{0}$ in $\beta$ equal to 3, and coefficients of $x_{1}$, $y_{1}$, $z_{1}$ less than or equal to 1,1 and 2 respectively; but then

$$\langle \beta, x_{0} \rangle \geq 3 + \left(-\frac{1}{2}\right)1 + \left(-\frac{1}{2}\right)1 + \left(-\frac{1}{2}\right)2 = 1,$$

forcing $\beta \in \Delta$ and thus $\alpha \in \Delta$, a contradiction. Thus $\mu_{m-1} = 2$, and symmetrically also $\nu_{n-1} = 2$, and by (6.73)(i) it is clear that $\alpha$ is of type (1).

Next suppose that $x_{0}$ has coefficient 4 in $\alpha$, and let $\beta$ be of maximal depth with $\sigma_{m,n} \preceq \beta \preceq \alpha$ such that the coefficient of $x_{0}$ in $\beta$ is less than 4. Then maximality of $\beta$ implies that $\beta \prec r_{0} \cdot \beta \preceq \alpha$ and that the coefficient of $x_{0}$ in $r_{0} \cdot \beta$ equals 4. As $\alpha \succeq r_{0} \cdot \beta$, it is clear that $r_{0} \cdot \beta$ is elementary, and by (6.64) we deduce that $\langle \beta, x_{0} \rangle = -\frac{1}{2}$ and that the coefficient of $x_{0}$ in $\beta$ equals 3. As $\alpha \succeq \beta \succeq r_{0} \cdot \beta$, we can further conclude that $\beta$ is in $\mathcal{E}_{\Delta}^{(s_{m},t_{n})}$. The previous paragraph now yields that $\beta$ is of type (1). If $j, k \geq 2$, then

$$\langle \beta, x_{0} \rangle = 3 + \left(-\frac{1}{2}\right)2 + \left(-\frac{1}{2}\right)3 + \left(-\frac{1}{2}\right)3 \leq -1,$$

contradicting our conclusion that $\langle \beta, x_{0} \rangle = -\frac{1}{2}$. So by symmetry we may assume without loss of generality that $k = 1$, and thus $j > 1$ (since $\langle \beta, x_{0} \rangle$ equals $-\frac{1}{2}$). Set $j(1) = j$; then

$$r_{0} \cdot \beta = 2x_{1} + y_{m} + 2(y_{m-1} + \cdots + y_{j(1)}) + 3(y_{j(1)-1} + \cdots + y_{1})$$

$$+ 4x_{0} + 2(z_{1} + \cdots + z_{n-1}) + z_{n};$$

since $\langle r_{0} \cdot \beta, z_{1} \rangle = -1$, Lemma (3.38) implies that the coefficient of $z_{1}$ in $\alpha$ equals 2, and (6.73)(i) forces the coefficient of $z_{k}$ to be equal to 2 for $k \in \{1, \ldots, n-1\}$. Now let $j(2) < j(1)$ be maximal such that $\alpha$ is preceded by $\gamma = (s_{j(2)} \cdots s_{j(1)} r_{0}) \cdot \beta$; then

$$\gamma = 2x_{1} + y_{m} + 2(y_{m-1} + \cdots + y_{j(1)}) + 3(y_{j(1)-1} + \cdots + y_{j(2)})$$

$$+ 4(y_{j(2)-1} + \cdots + y_{1}) + 4x_{0} + 2(z_{1} + \cdots + z_{n-1}) + z_{n}.$$
Note that $\gamma \prec r \cdot \gamma$ for $r \in Y$ only if $r = s_{j(1)}$ and $j(1) < n$, or $r = s_{j(2)}$ and $j(2) < j(1) - 1$, or $r = t_1$. But $\alpha$ cannot be preceded by $t_1 \cdot \gamma$ since the coefficient of $z_1$ in $\alpha$ equals 2, and by maximality of $\beta$ and $j(2)$, $\alpha$ can also not be preceded by $s_{j(1)} \cdot \gamma$ (if $j(1) < n$) or $s_{j(2)} \cdot \gamma$ (if $j(2) < j(1) - 1$); thus $\alpha = \gamma$, as required.

Now assume for a contradiction that there exists an $\alpha \in E^{\{s_{m}, t_{n}\}}_Y$ such that the coefficient of $x_0$ in $\alpha$ is greater than or equal to 5. We may assume without loss of generality that $\alpha$ is of minimal depth with this property. Then $\alpha \succ \sigma_{m,n}$, and thus there exists an $r \in Y$ such that $\alpha \succ r \cdot \alpha \succeq \sigma_{m,n}$. Since $r \cdot \alpha$ precedes an elementary root, it must be elementary, and as $r \cdot \alpha$ lies between $\alpha$ and $\sigma_{m,n}$, both of which are in $E^{\{s_{m}, t_{n}\}}_Y$, further $r \cdot \alpha \in E^{\{s_{m}, t_{n}\}}_Y$. Thus $r = r_0$ by minimality of $\alpha$, and moreover, the coefficient of $x_0$ in $r \cdot \alpha$ is less than 5. Lemma (6.64) yields that the coefficient of $x_0$ in $r_0 \cdot \alpha$ equals 4, and thus $r_0 \cdot \alpha$ is of type (2) or (3) by our earlier conclusion. But then

$$\langle r_0 \cdot \alpha, x_0 \rangle \geq 4 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = 0,$$

contradicting $r_0 \cdot \alpha \prec \alpha$. Thus there are no roots in $E^{\{s_{m}, t_{n}\}}_Y$ with coefficient for $x_0$ greater than or equal to 5, and this finishes the proof.

(6.80) Lemma Suppose $Y$ is of the shape described in (6.78) with $m, n \geq 3$. Then $E^{X}_Y = \emptyset$ if $X$ equals $\{s_{m}\}$, $\{t_{n}\}$ or $\emptyset$.

Proof. Assume for a contradiction that $E^{\{s_{m}\}}_Y \neq \emptyset$, and let $\alpha$ be an element of minimal depth. Then (6.75) yields that there exists an $r \in \{r_1, t_n\}$ with $r \cdot \alpha \in E^{\{s_{m}, r\}}_Y$. Since $E^{\{s_{m}, r\}}_Y$ is empty by (6.74), we find that $r = t_n$ and $t_n \cdot \alpha \in E^{\{s_{m}, t_n\}}_Y$; but (6.79) now forces the coefficient of $z_{n-1}$ in $t_n \cdot \alpha$ to be equal to 2, and thus

$$\langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -(1 + \left(-\frac{1}{2}\right) \cdot 2) = 0,$$

contradicting $t_n \cdot \alpha \prec \alpha$. So $E^{\{s_{m}\}}_Y = \emptyset$, and symmetrical also $E^{\{t_{n}\}}_Y = \emptyset$. Since $E^{\{r_1\}}_Y$ is also empty by (6.74), Corollary (6.76) implies that $E^{\emptyset}_Y = \emptyset$. \qed
From now on we only need to determine $E^X_Y$ for $Y$ of the shape

\[ \begin{array}{ccccccc}
  & & & & & & \\
  & & & & & & \\
 s_2 & s_1 & r_0 & t_1 & \cdots & t_{n-1} & t_n \\
  & & & & & & \\
\end{array} \]

with $n \geq 2$ and $X \subseteq \{s_2, t_n\}$.

Lemma Suppose $Y$ is of the shape described in (6.81), and let $\alpha$ be a root in $\Phi_Y$. Denote the coefficient of $x_0, y_1$ and $y_2$ in $\alpha$ by $\lambda, \mu_1$ and $\mu_2$ respectively, and suppose that $\lambda \geq 3$. Then $\mu_2 < \mu_1 < \lambda$.

Proof. Assume for a contradiction that $\mu_1 = \mu_2$ or $\lambda = \mu_1$. Then the support of $s_1 \cdot \alpha$ or $s_2 s_1 \cdot \alpha$ is a subset of

\[ \begin{array}{ccccccc}
  & & & & & & \\
  & & & & & & \\
 y_1 & x_0 & z_1 & \cdots & z_{n-1} & z_n \\
  & & & & & & \\
\end{array} \]

and it can be easily checked using (6.77) that the coefficients in roots with this support are at most 2, contradicting $\lambda \geq 3$.

Proposition Suppose $Y$ is of the shape described in (6.81). Then the elements of $E_Y^{\{s_2, t_n\}}$ are:

\[ \begin{array}{c}
  \lambda x_1 + y_2 + \mu y_1 + Mx_0 \\
  + M (z_1 + \cdots + z_{k(M-1)-1}) + (M-1) (z_{k(M-1)} + \cdots + z_{k(M-2)-1}) + \cdots \\
  \cdots + 3 (z_{k(3)} + \cdots + z_{k(2)-1}) + 2 (z_{k(2)} + \cdots + z_{k(1)}) + z_n, \\
\end{array} \]

where

(i) $M \in \{5, \ldots, n+1\}$ is odd, and $\lambda = \frac{M-1}{2}$, $\mu = \frac{M+1}{2}$, or
(ii) $M \in \{3, \ldots, n+1\}$ is odd and $\lambda = \mu = \frac{M+1}{2}$, or
(iii) $M \in \{4, \ldots, n+1\}$ is even and $\lambda = \mu = \frac{M}{2}$, or
(iv) $M \in \{4, \ldots, n+1\}$ is even and $\lambda = \frac{M}{2}$ and $\mu = \frac{M}{2} + 1$, and

\[ 0 < k(M-1) < k(M-2) < \ldots < k(3) < k(2) < n. \]
Chapter 6  The set of elementary roots

\[ |E_Y^{(s_2,t_n)}| = \left(\begin{array}{c} n-1 \\ 1 \end{array}\right) + 2 \sum_{M=4}^{n+1} \left(\begin{array}{c} n-1 \\ M-2 \end{array}\right) = 2^n - n - 1. \]

**Proof.** We show first that \( \alpha \) is elementary if \( \alpha \) is a vector of the above type, and it follows trivially that \( \alpha \in E_Y^{(s_2,t_n)} \). Denote the sum of the coefficients of \( \alpha \) by \( S \); then

\[ S \geq 2 + 1 + 2 + 3 + 2(n - 1) + 1 = 2n + 7. \]

If \( S = 2n + 7 \), then \( \alpha = \sigma_{2,n} \) is elementary. Suppose next that \( S > 2n + 7 \), and proceed by induction. If \( k(M-1) > 1 \), then

\[ \langle \alpha, z_{k(M-1)-1} \rangle = M + \left(\frac{-1}{2}\right)M + \left(\frac{-1}{2}\right)(M-1) = \frac{1}{2}, \]

and thus \( t_{k(M-1)-1} \cdot \alpha \) is of the form described above with \( k(M-1) - 1 \) in place of \( k(M-1) \). By induction this is an elementary root and (3.37) yields that \( \alpha \in E \).

Next assume that \( k(M-1) = 1 \). Then \( M \geq 4 \) since \( S > 2n + 7 \), and thus \( M - 1 \in \{3, \ldots, n+1\} \). The coefficient of \( z_1 \) in \( \alpha \) equals \( M - 1 \), and \( \lambda + \mu \) equals \( M \) or \( M + 1 \).

If \( \lambda + \mu \) equals \( M \) (that is, in cases (i) and (iii)), we find that

\[ \langle \alpha, x_0 \rangle = M + \left(\frac{-1}{2}\right)M + \left(\frac{-1}{2}\right)(M-1) = \frac{1}{2}, \]

and the coefficient of \( x_0 \) in \( r_0 \cdot \alpha \) equals \( M - 1 \). Since \( \langle r_0 \cdot \alpha, x_0 \rangle = -\frac{1}{2} \), it suffices to show that \( r_0 \cdot \alpha \) is of the form described in the assertion; for then \( r_0 \cdot \alpha \in E \) by induction, and (3.37) yields that \( \alpha \in E \). If \( M \) is odd, \( M - 1 \) is even; furthermore,

\[ \lambda = \frac{M - 1}{2} \quad \text{and} \quad \mu = \frac{M + 1}{2} = \frac{M - 1}{2} + 1, \]

and these satisfies (iv) for \( M - 1 \) in place of \( M \), as required. Suppose next that \( M \) is even, and thus \( M - 1 \) is odd; then

\[ \lambda = \mu = \frac{M}{2} = \frac{(M - 1) + 1}{2}, \]
Chapter 6  The set of elementary roots  83

and these satisfies (ii) for $M - 1$ in place of $M$, and $r_0 \cdot \alpha$ is of the required form.

Finally, assume that $\lambda + \mu$ equals $M + 1$. If $M$ is even, then $\mu = \frac{M}{2} + 1$, $\lambda = \frac{M}{2}$ and $\langle \alpha, y_1 \rangle = \frac{1}{2}$. The coefficient of $y_1$ in $s_1 \cdot \alpha$ equals $\frac{M}{2}$, while the coefficient of $x_1$ in $s_1 \cdot \alpha$ equals $\lambda = \frac{M}{2}$; so the coefficients of $s_1 \cdot \alpha$ satisfy (iii). By induction $s_1 \cdot \alpha$ is an elementary root, and (3.37) yields that $\alpha \in \mathcal{E}$. If $M$ is odd, then $\lambda = \mu = \frac{M+1}{2}$, and $\langle \alpha, x_1 \rangle = \frac{1}{2}$. The coefficient of $x_1$ in $r_1 \cdot \alpha$ equals $\frac{M-1}{2}$, whence the coefficients of $r_1 \cdot \alpha$ satisfy (i). By induction, $r_1 \cdot \alpha \in \mathcal{E}$, and since $\langle r_1 \cdot \alpha, x_1 \rangle = -\frac{1}{2}$, Lemma (3.37) once again implies that $\alpha$ is elementary.

It remains to show that all the elements of $\mathcal{E}_Y^{\{s_2, \ldots, n\}}$ have been enumerated. To do so, we again take the scenic route. So let

$$\alpha = \lambda_1 x_1 + y_2 + \mu_1 y_1 + \lambda_0 x_0 + \nu_1 + \cdots + \nu_{n-1} z_{n-1} + z_n$$

with $\lambda_1, \mu_1, \lambda_0, \nu_1, \ldots, \nu_{n-1} \geq 2$ be an elementary root. Assume for a contradiction that $\nu_k \geq \nu_{k+1} + 2$ for some $k \in \{0, \ldots, n-1\}$ (where $\nu_0 = \lambda_0$ and $\nu_n = 1$). Let $\beta \preceq \alpha$ be according to (6.73)(iii) such that the coefficient of $x_0$ in $\beta$ equals $\lambda_0$, and the coefficient of $z_1$ in $\beta$ is less than or equal to $\lambda_0 - 2$. If the coefficient $\lambda'$ of $x_1$ in $\beta$ is greater than $\frac{\lambda_0}{2}$, then

$$\langle \beta, x_0 \rangle > \frac{\lambda_0}{2} + \left( -\frac{1}{2} \right) \lambda_0 = 0$$

and thus $r_0 \cdot \beta \prec \beta$; moreover, the coefficient of $x_1$ in $r_1 \cdot \beta$ equals $-\lambda'+ \lambda_0$, and this is clearly less than $\frac{\lambda_0}{2}$. Thus we may replace $\beta$ by $r_1 \cdot \beta$, which also precedes $\alpha$ and has coefficient for $x_1$ less than or equal to $\frac{\lambda_0}{2}$. If the coefficient of $y_1$ in $\beta$ is greater than $\frac{\lambda_0}{2} + \frac{1}{2}$, then

$$\langle \beta, y_1 \rangle > \frac{\lambda_0}{2} + \frac{1}{2} + \left( -\frac{1}{2} \right) \lambda_0 + \left( -\frac{1}{2} \right) \lambda_0 = 0$$

(since the coefficient of $y_2$ in $\beta$ is less than or equal to the coefficient of $y_2$ in $\alpha$, and thus less than or equal to 1). So $\beta \succ s_1 \cdot \beta$, and we may replace $\beta$ by $s_1 \cdot \beta$, which also precedes $\alpha$ and has coefficient for $y_1$ less than or equal to $\frac{\lambda_0}{2} + \frac{1}{2}$. So assume without loss of generality that the sum of the coefficients of $x_1$ and $y_1$ in $\beta$ is less than or equal to $\lambda_0 + \frac{1}{2}$, and thus less than or equal to $\lambda_0$ (as the coefficients of $x_1$ and $y_1$ in $\beta$ are integers). Therefore

$$\langle \beta, x_0 \rangle \geq \lambda_0 + \left( -\frac{1}{2} \right) (\lambda_0 - 2) + \left( -\frac{1}{2} \right) (\lambda_1 + \mu_1) \geq \lambda_0 + \left( -\frac{1}{2} \right) (\lambda_0 - 2) + \left( -\frac{1}{2} \right) \lambda_0,$$
which equals 1, forcing $\beta \in \Delta$, and contradicting $\alpha \in \mathcal{E}$. So $\nu_k$ equals $\nu_{k+1}$ or $\nu_{k+1}+1$ for all $k$ by (6.73)(i). In particular, $\lambda_0 \leq n + 1$, and we define $M$ to be $\lambda_0$. Since $\lambda_1 \geq 2$ further $M \geq 3$ by (6.73)(i).

It remains to show that $\lambda_1$ and $\mu_1$ satisfy (i), (ii), (iii) or (iv). First note that since $\langle \alpha, x_1 \rangle = \lambda_1 + (-\frac{1}{2})M$ and $\langle \alpha, y_1 \rangle = \mu_1 + (-\frac{1}{2})1 + (-\frac{1}{2})M$ have to be less than or equal to $\frac{1}{2}$,

$$\lambda_1 \leq \frac{M+1}{2} \quad \text{and} \quad \mu_1 \leq \frac{M}{2} + 1.$$  \hfill (*)

Next, let $\gamma \preceq \alpha$ be according to (6.73)(ii) such that the coefficient of $x_0$ in $\gamma$ equals $M$, while the coefficient of $z_1$ in $\gamma$ is less than or equal to $M - 1$, and denote the coefficients of $x_1, y_1$ in $\gamma$ by $\lambda'$ and $\mu'$ respectively. By (3.35), $\lambda' \leq \lambda_1$ and $\mu' \leq \mu_1$, and as $\gamma$ cannot dominate $x_0$,

$$\frac{1}{2} \geq \langle \gamma, x_0 \rangle \geq M + \left(-\frac{1}{2}\right)(\lambda'+\mu') + \left(-\frac{1}{2}\right)(M-1) \geq \frac{1}{2} + \left(-\frac{1}{2}\right)(\lambda_1+\mu_1) + \frac{M}{2};$$

this yields that

$$\lambda_1 + \mu_1 \geq M.$$  \hfill (***)

Suppose first that $M$ is even; then $\lambda_1 \leq \frac{M}{2}$ by (**) since $\lambda_1$ is an integer, and thus $\mu_1$ equals $\frac{M}{2}$ or $\frac{M}{2} + 1$ by (*) and (**). If $\lambda_1 = \frac{M}{2}$, then $\lambda_1, \mu_1$ satisfy (iii) or (iv). Assume for a contradiction that $\lambda_1 \leq \frac{M}{2} - 1$, and thus $\mu_1 = \frac{M}{2} + 1$ and $\lambda_1 = \frac{M}{2} - 1$ by the above. So

$$\langle \gamma, x_0 \rangle \geq M + \left(-\frac{1}{2}\right)(\lambda'+\mu') + \left(-\frac{1}{2}\right)(M-1) \geq \frac{1}{2} + \left(-\frac{1}{2}\right)(\lambda_1+\mu_1) + \frac{M}{2} = \frac{1}{2},$$

and since $\frac{1}{2} \geq \langle \gamma, x_0 \rangle$, we must have equality everywhere; hence $\lambda'$ must be equal to $\lambda = \frac{M}{2} - 1$ and $\mu' = \mu_1 = \frac{M}{2} + 1$. Moreover, the coefficient of $y_2$ in $\gamma$ is less than or equal to 1. Therefore $\langle \gamma, y_1 + x_0 \rangle = \langle \gamma, y_1 \rangle + \langle \gamma, x_0 \rangle$ is greater than or equal to

$$\left(\frac{M}{2} + 1\right) + \left(-\frac{1}{2}\right)1 + \left(-\frac{1}{2}\right)M$$

$$+ M + \left(-\frac{1}{2}\right)\left(\frac{M}{2} + 1\right) + \left(-\frac{1}{2}\right)\left(\frac{M}{2} - 1\right) + \left(-\frac{1}{2}\right)(M - 1),$$

which equals 1, forcing $\gamma \in \Delta$, a contradiction.
Finally assume that $M$ is odd; then $\mu_1 \leq \frac{M+1}{2}$ by (*), and by (**) we know that $\lambda_1$ equals $\frac{M+1}{2}$ or $\frac{M+1}{2}$. Now $\lambda_1, \mu_1$ satisfy (i) or (ii) if $\mu_1 = \frac{M+1}{2}$, and we assume for a contradiction that $\mu_1 \leq \frac{M-1}{2}$. Then $\lambda_1 = \frac{M+1}{2}$ and $\mu_1 = \frac{M-1}{2}$ by the above; hence

$$\langle \gamma, x_0 \rangle \geq M + \left( -\frac{1}{2} \right) (\lambda' + \mu') + \left( -\frac{1}{2} \right) (M - 1) = \frac{1}{2} + \left( -\frac{1}{2} \right) (\lambda_1 + \mu_1) + \frac{M}{2} = \frac{1}{2},$$

and since $\frac{1}{2} \geq \langle \gamma, x_0 \rangle$, we must once again have equality everywhere. Thus $\lambda' = \lambda_1 = \frac{M}{2} - 1$ and $\mu' = \mu_1 = \frac{M}{2} + 1$. Now $\langle \gamma, x_1 + x_0 \rangle = \langle \gamma, x_1 \rangle + \langle \gamma, x_0 \rangle$ is greater than or equal to

$$\frac{M + 1}{2} + \left( -\frac{1}{2} \right) M + M + \left( -\frac{1}{2} \right) \frac{M - 1}{2} + \left( -\frac{1}{2} \right) \frac{M + 1}{2} + \left( -\frac{1}{2} \right) (M - 1),$$

and this equals 1, again forcing $\gamma \in \Delta$, a contradiction. 

(6.84) Lemma Suppose $Y$ is of the shape described in (6.81). Then $E_Y^{\{s_2\}}$ is empty.

Proof. Assume for a contradiction that $E_Y^{\{s_2\}}$ is not empty, and let $\alpha$ be an element of minimal depth. By (6.75) there exists an $r \in \{r_1, t_n\}$ such that $r \cdot \alpha \in E_Y^{\{s_2, r\}}$, and since $E_Y^{\{r_1, s_2\}}$ is empty by (6.74), we are left with $r = t_n$. The previous proposition now forces the coefficient of $z_{n-1}$ in $t_n \cdot \alpha$ to be equal to 2, and since the coefficient of $z_n$ in $t_n \cdot \alpha$ is 1, this yields

$$\langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -(1 + \left( -\frac{1}{2} \right) \cdot 2) = 0,$$

contradicting $t_n \cdot \alpha < \alpha$. 

In particular, if $m = n = 2$, symmetrical arguments yield that $E_Y^{\{t_2\}}$ is also empty; since $E_Y^{\{r_1\}}$ is empty by (6.74), Corollary (6.76) now implies that $E_Y^X$ is empty. Alternatively, it can be easily checked that $\sigma_{2,2}$ is the only successor of $\sigma_{2,2}$ if $m = n = 2$, and we get:

(6.85) Lemma Suppose $Y$ equals

Then $E_Y^{\{s_2, t_2\}} = \{\sigma_{2,2}\}$ and $E_Y^X = \emptyset$ if $X = \{s_2\}$, $\{t_2\}$ or $\emptyset$. 
This leaves us to determine $\mathcal{E}_Y^X$ for $Y$ of the shape

$$s_2 \quad s_1 \quad r_0 \quad t_1 \quad \ldots \quad t_{n-1} \quad t_n$$

with $n \geq 3$ and $X \subseteq \{t_n\}$.

Define $\tau_n$ to be

$$(s_2 s_1 r_0 t_1) \cdot \sigma_{2,n} = 2x_1 + 2y_2 + 3y_1 + 4x_0 + 3z_1 + 2(z_2 + \cdots + z_{n-1}) + z_n.$$ 

Since $\sigma_{2,n}$ is elementary, and

$$\langle \sigma_{2,n}, z_1 \rangle = \langle t_1 \cdot \sigma_{2,n}, x_0 \rangle = \langle (r_0 t_1) \cdot \sigma_{2,n}, y_1 \rangle = \langle (s_1 r_0 t_1) \cdot \sigma_{2,n}, y_2 \rangle = -\frac{1}{2},$$ 

(3.37) yields that $\tau_n$ is an elementary root, and it follows easily that $\tau_n$ is an element of $\mathcal{E}_Y^{\{t_n\}}$.

We show now that each root in $\mathcal{E}_Y^X$ with $X \subseteq \{t_n\}$ is a successor of $\tau_n$.

So let

$$\alpha = \lambda_1 x_1 + \mu_2 y_2 + \mu_1 y_1 + \lambda_0 x_0 + \nu_1 z_1 + \cdots + \nu_n z_n$$

be an element of $\mathcal{E}_Y^X$. Then $\lambda_1 \geq 2$ by hypothesis, and thus in $\lambda_0 > \lambda_1$ by (6.73)(i); hence in particular, $\lambda_0 \geq 3$, and thus $\mu_2 < \mu_1 < \lambda_0$ by (6.82). Since $\mu_2 \geq 2$ this yields in particular that $\mu_1 \geq 3$ and $\lambda_0 \geq 4$. Clearly $\alpha \succ \sigma_{2,n}$, and an easy calculation yields that $t_1$ is the only element of $Y$ with $t_1 \cdot \sigma_{2,n} \succ \sigma_{2,n}$; therefore $\alpha \succeq t_1 \cdot \sigma_{2,n}$ and $\nu_1 \geq 3$. Hence $\alpha \succeq \tau_n$ by (6.69), as desired.

It can be easily verified that the roots listed in the following two lemmas are the only roots preceded by $\tau_n$ for $n = 3$, 4, and that these are elementary.

(6.87) **Lemma** Suppose $Y$ equals

$$s_2 \quad s_1 \quad r_0 \quad t_1 \quad t_2 \quad t_3$$

Then $\mathcal{E}_Y^{\{t_3\}} = \{\tau_3\}$ and $\mathcal{E}_Y^\emptyset = \emptyset$. 

(6.88) **Lemma**  Suppose $Y$ equals

$$r_1$$

Then $E_Y^{(t_1)}$ consists of the following roots

$$\tau_4 = 2x_1 + 2y_2 + 3y_1 + 4x_0 + 3z_1 + 2z_2 + 2z_3 + z_4,$$

$$t_2 \cdot \tau_4 = 2x_1 + 2y_2 + 3y_1 + 4x_0 + 3z_1 + 3z_2 + 2z_3 + z_4,$$

$$(t_1 t_2) \cdot \tau_4 = 2x_1 + 2y_2 + 3y_1 + 4x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,$$

$$(r_0 t_1 t_2) \cdot \tau_4 = 2x_1 + 2y_2 + 3y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,$$

$$(r_1 r_0 t_1 t_2) \cdot \tau_4 = 3x_1 + 2y_2 + 3y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,$$

$$(s_1 r_0 t_1 t_2) \cdot \tau_4 = 2x_1 + 2y_2 + 4y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,$$

$$(r_1 s_1 r_0 t_1 t_2) \cdot \tau_4 = 3x_1 + 2y_2 + 2y_1 + 5x_0 + 3z_2 + 2z_3 + z_4,$$

$$(s_1 r_0 t_1 t_2) \cdot \tau_4 = 2x_1 + 2y_2 + 4y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,$$

and $E_Y^0$ has exactly one element, namely

$$(t_4 t_3 t_2 t_1 t_0 r_1 s_1 r_0 t_1 t_2) \cdot \tau_4 = 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 3z_3 + 2z_4.$$

(6.89)  It remains to determine $E_Y^X$ for $Y$ equal to

$$r_1$$

with $n \geq 5$, and $X \subseteq \{t_n\}$.

(6.90) **Proposition**  Suppose $Y$ is of the shape described in (6.89). Then $E_Y^{(t_n)}$ equals the set of vectors of the following six types:

$$2x_1 + 2y_2 + 3y_1 + 4x_0 + 4(z_1 + \cdots + z_{k(2)} - 1)$$

$$+ 3(z_{k(2)} + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n,$$
where \(0 < k(2) < k(1) < n\),
\[
\lambda x_1 + 2y_2 + \mu y_1 + 5x_0 \\
+ 5(z_1 + \cdots + z_{k(3)-1}) + 4(z_{k(3)} + \cdots + z_{k(2)-1}) \\
+ 3(z_{k(2)} + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n,
\]
where \(0 < k(3) < k(2) < k(1) < n\) and \(\lambda \in \{2, 3\}, \mu \in \{3, 4\}\),
\[
\lambda x_1 + 2y_2 + \mu y_1 + 6x_0 \\
+ 6(z_1 + \cdots + z_{k(4)-1}) + 5(z_{k(4)} + \cdots + z_{k(3)-1}) \\
+ 4(z_{k(3)} + \cdots + z_{k(2)-1}) + 3(z_{k(2)} + \cdots + z_{k(1)-1}) \\
+ 2(z_{k(1)} + \cdots + z_{n-1}) + z_n,
\]
where \(0 < k(4) < k(3) < k(2) < k(1) < n\) and \((\lambda, \mu) = (3, 3)\) or \((2, 4)\),
\[
3x_1 + 2y_2 + 4y_1 + 6x_0 + 4(z_1 + \cdots + z_{k(2)-1}) \\
+ 3(z_{k(2)} + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n,
\]
where \(1 < k(2) < k(1) < n\),
\[
3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 3(z_2 + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + 2z_n,
\]
where \(2 < k(1) < n\) and
\[
3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2(z_3 + \cdots + z_{n-1}) + z_n.
\]
Whence \(|E_Y^{(t_n)}| = 2^{(n+1)}\).

**Proof.** We show first that \(\alpha\) is elementary if it is of one of the types (1)-(6); since the coefficients of \(\alpha\) certainly satisfy the required conditions, this yields that \(\alpha\) is in \(E_Y^{(t_n)}\).

Suppose first that \(\alpha\) is of type (1); then \(k(1) + k(2) \geq 3\). If \(k(1) = 2\) and \(k(2) = 1\), then \(\alpha = \tau_n \in E\). Suppose next that \(k(1) + k(2) > 3\), and proceed by induction. If \(k(2) > 1\) we find that \(\langle \alpha, y_{k(2)-1} \rangle = \frac{1}{2}\), and that \(t_{k(2)-1} \cdot \alpha\) is of type (1) with \(k(2) - 1\) in place of \(k(2)\). By induction, this is an elementary root, and (3.37) implies that \(\alpha\) is elementary. Now suppose that \(k(2) = 1\), and thus \(k(1) > 2\). Then \(\langle \alpha, y_{k(1)-1} \rangle = \frac{1}{2}\) and \(t_{k(1)-1} \cdot \alpha\) is of type (1) with
Chapter 6  The set of elementary roots

\( k(1) - 1 \) in place of \( k(1) \). By induction, this is an elementary root, and since \( \langle t_{k(1)-1} \cdot \alpha, z_{(1)-1} \rangle = -\frac{1}{2} \), Lemma (3.37) yields again that \( \alpha \in \mathcal{E} \), and this finishes the induction.

Suppose next that \( \alpha \) is of type (2), and denote the sum of the coefficients of \( \alpha \) by \( S \); then

\[
S \geq 2 + 2 + 3 + 5 + 4 + 3 + 2(n - 3) + 1 = 2n + 14.
\]

If \( S = 2n + 14 \), then \( \alpha \) equals

\[
(r_0 t_1 t_2) \cdot \tau_n = 2x_1 + 2y_2 + 3y_1 + 5x_0 + 4z_1 + 3z_2 + 2(z_3 + \cdots + z_{n-1}) + z_n,
\]

and since \( \tau_n \) is elementary, it can be easily verified using (3.37) that this is an elementary root. So suppose next that \( S > 2n + 15 \), and proceed by induction. If \( k(3) > 1 \) we find that \( \langle \alpha, y_{k(3)-1} \rangle = \frac{1}{2} \), and \( t_{k(3)-1} \cdot \alpha \) is of type (2) with \( k(3) - 1 \) in place of \( k(3) \). By induction, this is an elementary root, and it follows by (3.37) that \( \alpha \) is elementary. If \( \lambda = 3 \), then \( \langle \alpha, x_1 \rangle = \frac{1}{2} \) and \( r_1 \cdot \alpha \) is of type (2) with 2 in place of \( \lambda \); this is an elementary root by inductive hypothesis, and (3.37) yields that \( \alpha \in \mathcal{E} \). If \( \mu = 4 \), then \( \langle \alpha, y_1 \rangle = \frac{1}{2} \) and \( s_1 \cdot \alpha \) is also of type (2); by induction \( s_1 \cdot \alpha \) is in \( \mathcal{E} \), and (3.37) implies again that \( \alpha \) is elementary.

Suppose now that \( \lambda = 2, \mu = 3 \) and \( k(3) = 1 \). Then

\[
\langle \alpha, x_0 \rangle = 5 + \left( -\frac{1}{2} \right) 3 + \left( -\frac{1}{2} \right) 2 + \left( -\frac{1}{2} \right) 4 = \frac{1}{2},
\]

and thus \( r_0 \cdot \alpha \) is of type (1). So \( r_0 \cdot \alpha \in \mathcal{E} \) by the above, and since \( \langle r_0 \cdot \alpha, x_0 \rangle \) equals \( -\frac{1}{2} \), Lemma (3.37) once again yields that \( \alpha \in \mathcal{E} \), and this finishes the induction.

Now let \( \alpha \) be of type (3) and denote the sum of the coefficients of \( \alpha \) by \( S \); then \( S \) is greater than or equal to

\[
(\lambda + \mu) + 2 + 6 + 5 + 4 + 3 + 2(n - 4) + 1 = 6 + 2 + 6 + 5 + 4 + 3 + 2(n - 4) + 1 = 2n + 19.
\]

If \( S = 2n + 19 \), then \( \alpha \) is equal to one of the following two roots:

\[
(r_0 s_1 t_1 r_0 t_2 t_1 t_2) \cdot \tau_n
= 2x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 3z_3 + 2(z_4 + \cdots + z_{n-1}) + z_n,
\]
As \( \tau \in E \) then \( 2 \leq n \) and \( r \alpha \) this implies that \( \alpha \) assume that by the above. As before, this yields that \( \alpha \) is greater than 4. We first show that \( t \lambda \). Since \( y \) equals 6, we have

\[
\langle \alpha, x_0 \rangle = 6 + \left( \frac{1}{2} \right) 6 + \left( -\frac{1}{2} \right) 5 = \frac{1}{2},
\]

and \( r_0 \cdot x_0 \) is of type (2). This is an elementary root by the above, and since \( \langle r_0 \cdot x_0 \rangle = -\frac{1}{2} \), this implies that \( \alpha \) is elementary.

Next, assume that \( \alpha \) is of type (4); then \( \langle \alpha, x_0 \rangle = \frac{1}{2} \), and \( r_0 \cdot \alpha \) is of type (2) with \( k(3) = 1 \). By the above, this is an element of \( E \), and since \( \langle r_0 \cdot \alpha, x_0 \rangle = -\frac{1}{2} \) we can deduce that \( \alpha \in E \). Further, if \( \alpha \) is of type (5), \( \langle \alpha, z_1 \rangle = \frac{1}{2} \) and \( t_1 \cdot \alpha \) is of type (4) with \( k(2) = 2 \); therefore \( t_1 \cdot \alpha \) is in \( E \) by the above. As before, this yields that \( \alpha \) is an elementary root. Finally, assume that \( \alpha \) equals

\[
3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2(z_3 + \cdots + z_{n-1}) + z_n.
\]

Then \( \langle \alpha, z_2 \rangle = \frac{1}{2} \) and \( t_2 \cdot \alpha \) is of type (5) with \( k(1) = 3 \). By the above, \( t_2 \cdot \alpha \in E \), and we can once more conclude that \( \alpha \) is an elementary root.

It remains to show that we have listed all the roots in \( E_Y^{(t_n)} \). So let \( \alpha \in E_Y^{(s_m,t_n)} \). Since \( \alpha \succeq \tau_n \) by an earlier remark, we know that the coefficient of \( x_0 \) in \( \alpha \) is greater than 4. We first show that \( \alpha \) is of one of the types (1)-(6) if the coefficient of \( x_0 \) in \( \alpha \) equals 4, 5 or 6. So let \( \lambda_1, \mu_2, \mu_1, \nu_1, \ldots, \nu_{n-1} \geq 2 \) such that

\[
\alpha = \lambda_1 x_1 + \mu_2 y_2 + \mu_1 y_1 + 4x_0 + \nu_1 z_1 + \cdots + \nu_{n-1} z_{n-1} + z_n.
\]

Then \( 2 \leq \lambda_1 \leq 3 \), and since \( \langle \alpha, x_1 \rangle \leq \frac{1}{2} \), we deduce that \( \lambda_1 = 2 \). Further, \( 2 \leq \mu_2 < \mu_1 < 4 \), and thus \( \mu_2 = 2 \) and \( \mu_1 = 3 \).

Assume for a contradiction that \( \nu_k \geq \nu_{k+1} + 2 \) for some \( k \in \{0, \ldots, n-1\} \) (where \( \nu_0 = 4 \) and \( \nu_n = 1 \)), and let \( \beta \preceq \alpha \) be according to (6.73)(iii) such
that the coefficient of \( x_0 \) in \( \beta \) equals 4, and the coefficient of \( z_1 \) in \( \beta \) is less than or equal to 2. Denote the coefficients of \( x_1 \) and \( y_1 \) in \( \beta \) by \( \lambda' \) and \( \mu' \) respectively. Then \( \lambda' \leq \lambda_1 \) and \( \mu' \leq \mu_1 \), since \( \beta \leq \alpha \), and thus \( \langle \beta, x_0 \rangle \) is greater than or equal to

\[
4 + \left(-\frac{1}{2}\right)\mu' + \left(-\frac{1}{2}\right)\lambda' + \left(-\frac{1}{2}\right)2 \geq 4 + \left(-\frac{1}{2}\right)3 + \left(-\frac{1}{2}\right)2 + \left(-\frac{1}{2}\right)2 = \frac{1}{2};
\]

hence \( r_0 \cdot \beta < \beta \). It is clear that \( \beta \) is an elementary root, and (6.64) yields that \( \langle \beta, x_0 \rangle = \frac{1}{2} \) and that the coefficient of \( x_0 \) in \( r_0 \cdot \beta \) equals 3. But \( \langle \beta, x_0 \rangle = \frac{1}{2} \) now forces equality in the above inequality; in particular, \( \mu' = 3 \), and thus \( r_0 \cdot \beta \) has coefficient 3 for \( x_0 \) and \( y_1 \), contradicting (6.82). So \( \nu_k \in \{\nu_{k+1},\nu_{k+1}+1\} \) for all \( k \in \{0,\ldots,n-1\} \) by (6.73)(i), and it follows that \( \alpha \) is of type (1).

Next let \( \lambda_1, \mu_2, \mu_1, \nu_1, \ldots, \nu_{n-1} \geq 2 \) such that

\[
\alpha = \lambda_1 x_1 + \mu_2 y_2 + \mu_1 y_1 + 5x_0 + \nu_1 z_1 + \cdots + \nu_{n-1} z_{n-1} + z_n.
\]

Then \( 2 \leq \lambda_1 \leq 4 \), and since \( \langle \alpha, x_1 \rangle \leq 2 \) we know that \( \lambda_1 \in \{2, 3\} \). Furthermore, \( 2 \leq \mu_2 < \mu_1 < 5 \), and thus \( \mu_2 = 2 \) and \( \mu_1 \in \{3, 4\} \), or \( \mu_2 = 3 \) and \( \mu_1 = 4 \). But in the latter case \( \langle \alpha, y_2 \rangle = 3 + (-\frac{1}{2})4 = 1 \), contradicting \( \alpha \in \mathcal{E} \), and thus \( \mu_2 = 2 \) and \( \mu_1 \in \{3, 4\} \).

Assume for a contradiction that \( \nu_k \geq \nu_{k+1}+2 \) for some \( k \), and let \( \beta \preceq \alpha \) be according to (6.73)(iii) such that the coefficient of \( x_0 \) in \( \beta \) equals 5, while the coefficient of \( z_1 \) in \( \beta \) is less than or equal to 3. If the coefficient of \( x_1 \) in \( \beta \) is 3, \( \alpha \) is also preceded by \( r_1 \cdot \beta \), and so we may assume without loss of generality that the coefficient of \( x_1 \) in \( \beta \) is less than or equal to 2; similarly, if the coefficient of \( y_1 \) in \( \beta \) is 4, \( \alpha \) is also preceded by \( s_1 \cdot \beta \), and so we may further assume without loss of generality that the coefficient of \( y_1 \) in \( \beta \) is less than or equal to 3. Thus

\[
\langle \beta, x_0 \rangle \geq 5 + \left(-\frac{1}{2}\right)3 + \left(-\frac{1}{2}\right)2 + \left(-\frac{1}{2}\right)3 = 1,
\]

forcing \( \beta \in \Delta \), and thus \( \alpha \in \Delta \), a contradiction. Therefore \( \nu_k \) equals either \( \nu_{k+1} \) or \( \nu_{k+1} + 1 \) for all \( k \) by (6.73)(i), and \( \alpha \) is of type (2).

Suppose now that \( \alpha \in \mathcal{E}^{(t_n)}_Y \) has coefficient 6 for \( x_0 \), and let \( \beta \) be of maximal depth with \( \tau_n \preceq \beta \preceq \alpha \) such that the coefficient of \( x_0 \) in \( \beta \) is less than 6. It is clear that \( \beta \) is elementary, and since \( \tau_n \preceq \beta \preceq \alpha \), it follows that \( \beta \in \mathcal{E}^{(t_n)}_Y \). By maximality of \( \beta \), we deduce that \( \beta < r_0 \cdot \beta \preceq \alpha \) and that the
coefficient of \( x_0 \) in \( r_0 \cdot \beta \) equals 6; now (6.64) implies that \( \langle r_0 \cdot \beta, x_0 \rangle = \frac{1}{2} \) and that the coefficient of \( x_0 \) in \( \beta \) equals 5. Since \( \beta \in \mathcal{E}_Y(t_n) \), the above yields that \( \beta \) is of type (2); that is, \( \beta \) equals

\[
\lambda x_1 + 2y_2 + \mu y_1 + 5x_0 + 5(z_1 + \cdots + z_{k(3)-1}) + \cdots + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n,
\]

for some \( 0 < k(3) < k(2) < k(1) < n \) and \( \lambda \in \{2, 3\}, \mu \in \{3, 4\} \). As \( \langle \beta, x_0 \rangle = -\frac{1}{2} \), we are left with \((\lambda, \mu) \in \{(2, 4), (3, 3)\}\) and \(k(3) > 1\), or \((\lambda, \mu) = (3, 4)\) and \(k(3) = 1\).

If \((\lambda, \mu) \in \{(2, 4), (3, 3)\}\) and \(k(3) > 1\), then \(r_0 \cdot \beta\) equals

\[
\lambda x_1 + 2y_2 + \mu y_1 + 6x_0 + 5(z_1 + \cdots + z_{k(3)-1}) + 4(z_{k(3)} + \cdots + z_{k(2)-1}) + 3(z_{k(2)} + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n.
\]

Let \(k(4) < k(3)\) be the maximal such that \(\alpha\) is preceded by \((t_{k(4)-1} \cdots t_1)r_0 \cdot \beta\), and call this root \(\gamma\). Then

\[
\gamma = \lambda x_1 + 2y_2 + \mu y_1 + 6x_0 + 5(z_1 + \cdots + z_{k(4)-1}) + 4(z_{k(4)} + \cdots + z_{k(3)-1}) + 3(z_{k(3)} + \cdots + z_{k(2)-1}) + 2(z_{k(2)} + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n.
\]

Now assume for a contradiction that \(\alpha\) does not equal \(\gamma\), and let \(r \in Y\) such that \(\alpha \succeq r \cdot \gamma \succ \gamma\); then \(\langle \gamma, r \alpha \rangle = -\frac{1}{2}\) by (6.64). Since \(\lambda \in \{2, 3\}\) and \(\mu \in \{3, 4\}\) clearly \(r \neq r_1, s_2, s_1\), and as

\[
\langle \gamma, x_0 \rangle \geq 6 + \left(-\frac{1}{2}\right) 6 + \left(-\frac{1}{2}\right)(\lambda + \mu) \geq 6 + \left(-\frac{1}{2}\right) 6 + \left(-\frac{1}{2}\right) 6 \geq 0,
\]

furthermore \(r \neq r_0\). Now \(\langle \gamma, z_{k(4)} \rangle = 0\) if \(k(4) = k(3) - 1\) and \(r \neq t_{k(4)}\) by maximality of \(k(4)\) if \(k(4) < k(3) - 1\); hence we are left with \(r = t_{k(3)}\) and \(k(3) < k(2) - 1\), or \(r = t_{k(2)}\) and \(k(2) < k(1) - 1\), or \(r = t_{k(1)}\) and \(k(1) < n\). But then an easy calculation yields that \(\alpha\) is preceded by \(r \cdot \beta \succ \beta\), contradicting the maximality of \(\beta\). So \(\alpha = \gamma\), and this is of type (3).

Suppose next that \((\lambda, \mu) = (3, 4)\) and \(k(3) = 1\). Then \(\gamma = r_0 \cdot \beta\) equals

\[
3x_1 + 2y_2 + 4y_1 + 6x_0 + 4(z_1 + \cdots + z_{k(2)-1}) + 3(z_{k(2)} + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n.
\]
and this is of type (4). If \( \alpha \) is not equal to \( \gamma \), let \( r \in Y \) such that \( \alpha \succeq r \cdot \gamma \succ \gamma \). Then \( \langle \gamma, \alpha_r \rangle = -\langle r \cdot \gamma, \alpha_r \rangle = -\frac{1}{2} \) by (6.64), and since

\[
\langle \gamma, x_1 \rangle = \langle \gamma, y_2 \rangle = \langle \gamma, y_1 \rangle = \langle \gamma, x_0 \rangle = 0,
\]

it follows that \( r \in \{t_1, \ldots, t_n\} \), and thus \( r \in \{t_1, t_{k(2)}, t_{k(1)}\} \); maximality of \( \beta \) now only leaves us with \( r = t_1 \). Since \( \langle \gamma, z_1 \rangle = -\frac{1}{2} \), we find that \( k(2) = 2 \); hence \( \delta = t_1 \cdot \gamma \) equals

\[
3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 3(z_2 + \cdots + z_{k(1)-1}) + 2(z_{k(1)} + \cdots + z_{n-1}) + z_n,
\]

which is of type (5). If \( \alpha \neq \delta \), let \( s \in Y \) such that \( \alpha \succeq r \cdot \delta \succ \delta \). Then \( \langle \delta, \alpha_s \rangle = -\frac{1}{2} \), and since

\[
\langle \delta, x_1 \rangle = \langle \delta, y_2 \rangle = \langle \delta, y_1 \rangle = \langle \delta, x_0 \rangle = 0
\]
as well as \( \langle \delta, y_1 \rangle > 0 \), we deduce that \( s \in \{t_2, t_{k(1)}\} \); thus \( s = t_2 \) by maximality of \( \beta \). Since \( \langle \delta, z_2 \rangle = -\frac{1}{2} \), moreover \( k(1) = 3 \), and

\[
t_2 \cdot \delta = 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2(z_3 + \cdots + z_{n-1}) + z_n.
\]

As \( n \geq 5 \), it can be easily verified that there are no \( t \in Y \) such that \( \langle t_2 \cdot \delta, \alpha_t \rangle \) lies in the open interval \((-1, 0)\), and we can deduce from (3.38) that \( \alpha \) must be equal to \( t_2 \cdot \delta \); that is, \( \alpha \) is of type (6).

Now assume for a contradiction that there exists an \( \alpha \in \mathcal{E}_Y^{(t_n)} \) such that the coefficient of \( x_0 \) in \( \alpha \) is greater than or equal to 7. We may assume without loss of generality that \( \alpha \) is of minimal depth with this property. Then \( \alpha \succ \tau_n \), and hence there exists an \( r \in Y \) such that \( \alpha \succeq r \cdot \alpha \succeq \tau_n \). It is clear that \( r \cdot \alpha \in \mathcal{E}_Y^{(t_n)} \), and minimality of \( \alpha \) yields that \( r = r_0 \) and that the coefficient of \( x_0 \) in \( r \cdot \alpha \) is less than or equal to 6. By (6.64) the coefficient of \( x_0 \) in \( r_0 \cdot \alpha \) equals 6, and thus \( r \cdot \alpha \) is of type (3), (4), (5) or (6) by the above; but then \( \langle r_0 \cdot \alpha, x_0 \rangle \geq 0 \), contradicting \( \alpha \succ r_0 \cdot \alpha \), and this finishes the proof.

(6.91) Proposition. Suppose \( Y \) is of the shape described in (6.89). Then \( \mathcal{E}_Y^\emptyset = \emptyset \).

Proof. Assume for a contradiction that \( \mathcal{E}_Y^\emptyset \neq \emptyset \), and let \( \alpha \) be an element of minimal depth. By (6.75), there exists an \( r \in \{r_1, s_2, t_n\} \) such that \( r \cdot \alpha \) is in
Chapter 6  The set of elementary roots

\[ E_Y^{\{r_1\}} = E_Y^{\{s_1\}} = \emptyset \] by (6.74) and (6.84) respectively, this leaves us with \( t_n \cdot \alpha \in E_Y^{\{t_n\}} \). The elements of \( E_Y^{\{t_n\}} \) are of types (1)-(6) stated in the previous proposition, and so the coefficient of \( z_{n-1} \) in \( t_n \cdot \alpha \) equals 2, while the coefficient of \( z_n \) in \( t_n \cdot \alpha \) equals 1; whence

\[ \langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -(1 + \left(-\frac{1}{2}\right)2) = 0, \]

contradicting \( t_n \cdot \alpha \prec \alpha \).

\[ \square \]

§6b One non-simple bond

Henceforth assume that \( X, Y \subseteq R \) satisfy (6.61), and that \( Y \) contains exactly one non-simple bond of finite weight \( m \). Let \( r_1, s_1 \in Y \) be the vertices of the non-simple bond, and denote the simple roots corresponding to \( r_1, s_1 \) by \( x_1 \) and \( y_1 \) respectively. Further, let \( Y_1 \) and \( Y_2 \) be the connected components of the graph obtained from \( Y \) by deleting the non-simple bond, and assume that \( r_1 \in Y_1 \) and \( s_1 \in Y_2 \).

We denote \( 2 \cos(\pi/m) \) by \( c_m \); then (2.26) yields that the coefficient of a simple root in any element of \( \Phi_+^Y \) equals 0, 1 or \( c_m \), or is greater than or equal to 2.

Each element of \( E_Y \) is preceded by some simple root \( \alpha_r \) with \( r \in Y \). Suppose \( \alpha \in E_Y \) is preceded by \( \alpha_r \) for \( r \in Y_1 \), and let \( \beta \) be of maximal depth with \( \alpha \succeq \beta \succeq \alpha_r \), such that \( I(\beta) \subseteq Y_1 \). Since \( I(\alpha) \not\subseteq Y_1 \), maximality of \( \beta \) yields that \( \alpha \succeq s_1 \cdot \beta \succeq \beta \); then \( \langle \beta, y_1 \rangle < 0 \) and thus \( \langle \beta, y_1 \rangle \in (-1, 0) \) by (3.38). If \( \lambda \) denotes the coefficient of \( x_1 \) in \( \beta \), we find that \( \langle \beta, y_1 \rangle = (-\frac{c_m}{2})\lambda \), and hence \( 0 < \lambda < 2/c_m \leq \sqrt{2} \); thus \( \lambda = 1 \) by (2.26). If \( s \in Y_2 \setminus \{s_1\} \), the coefficient of \( \alpha_s \) in \( s_1 \cdot \beta \) equals 0, therefore (6.58) yields that

\[ \alpha \succeq s_1 \cdot x_1 = x_1 + c_m y_1 > x_1. \]

Symmetrical arguments give \( \alpha \succeq c_m x_1 + y_1 > y_1 \) if \( \alpha \) is preceded by \( \alpha_r \) with \( r \in Y_2 \).
Observe that $\mathcal{E}_Y^X$ does not depend on $R$ as long as $Y \subseteq R$, hence we may assume without loss of generality that there exists a $t \in R \setminus Y$ such that
\[
m_{tr} = m_{rt} = \begin{cases} 3 & \text{if } r = s_1, \\ 2 & \text{if } r \neq s_1, \end{cases}
\]
for $r \in Y$.

(6.93) Proposition

Suppose $R$ satisfies (6.92). Then $\phi: \beta \mapsto c_m(\beta - \alpha_t) + x_1$ defines a one-one correspondence between the set of roots in $\Phi_{Y_2 \cup \{t\}}^+$ with coefficient 1 for $\alpha_t$, and the set of roots in $\Phi_{\{r_1\} \cup Y_2}^+$ with coefficient 1 for $x_1$. Moreover, $\phi$ restricts to a one-one correspondence between the set of roots in $\mathcal{E}_{\{r_1\} \cup Y_2}$ with coefficient 1 for $\alpha_t$, and the set of roots in $\mathcal{E}_{\{r_1\} \cup Y_2}$ with coefficient 1 for $x_1$.

Proof. Observe that for $s \in Y_2$,
\[
\langle x_1 - c_m \alpha_t, \alpha_s \rangle = \langle x_1, \alpha_s \rangle - c_m \langle \alpha_t, \alpha_s \rangle
\]
\[
= \begin{cases} 0 - c_m \times 0 & \text{if } s \in Y_2 \setminus \{s_1\}, \\ - \frac{c_m}{2} - c_m\left(-\frac{1}{2}\right) & \text{if } s = s_1, \end{cases}
\]
\[
= 0.
\]
Hence for $\beta \in \Phi$ and $\gamma \in \Phi_{Y_2}$,
\[
\langle x_1 + c_m(\beta - \alpha_t), \gamma \rangle = c_m \langle \beta, \gamma \rangle; \quad (*)
\]
moreover, for $s \in Y_2$,
\[
s \cdot (x_1 + c_m(\beta - \alpha_t)) = x_1 + c_m(\beta - \alpha_t) - 2\langle x_1 + c_m(\beta - \alpha_t), \alpha_s \rangle
\]
\[
= x_1 + c_m(\beta - \alpha_t) - 2c_m(\beta, \alpha_s)
\]
\[
= x_1 + c_m(\beta - 2(\beta, \alpha_s) - \alpha_t) \quad (**) \]
\[
= x_1 + c_m(s \cdot \beta - \alpha_t).
\]

Now let $\beta \in \Phi_{Y_2 \cup \{t\}}^+$ with coefficient 1 for $\alpha_t$; then (6.59) implies that there exists a $w \in \mathcal{W}_{Y_2}$ with $l(w) = dp(\beta) - 1$ and $\beta = w \cdot \alpha_t$. The coefficient
of $x_1$ in $w \cdot x_1$ equals 1, and since $x_1 = x_1 + c_m(\alpha_t - \alpha_t)$, a straightforward
induction on $l(w)$ using (***) yields that $w \cdot x_1 = x_1 + c_m(w \cdot \alpha_t - \alpha_t)$. Hence
$\phi$ is well defined. By (6.59), every element of $\Phi^+_w \cup Y_2$ with coefficient 1 for
$x_1$ can be written as $w \cdot x_1$ for some $w \in W_{Y_2}$, so the above also shows that $\phi$
is onto. Since $\phi$ is certainly one-one, $\phi$ is an one-one correspondence between
the given sets, and by the above construction it remains to show for $w \in W_{Y_2}$
that $w \cdot x_1 \in \Delta$ if and only if $w \cdot \alpha_t \in \Delta$.

First, suppose that there exists a $\gamma \in \Phi^+ \setminus \{w \cdot x_1\}$ such that $w \cdot x_1$
dominates $\gamma$. Then $w^{-1} \cdot \gamma \in \Phi^-$, since $x_1$ is not in $\Delta$, and thus $\gamma$ is an
element of $N(w^{-1})$, which is a subset of $\Phi^+_2$. Now $\langle w \cdot x_1, \gamma \rangle \geq 1$ by (3.32),
and thus by (*)

$$\langle w \cdot \alpha_t, \gamma \rangle = \frac{1}{c_m} \langle \alpha, \gamma \rangle \geq \frac{1}{c_m} > \frac{1}{2}.$$ 

Since $I(w \cdot \alpha_t) \cup I(\gamma)$ contains only simple bonds, $\langle w \cdot \alpha_t, \gamma \rangle$ is an integer
multiple of $\frac{1}{2}$, and thus $\langle w \cdot \alpha_t, \gamma \rangle \geq 1$. So $w \cdot \alpha_t \text{ dom } \gamma$ or $\gamma \text{ dom } w \cdot \alpha_t$ by (3.32);
but $\gamma$ cannot dominate $w \cdot \alpha_t$, as $w^{-1} \cdot \gamma$ is negative, while $w^{-1} \cdot (w \cdot \alpha_t) = \alpha_t$
is positive, and thus $w \cdot \alpha_t \in \Delta$.

For the converse, suppose that there exists a $\gamma \in \Phi^+ \setminus \{w \cdot \alpha_t\}$ such that
$(w \cdot \alpha_t) \text{ dom } \gamma$. Then $\langle w \cdot x_1, \gamma \rangle = c_m \langle w \cdot \alpha_t, \gamma \rangle \geq c_m \geq 1$ by (**) and
(3.32); since $w^{-1} \cdot \gamma$ is negative and $w^{-1} \cdot (w \cdot x_1) = x_1$ is not, we deduce that
$w \cdot x_1 \in \Delta$.

(6.94) Proposition Suppose $R$ satisfies (6.92) and $Y_1 = \{r_1\}$. Then

$$\mathcal{E}^{\{r_1\}}_Y = \{x_1 + c_m(\beta - \alpha_t) \mid \beta \in \mathcal{E}_{Y_2 \cup \{t\}} \text{ has coefficient 1 for } \alpha_t\}$$

$$= \{x_1 + c_m(\beta - \alpha_t) \mid \beta \in \bigcup_{I \subseteq Y_2} \mathcal{E}^{I \cup \{t\}}_{Y_2 \cup \{t\}}\}.$$ 

This leaves us to determine $\mathcal{E}^X_Y$ for $X$ with $r_1, s_1 \notin X$; that is, we only
need to consider the set of elementary roots with coefficients greater than 1 for $x_1$ and $y_1$. 

(6.95) **Lemma** Suppose \( Y = \{r_1, s_1\} \). Then

\[
E^\emptyset_Y = \left\{ \frac{\sin((n+1)\pi/m)}{\sin(\pi/m)} x_1 + \frac{\sin(n\pi/m)}{\sin(\pi/m)} y_1 \mid n \in \{2, \ldots, \frac{m}{2} - 1\} \right\} \\
\cup \left\{ \frac{\sin(n\pi/m)}{\sin(\pi/m)} x_1 + \frac{\sin((n+1)\pi/m)}{\sin(\pi/m)} y_1 \mid n \in \{2, \ldots, \frac{m}{2} - 1\} \right\}
\]

if \( m \) is even, and if \( m \) is odd,

\[
E^\emptyset_Y = \left\{ \frac{\sin((n+1)\pi/m)}{\sin(\pi/m)} x_1 + \frac{\sin(n\pi/m)}{\sin(\pi/m)} y_1 \mid n \in \{2, \ldots, m-3\} \right\}.
\]

(6.96) **Proposition** Suppose that \( m \geq 6 \), \( |Y_2| \geq 2 \) and \( r_1 \notin X \). Then \( E^X_Y \) is empty unless \( Y_1 = \{r_1\} \) and \( X = \emptyset \). Moreover, if \( m \geq 7 \) and \( Y_1 = \{r_1\} \), then

\[
E^\emptyset_Y = \left\{ (c^2_m - 1)x_1 + c_m \beta \mid \beta \in E_{Y_2} \text{ has coefficient 1 for } y_1 \right\}
\]

\[
= \left\{ (c^2_m - 1)x_1 + c_m \beta \mid \beta \in \bigcup_{J \subseteq Y_2 \setminus \{s_1\}} E^J_{Y_2} \right\}.
\]

Suppose next that \( m = 6 \) and \( Y_1 = \{r_1\} \). We may assume that there exist \( t_1, t_2 \in R \setminus Y \) such that

\[
m_{rt_1} = m_{rt_2} = \begin{cases} 
3 & \text{if } r = s_1, \\
2 & \text{if } r \neq s_1,
\end{cases}
\]

for all \( r \in Y \). Denote the simple roots corresponding to \( t_1 \) and \( t_2 \) by \( z_1 \) and \( z_2 \) respectively. Then

\[
E^\emptyset_Y = \left\{ 2x_1 + \sqrt{3}(\beta - z_1 - z_2) \mid \beta \in E_{\{t_1, t_2\} \cup Y_2} \text{ has coeff. 1 for } z_1 \text{ and } z_2 \right\}
\]

\[
= \left\{ 2x_1 + \sqrt{3}(\beta - z_1 - z_2) \mid \beta \in \bigcup_{J \subseteq Y_2} E^{\{t_1, t_2\} \cup J}_{Y_2} \right\}.
\]

**Proof.** We show first that if \( \alpha \) is an elementary root in \( \Phi_Y \) preceded by \( x_1 + c_m y_1 \) with coefficient for \( x_1 \) greater than 1, then \( \alpha \) is a successor of \( (c^2_m - 1)x_1 + c_m y_1 \) and \( I(\alpha) \subseteq \{r_1\} \cup Y_2 \).
Chapter 6  The set of elementary roots

Let $\beta$ be of maximal depth with $\alpha \succeq \beta \succeq x_1 + c_m y_1$ such that the coefficient of $x_1$ in $\beta$ equals 1. Then $\alpha \succeq r_1 \cdot \beta \succeq \beta$ by maximality of $\beta$; hence $\langle \beta, y_1 \rangle < 0$, and thus $\langle \beta, x_1 \rangle \in (-1, 0)$ by (3.38). Since the coefficient of $x_1$ in $\beta$ equals 1, (6.57) together with (6.93) imply that the coefficient of $y_1$ in $\beta$ equals $kc_m$ for some $k \in \mathbb{N}$. Now suppose that $r \in Y_1$ is adjoined to $r_1$, and denote the coefficient of $\alpha$, in $\beta$ by $\lambda$; then $\lambda = 0$ or $\lambda \geq 1$ by (2.26). Further

$$\langle \beta, x_1 \rangle \leq 1 + \langle \alpha_r, x_1 \rangle \lambda + \left( -\frac{c_m}{2} \right) \mu \leq 1 - \frac{\lambda}{2} - k \frac{c_m^2}{2} \leq 1 - \frac{\lambda}{2} - \frac{3k}{2}$$

as $c_m \geq c_6 = \sqrt{3}$; since $\langle \beta, x_1 \rangle > -1$, we deduce that $k = 1$ and $\lambda = 0$. So the coefficient of $y_1$ in $r_1 \cdot \beta$ equals $c_m$, and $r_1$ is only adjoined to $s_1$ in $I(\beta)$; therefore $r_1 \cdot \beta \succeq (c_m^2 - 1)x_1 + c_my_1$ by (6.58), and transitivity of $\succeq$ yields that

$$\alpha \succeq (c_m^2 - 1)x_1 + c_my_1.$$ 

Note that connectedness of the support of $\beta$ implies that $I(\beta) \subseteq \{r_1\} \cup Y_2$, and thus $I(r_1 \cdot \beta) \subseteq \{r_1\} \cup Y_2$. If $r \in Y_1$ is adjoined to $r$, then

$$\langle r_1 \cdot \beta, \alpha_r \rangle = 0 + \left( -\frac{1}{2} \right) (c_m^2 - 1) \leq -1,$$

and since $\alpha \succeq r_1 \cdot \beta$ and $\alpha \in \mathcal{E}$, Lemma (3.38) yields that $\alpha_r \not\in \text{supp}(\alpha)$; by the connectedness of the support of $\alpha$ we deduce that $I(\alpha)$ is a subset of $\{r_1\} \cup Y_2$.

Now let $\alpha \in \mathcal{E}_Y^\emptyset$. Since $I(\alpha) \not\subseteq Y_1 \cup \{s_1\}$, the above yields that $\alpha$ cannot be preceded by $c_m x_1 + y_1$. So $\alpha$ is preceded by $c_m x_1 + y_1$, and thus by $(c_m^2 - 1)x_1 + c_my_1$; moreover, $I(\alpha) \subseteq \{r_1\} \cup Y_2$. Hence $\mathcal{E}_Y^\emptyset = \emptyset$ unless $Y_1 = \{r_1\}$. By (6.56), the coefficient of $\alpha_s$ in $\alpha$ is greater than or equal to $c_m$ for $s \in Y_2$, and thus $X = \emptyset$. This leaves us to determine $\mathcal{E}_{\{r_1\} \cup Y_2}^\emptyset$.

Suppose first that $m \geq 7$ and $Y_1 = \{r_1\}$. Let $\alpha$ be an element of $\mathcal{E}_Y^\emptyset$; then $\alpha \succeq (c_m^2 - 1)x_1 + c_my_1$ by the above. We now show that the coefficient of $y_1$ in $\alpha$ equals $c_m$. Let $\gamma$ be of maximal depth with $\alpha \succeq \gamma \succeq (c_m^2 - 1)x_1 + c_my_1$ such that $\gamma$ has coefficient $c_m$ for $y_1$; then $\alpha = \gamma$ or $\alpha \succeq s_1 \cdot \gamma \succ \gamma$ by maximality of $\gamma$.

Assume for a contradiction that $I(\gamma) \subseteq \{r_1, s_1\}$. It follows that $\gamma$ equals $(c_m^2 - 1)x_1 + c_my_1$, and since $Y \not\subseteq \{r_1, s_1\}$ clearly $\alpha \not\neq \gamma$; therefore

$$\alpha \succeq s_1 \cdot \gamma = (c_m^2 - 1)x_1 + c_m(c_m^2 - 2)y_1.$$
But since \( c_m(c_m^2 - 2) \geq c_7(c_7^2 - 1) \geq 2 \), we deduce that \( \langle s_1 \cdot \gamma, \alpha_\gamma \rangle \leq -1 \) for \( s \in Y_2 \) adjacent to \( s_1 \). So by (3.38), the coefficients of \( \alpha_\gamma \) in \( s_1 \cdot \gamma \) and \( \alpha \) coincide, and by connectedness of the support of \( \alpha \) we find that \( I(\alpha) \) is contained in \{\( r_1, s_1 \)\}, contradicting \( |Y_2| \geq 2 \). Thus \( I(\gamma) \not\subseteq \{r_1, s_1\} \), and by connectedness of the support of \( \gamma \) there exists an \( s \in I(\gamma) \setminus \{r_1, s_1\} \) adjacent to \( s_1 \). Denote the coefficient of \( \alpha_\gamma \) in \( \gamma \) by \( \mu \). Then \( \mu \geq c_m \) by (6.56), and thus

\[
\langle \gamma, y_1 \rangle \leq c_m + \left( -\frac{c_m}{2} \right)(c_m^2 - 1) + \left( -\frac{1}{2} \right)c_m = \frac{c_m}{2}(2 - c_m^2) \leq \frac{c_7}{2}(2 - c_7^2) \leq -1.
\]

So Lemma (3.38) implies again that the coefficients of \( y_1 \) in \( \gamma \) and \( \alpha \) coincide; that is, the coefficient of \( y_1 \) in \( \alpha \) equals \( c_m \).

Note that since the coefficient of \( y_1 \) in \( \alpha \) equals \( c_m \), and \( \alpha \) is a successor of \( (c_m^2 - 1)x_1 + c_my_1 \), we can deduce that the coefficient of \( x_1 \) in \( \alpha \) is \( c_m^2 - 1 \). Then \( r_1 \cdot \alpha \) is an element of \( \mathcal{E}_Y^{(r_1)} \), and (6.94) yields that \( r_1 \cdot \alpha = x_1 + c_m(\beta' - \alpha_t) \) for some \( \beta' \in \mathcal{E}_{Y_2 \cup \{t\}} \) with coefficient 1 for \( \alpha_t \) (where \( t \in R \setminus Y \) according to (6.92)). Since the coefficient of \( y_1 \) in \( r_1 \cdot \alpha \) equals \( c_m \), we also know that the coefficient of \( y_1 \) in \( \beta' \) has to be equal to 1, and as \( t \) is only adjoined to \( s_1 \) in \( Y \), we deduce that \( t \cdot \beta' = \beta' - \alpha_t \). It is clear that \( t \cdot \beta' \) is an element of \( \mathcal{E}_{Y_2} \) with coefficient 1 for \( y_1 \); moreover, \( r_1 \cdot \alpha = x_1 + c_m(t \cdot \beta') \), and we conclude that \( \alpha \) equals \( (c_m^2 - 1)x_1 + c_m(t \cdot \beta') \), as required.

For the converse, let \( \beta \in \mathcal{E}_{Y_2} \) with coefficient 1 for \( y_1 \). Since \( t \) is only adjoined to \( s_1 \) (and, moreover, \( s_1 \) and \( t \) are adjoined by a simple bond), we find that \( \langle \beta, \alpha_t \rangle = -\frac{1}{2} \). Now (3.37) implies that \( t \cdot \beta = \beta + \alpha_t \) is in \( \mathcal{E}_{Y_2 \cup \{t\}} \), and thus

\[
x_1 + c_m\beta = x_1 + c_m(t \cdot \beta - \alpha_t)
\]

is an elementary root by (6.94). Since \( \langle x_1 + c_m\beta, x_1 \rangle = 1 + \left( -\frac{c_m}{2} \right) \in (-1, 0) \), (3.37) yields further that

\[
r_1 \cdot (x_1 + c_m\beta) = (c_m^2 - 1)x_1 + c_m\beta
\]

is also elementary. The coefficients of this root certainly satisfy the required conditions, and thus \( (c_m^2 - 1)x_1 + c_m\beta \in \mathcal{E}_{Y_2}^\beta \); this finishes the proof for the case \( m = 7 \).

Suppose now that \( m = 6 \); then \( c_6 = \sqrt{3} \), and an easy induction yields that for each root in \( \Phi_Y \) preceded by \( x_1 \), the coefficient of \( \alpha_r \) in this root is
Chapter 6  The set of elementary roots

an integer if \( r \in Y_1 \), and an integer multiple of \( \sqrt{3} \) if \( r \in Y_2 \). Now let \( \alpha \in \mathcal{E}_Y^0 \); then \( \alpha \succeq (c_6^2 - 1)x_1 + c_6y_1 = 2x_1 + \sqrt{3}y_1 \) by the above.

Assume for a contradiction that the coefficient of \( x_1 \) in \( \alpha \) is greater than 2, and let \( \beta \) be of maximal depth with \( \alpha \succeq \beta \succeq 2x_1 + \sqrt{3}y_1 \) such that the coefficient of \( x_1 \) in \( \beta \) equals 2. Then \( \alpha \succeq r_1 \cdot \beta \succ \beta \) by maximality, and (3.38) gives \( \langle \beta, x_1 \rangle \in (-1, 0) \). But the coefficient of \( y_1 \) in \( \beta \) equals \( k\sqrt{3} \) for some \( k \in \mathbb{N} \), and \( r_1 \) is only adjacent to \( s_1 \) in \( I(\beta) \setminus \{r_1\} \), and hence

\[
\langle \beta, x_1 \rangle = 2 + k\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) = 2 - \frac{3k}{2} \begin{cases} 
\frac{1}{2} & \text{if } k = 1, \\
\leq -1 & \text{if } k \geq 2,
\end{cases}
\]

contradicting \( \langle \beta, x_1 \rangle \in (-1, 0) \). Hence the coefficient of \( x_1 \) in \( \alpha \) equals 2.

Now observe that if \( s \in Y_2 \), then

\[
\langle 2x_1 - \sqrt{3}(z_1 + z_2) + \alpha_s \rangle = 2\langle x_1, \alpha_s \rangle - \sqrt{3}\langle z_1 + z_2, \alpha_s \rangle
= \begin{cases} 
2 	imes 0 - \sqrt{3} \times 0 & \text{if } s \in Y_2 \setminus \{s_1\}, \\
-\sqrt{3} - \sqrt{3}\left(-\frac{1}{2} - \frac{1}{2}\right) & \text{if } s = s_1,
\end{cases}
= 0.
\]

Then for \( \gamma \in \Phi \) and \( \delta \in \Phi_{Y_2} \),

\[
\langle 2x_1 + \sqrt{3}(\gamma - (z_1 + z_2)) + \delta \rangle = \sqrt{3}\langle \gamma, \delta \rangle; 
\]

moreover, for \( s \in Y_2 \),

\[
s \cdot \langle 2x_1 + \sqrt{3}(\gamma - z_1 - z_2) \rangle
= 2x_1 + \sqrt{3}(\gamma - z_1 - z_2) - 2\langle 2x_1 + \sqrt{3}(\gamma - z_1 - z_2) + \alpha_s \rangle
= 2x_1 + \sqrt{3}(\gamma - z_1 - z_2) - 2\sqrt{3}\langle \gamma, \alpha_s \rangle \quad (**)
= 2x_1 + \sqrt{3}(\gamma - 2\langle \gamma, \alpha_s \rangle - z_1 - z_2)
= 2x_1 + \sqrt{3}(s \cdot \gamma - z_1 - z_2).
\]

Recall now that \( \alpha \succeq 2x_1 + \sqrt{3}y_1 \), and the coefficient of \( x_1 \) in \( \alpha \) equals 2 for \( \alpha \in \mathcal{E}_Y^0 \). Hence there exists a \( w \in W_{Y_2} \) such that \( \alpha = w \cdot (2x_1 + \sqrt{3}y_1) \).
So

\[
\alpha = w \cdot (2x_1 + \sqrt{3}((z_1 + z_2 + y_1) - z_1 - z_2)),
\]
and an easy induction on \( l(w) \) using (**) yields that \( \alpha \) equals

\[
2x_1 + \sqrt{3}(w \cdot (z_1 + z_2 + y_1) - z_1 - z_2) = 2x_1 + \sqrt{3}(wt_1t_2 \cdot y_1 - z_1 - z_2).
\]

Set \( \gamma = (wt_1t_2) \cdot y_1 \); then clearly \( I(\gamma) = \{t_1, t_2\} \cup Y_2 \). Assume for a contradiction that \( \gamma \) is not an elementary root, and let \( \delta \in \Phi^+ \setminus \{\gamma\} \) be dominated by \( \gamma \). Then \( \langle \gamma, \delta \rangle \geq 1 \) by (3.32). Further, \( w^{-1} \cdot \delta \in \Phi^- \) since \( z_1 + z_2 + y_1 \) is elementary, and hence \( \delta \in \Phi_{Y_2} \). Now (*) yields that

\[
\langle \alpha, \delta \rangle = \sqrt{3}\langle \gamma, \delta \rangle \geq \sqrt{3} \geq 1,
\]

and thus \( \alpha \) dom \( \delta \) or \( \alpha \) dom \( \alpha \) by (3.32). Since \( w^{-1} \cdot \delta \) is negative, and \( w^{-1} \cdot \alpha \) is not, this forces \( \alpha \in \Delta \), a contradiction.

Next, let \( \beta \in E_{Y_2 \cup \{t_1, t_2\}} \) with coefficient 1 for \( z_1 \) and \( z_2 \), and define

\[
\alpha = 2x_1 + \sqrt{3}(\beta - z_1 - z_2);
\]

we show first that \( \alpha \) is in fact a root. Since \( I(\beta) = Y_2 \cup \{t_1, t_2\} \) contains only simple bonds, Proposition (6.69) yields that \( \beta \) is a successor of \( z_1 + z_2 + y_1 \); hence \( \beta = w \cdot (z_1 + z_2 + y_1) \) with \( dp(\beta) - dp(z_1 + z_2 + y_1) = l(w) \) for some \( w \in W \). The coefficients of \( z_1 \) and \( z_2 \) in \( \beta \) and \( z_1 + z_2 + y_1 \) coincide, and since \( I(\beta) = Y_2 \cup \{t_1, t_2\} \) we know that \( w \in W_{Y_2} \). Now a straightforward induction on \( l(w) \) using (**) yields that

\[
(wr_1s_1) \cdot x_1 = w \cdot (2x_1 - \sqrt{3}(z_1 + z_2) + \sqrt{3}(y_1 + z_1 + z_2))
\]

\[
= 2x_1 + \sqrt{3}(w \cdot (z_1 + z_2 + y_1) - z_1 - z_2)
\]

\[
= 2x_1 + \sqrt{3}(\beta - z_1 - z_2);
\]

therefore \( \alpha \) is in fact a root.

Assume for a contradiction that \( \alpha \) dominates some \( \delta \in \Phi^+ \setminus \{\alpha\} \). Then \( \delta \in N(w^{-1}) \), since \( (r_1s_1) : x_1 \notin \Delta \), and thus \( \delta \in \Phi_{Y_2}^+ \). Hence by (*) and (3.32),

\[
\langle \beta, \delta \rangle = \frac{1}{\sqrt{3}}\langle \alpha, \delta \rangle \geq \frac{1}{\sqrt{3}}.
\]

Since \( I(\beta) \cup I(\delta) = Y_2 \cup \{t_1, t_2\} \) contains only simple bonds, this forces \( \langle \beta, \delta \rangle \geq 1 \). As \( w^{-1} \cdot \delta \) is negative while \( w^{-1} \cdot \beta = \gamma \) is positive, \( \delta \) cannot dominate \( \beta \), and thus \( \beta \in \Delta \), contrary to our choice of \( \beta \). \( \square \)
It remains to discuss the cases \( m = 4, 5 \); before we do so, consider the following consequence of the previous result for \( m = 6 \): If \( Y_2 \) equals

\[
\begin{array}{ccccccc}
s_1 & s_2 & \cdots & s_{n-1} & s_n \\
\end{array}
\]

then \( \{t_1, t_2\} \cup Y_2 \) equals

\[
\begin{array}{ccccccc}
s_1 & s_2 & \cdots & s_{n-1} & s_n \\
t_1 & & & & \\
t_2 & & & & \\
\end{array}
\]

and if we denote the simple root corresponding to \( s_j \) by \( y_j \), we can deduce from (6.77) that the set of roots in \( E_{\{t_1, t_2\} \cup Y_2} \) with coefficient 1 for \( z_1 \) and \( z_2 \) is

\[
\{ z_1 + z_2 + 2(y_1 + \ldots + y_{j-1}) + y_j + \cdots + y_n \mid j \in \{1, \ldots, n\} \};
\]

therefore

\[
E_Y^\theta = \{ 2x_1 + \sqrt{3}(2(y_1 + \ldots + y_{j-1}) + y_j + \cdots + y_n) \mid j \in \{1, \ldots, n\} \},
\]

and thus \( |E_Y^\theta| = n \).

Next, suppose that \( Y_2 \) contains a vertex of valency greater than 2, and let \( n \geq 1 \) be maximal such that there exist \( s_2, \ldots, s_n \in Y_2 \) with \( s_j \) adjacent only to \( s_{j-1} \) and \( s_{j+1} \) in \( Y \) for \( j \in \{1, \ldots, n-1\} \) (where \( s_0 = r_1 \)). Denote the simple root corresponding to \( s_j \) by \( y_j \), and let \( \alpha \) be an element of \( E_{\{t_1, t_2\} \cup Y_2} \) with coefficient 1 for \( z_1 \) and \( z_2 \). Since the support of \( \alpha \) contains at least two vertices of valency greater than or equal to 3 (if \( n > 1 \), namely \( s_1 \) and \( s_n \)) or at least one vertex of valency greater than or equal to 4 (if \( n = 1 \), namely \( s_1 \)), (6.65) implies that there exists a \( j \in \{1, \ldots, n\} \) such that the coefficient of \( y_j \) in \( \alpha \) is 1. If we choose \( j \) minimal with this property, then (6.57) yields that \( \alpha \) equals \( \beta + \gamma - y_j \) for some \( \beta \in E_{\{t_1, t_2, s_1, \ldots, s_j\}} \) with coefficient 1 for \( y_j \) and coefficient greater than or equal to 2 for \( y_1, \ldots, y_{j-1} \), and \( \gamma \in E_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} \) with coefficient 1 for \( y_j \). Since the coefficients of \( z_1 \) and \( z_2 \) in \( \alpha \) equal 1, this yields that \( \beta \in E_{\{t_1, t_2, s_1, \ldots, s_j\}} \); so by (6.77),

\[
\beta = z_1 + z_2 + 2(y_1 + \cdots + y_{j-1}) + y_j,
\]
and thus every element of $\mathcal{E}_Y^\emptyset$ can be written as $2x_1 + \sqrt{3}(2(y_1 + \cdots + y_{j-1}) + \gamma)$ for some $\gamma \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}$ with coefficient 1 for $y_j$.

Since $z_1 + z_2 + 2(y_1 + \cdots + y_{j-1}) + y_j$ is elementary for $j \in \{1, \ldots, n\}$, it also follows from (6.57) that

$$2x_1 + \sqrt{3}(2(y_1 + \cdots + y_{j-1}) + \gamma)$$

is in $\mathcal{E}_Y^\emptyset$ for all $\gamma \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}$ with coefficient 1 for $y_j$; therefore $\mathcal{E}_Y^\emptyset$ is the set

$$\bigcup_{j=1}^n \left\{2x_1 + \sqrt{3}(2(y_1 + \cdots + y_{j-1}) + \gamma) \mid \gamma \in \bigcup_{J \subseteq Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}^{\{s_j\} \cup J}\right\}. $$

Note that by (6.66), $\mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} = \{y_j + \cdots + y_n\}$ if $|Y_2| = n$; hence the above also applies for the case $|Y_2| = n$.

If $m = 4, 5$ we can use similar arguments to the ones just demonstrated for the case $m = 6$. To do so, we need to develop some more tools. We start with the following variation of (6.57), which is clearly valid for all $m$.

(6.97) PROPOSITION Let $r \in Y$ and $J_0, \ldots, J_k \subseteq Y$ such that $Y \setminus \{r\}$ is the disjoint union of $J_0, \ldots, J_k$. Suppose that no element of $J_i$ is adjoined to any element $J_j$ for $i \neq j$, and set $I_j = J_j \cup \{r\}$ for all $j \in \{0, \ldots, k\}$. Assume further that $r_1, s_1 \in I_0$. Then

$$\phi: (\beta_0, \ldots, \beta_k) \mapsto \beta_0 + c_m(\beta_1 + \cdots + \beta_k - k\alpha_r)$$

is a one-one correspondence between the set of $(k+1)$-tuples in $\Phi_1^{I_0} \times \cdots \times \Phi_1^{I_k}$ such that the coefficient of $\alpha_r$ in the first component equals $c_m$, while for all other components the coefficient of $\alpha_r$ equals 1, and the set of roots in $\Phi_1^+$ with coefficient $c_m$ for $\alpha_r$. Moreover, $\phi$ restricts to a one-one correspondence between the set of $(k+1)$-tuples in $\mathcal{E}_{I_0} \times \cdots \times \mathcal{E}_{I_k}$ such that $\alpha_r$ has coefficient $c_m$ in the first component, and 1 in the others, and the set of roots in $\mathcal{E}_1^+$ with coefficient $c_m$ for $\alpha_r$.

Proof. We show first that $\phi$ is well defined. So let $(\beta_0, \ldots, \beta_k)$ be an element of $\Phi_1^{I_0} \times \cdots \times \Phi_1^{I_k}$ such that the coefficient of $\alpha_r$ in $\beta_0$ equals $c_m$, while the coefficient of $\alpha_r$ in $\beta_j$ equals 1 for $j \in \{1, \ldots, k\}$. Lemma (6.59) implies
that $\beta_j \succeq \alpha_r$ for $j \in \{1, \ldots, k\}$, and hence there exist $w_j \in W_{I_j}$ such that $\beta_j = w_j \cdot \alpha_r$ and $l(w_j) = dp(\beta_j) - 1$. Define $\alpha$ to be $(w_1 \cdots w_k) \cdot \beta_0$; then the coefficient of $\alpha_r$ in $\alpha$ equals $c_m$, and it can be easily seen that $\alpha$ equals $\phi(\beta_0, \ldots, \beta_k)$.

As $I(\beta_i) \cap I(\beta_j) = \{r\}$ if $i \neq j$, and the coefficient of $\alpha_r$ in $\beta_i$ is $c_m$ if $i = 0$, and 1 if $i \in \{1, \ldots, k\}$, it follows that $\phi$ is one-one, and we show now that $\phi$ is onto.

Suppose that $\alpha \in \Phi^+_Y$ has coefficient $c_m$ for $\alpha_r$, and let $\gamma \preceq \alpha$ be of minimal depth such that the coefficient of $\alpha_r$ in $\alpha$ equals $c_m$. Then $I(\gamma) \subseteq Y$, and as $Y \setminus J_0$ contains only simple bonds, we deduce that $I(\gamma) \cap J_0 \neq \emptyset$. Now $r \cdot \gamma \prec \gamma$ by minimality of $\gamma$, and it follows by (2.26) that the coefficient of $\alpha_r$ in $r \cdot \gamma$ equals 0 or 1. In the first case, connectedness of the support of $r \cdot \gamma$ yields that $I(r \cdot \gamma) \subseteq J_0$, and thus $I(\gamma) \subseteq I_0$.

Assume now that the coefficient of $\alpha_r$ in $r \cdot \gamma$ equals 1, and let $K_1, \ldots, K_n$ be the connected components of $I(r \cdot \gamma) \setminus \{r\}$. Assume for a contradiction that $n \geq 2$. By (6.57), there exist roots $\gamma_1, \ldots, \gamma_n$ with $I(\gamma_i) \subseteq K_i$, such that $r \cdot \gamma = \gamma_0 + \cdots + \gamma_n - (n - 1)\alpha_r$. For $i \in \{1, \ldots, n\}$, let $t_i \in K_i$ be adjoined to $r$, and denote the simple root corresponding to $t_i$ by $z_i$, and the coefficient of $z_i$ in $\gamma_i$ by $\nu_i$; then $\nu_i \geq 1$ by (2.26), as $t_i \in I(r \cdot \gamma)$. The coefficient of $\alpha_r$ in $r \cdot \gamma$ is

$$1 = c_m - 2\langle \gamma, \alpha_r \rangle = -c_m - 2\langle z_1, \alpha_r \rangle \nu_1 - \cdots - 2\langle z_k, \alpha_r \rangle \nu_k,$$

and we find that $1 \geq -c_m - (\nu_1 + \cdots + \nu_n) = n - c_m$. This forces $n \leq 2$, and thus $n = 2$ by our assumption. Further, $\nu_1 + \nu_2 \leq 1 + c_m$, and by symmetry of $K_1$ and $K_2$ we may assume without loss of generality that $\nu_1 \leq \nu_2$. Since $\nu_1$ equals 1, or is greater than or equal to $c_m$ by (2.26), we deduce that $\nu_1 = 1$ and $\nu_2 \in \{1, c_m\}$. Since the coefficients of both $z_1$ and $\alpha_r$ in $r \cdot \gamma$ equal 1, we deduce from (6.56) that $\langle \alpha_r, z_2 \rangle = -\frac{1}{2}$. So the coefficient of $\alpha_r$ in $\gamma$ equals

$$c_m = 1 - 2(1 + \left(-\frac{1}{2}\right)1 + \langle \alpha_r, z_2 \rangle \nu_2) = -2\langle \alpha_r, z_2 \rangle \nu_2.$$

If $\nu_2 = 1$, this forces $\langle \alpha_r, z_2 \rangle = -\frac{c_m}{2}$; but then the coefficients of $z_2$ and $\alpha_r$ in $r \cdot \gamma$ cannot both be 1 by (6.56) together with (2.26), contrary to our construction. If $\nu_2 = c_m$, the above yields that $\langle \alpha, z_2 \rangle = -\frac{1}{2}$; but then the coefficient of $\alpha_r$ in $r \cdot \gamma$ equals $c_m - 1 \in (0, 1)$, and this contradicts (2.26). So $I(r \cdot \gamma)$ has only one connected component, and since $I(r \cdot \gamma) \cap J_0 \neq \emptyset$, we deduce that $I(r \cdot \gamma) \subseteq I_0$, and hence $I(\gamma) \subseteq I_0$. 

Chapter 6: The set of elementary roots
Since $\gamma$ has coefficient $c_m$ for $\alpha_r$, there exists a $w \in W_{Y \setminus \{r\}}$ such that $\alpha = w \cdot \gamma$ and $\text{dp}(\alpha) - \text{dp}(\gamma) = l(w)$. As $W_{Y \setminus \{r\}}$ is the direct product of $W_{J_0}, \ldots, W_{J_k}$, there exist $w_j \in W_{I_j}$ for all $j \in \{0, \ldots, k\}$ such that $w = w_0 \cdots w_k$ with length adding. Define $\beta = w_0 \cdot \gamma$; this is an element of $\Phi^+_I$ with coefficient $c_m$ for $\alpha_r$, and clearly

$$\alpha = (w_1 \cdots w_k) \cdot \beta = \phi(\beta, w_1 \cdot \alpha_r, \ldots, w_k \cdot \alpha_r).$$

The above proves that $\phi$ is onto, and it remains to show for $\beta \in \Phi^+_I$ and $w_1 \in W_{I_1}, \ldots, w_k \in W_{I_k}$ that $(w_1 \cdots w_k) \cdot \beta \in \Delta$ if and only if $\beta \in \Delta$ or $w_j \cdot \alpha_r \in \Delta$ for some $j \in \{1, \ldots, k\}$.

Set $w = w_1 \cdots w_k$, and note that for $\delta \in \Phi^+_I$, clearly $\langle \delta, \alpha_s \rangle = 0$ for all $s \in Y \setminus I_1$; therefore $\langle \beta, \delta \rangle = c_m \langle \alpha_r, \delta \rangle$ and $\langle w_i \cdot \alpha_r, \delta \rangle = \langle \alpha_r, \delta \rangle$ for all $i$ in $\{2, \ldots, k\}$. This implies that

$$\langle w \cdot \beta, \delta \rangle = \langle \beta, \delta \rangle + c_m \langle w_1 \cdot \alpha_r, \delta \rangle + c_m \sum_{i=2}^{k} \langle w_i \cdot \alpha_r, \delta \rangle - k c_m \langle \alpha_r, \delta \rangle$$

$$= c_m \langle \alpha_r, \delta \rangle + c_m \langle w_1 \cdot \alpha_r, \delta \rangle + c_m \sum_{i=2}^{k} \langle \alpha_r, \delta \rangle - k c_m \langle \alpha_r, \delta \rangle \quad (*)$$

$$= c_m \langle w_1 \cdot \alpha_r, \delta \rangle.$$

Suppose now that $w \cdot \beta \in \Delta$, and let $\delta \in \Phi^+ \setminus \{w \cdot \beta\}$ such that $w \cdot \beta \text{ dom } \delta$. If $w^{-1} \cdot \delta$ is positive, it follows that $\beta \text{ dom } (w^{-1} \cdot \delta)$, and since clearly $\beta \neq w^{-1} \cdot \delta$, we find that $\beta \in \Delta$. Assume next that $\delta \in N(w^{-1})$. An easy calculation yields that

$$N(w^{-1}) = N((w_1 \cdots w_k)^{-1}) = N(w_1^{-1}) \cup \cdots \cup N(w_k^{-1}),$$

and by symmetry we may assume without loss of generality that $\delta \in N(w_1^{-1})$; then in particular, $I(\delta) \subseteq J_1$. Now (3.32) implies that $\langle w \cdot \beta, \delta \rangle \geq 1$, and thus

$$\langle w_1 \cdot \alpha_r, \delta \rangle \geq \frac{1}{c_m} > \frac{1}{2}$$

by $(*)$; since $I(w_1 \cdot \alpha_r) \cup I(\delta)$ contains only simple bonds, $\langle w_1 \cdot \alpha_r, \delta \rangle$ is an integer multiple of $\frac{1}{2}$, and hence $\langle w_1 \cdot \alpha_r, \delta \rangle \geq 1$. So $w_1 \cdot \alpha_r \text{ dom } \delta$ or
Chapter 6  The set of elementary roots

\(\delta\) is negative, and \(w_1 \cdot \alpha_r\) is not, and so \(\delta\) cannot dominate \(w_1 \cdot \alpha_r\); this forces \(w_1 \cdot \alpha_r \in \Delta\), as required.

For the converse, suppose first that \(\beta \in \Delta\). Since \(N(w) \subseteq \Phi_{Y \setminus I_0}^+\), while \(\beta \in \Phi_{I_0}^+\), it follows easily that \(N_+(w, \beta) = \emptyset\); so \(\alpha \geq \beta\) by (3.34), and thus \(\alpha \in \Delta\) by (3.36).

Now assume that \(w_1 \cdot \alpha_r \in \Delta\), and let \(\delta \in \Phi_+ \setminus \{w_1 \cdot \alpha_r\}\) such that \(w_1 \cdot \alpha_r \text{ dom } \delta\). Then \(w_1^{-1} \cdot \delta \in \Phi^-\) since \(\alpha_r \notin \Delta\), and thus \(\delta \in \Phi_{J_1}^+\). Further, \(\langle w_1 \cdot \alpha_r, \delta \rangle \geq 1\) as \(w_1 \cdot \alpha_r \text{ dom } \delta\), and (\#) yields that

\[
\langle w \cdot \beta, \delta \rangle = c_m \langle w_1 \cdot \alpha_r, \delta \rangle \geq c_m \geq 1.
\]

Since \(w_1^{-1} \cdot \delta\) is negative and \(w^{-1} \cdot (w \cdot \beta)\) is not, (3.32) this yields that \(w \cdot \beta \in \Delta\), as required. Symmetrical arguments apply if \(w \cdot \beta \in \Delta\) for \(j \in \{2, \ldots, k\}\), and this finishes the proof.

(6.98) Lemma Suppose \(Y\) contains the following subgraph

\[\text{and denote the simple roots corresponding to } s_j, t_k \text{ by } x, y_j \text{ and } z_k \text{ respectively. Let } \alpha \text{ be a root in } \Phi_Y^+ \text{ with coefficient for } x_1 \text{ greater than 1, coefficients for } y_1, \ldots, y_n \text{ greater than or equal to 2 and } z_1, z_2 \in \text{supp}(\alpha). \text{ Then } \alpha \in \Delta.\]

Proof. Let \(\beta \preceq \alpha\) be a positive root of minimal depth such that \(z_1\) and \(z_2\) are in the support of \(\beta\), the coefficient of \(x_1\) in \(\beta\) is greater than 1, and the coefficients of \(y_1, \ldots, y_n\) in \(\beta\) are greater than or equal to 2. By (3.36) it suffices to show that \(\beta\) is in \(\Delta\). Let \(s \in R\) such that \(s \cdot \beta \prec \beta\); we show that \(\langle s \cdot \beta, \alpha_s \rangle \leq -1\), which then implies \(\langle \beta, \alpha_s \rangle \geq 1\), and thus \(\beta \in \Delta\) by (3.32); (since \(\beta\) is clearly of depth greater than \(dp(\alpha_s) = 1\)).

Denote the coefficients of \(x_1, y_j, z_k\) in \(\beta\) by \(\lambda, \mu_j\) and \(\nu_k\) respectively. By minimality of \(\beta\) it follows that \(s\) equals \(r_1, s_j\) or \(t_k\). If \(s = r_1\), minimality of \(\beta\) also implies that the coefficient of \(x_1\) in \(r \cdot \beta\) is less than or equal to 1, and thus equals 0 or 1 by (2.26). If the coefficient of \(x_1\) in \(r \cdot \beta\) equals 0, then \(\langle r_1 \cdot \beta, x_1 \rangle \leq 0 + (-\frac{\mu_1}{2}) \mu_1 \leq -1\) since \(\mu_1 \geq 2\), as required. Suppose next that
the coefficient of \( x_1 \) in \( r_1 \cdot \alpha \) equals 1. Then \( \mu_1 \) is an integer multiple of \( c_m \) by (6.57) together with (6.93), and thus \( \mu_1 \geq 2c_m \); hence again

\[
\langle r_1 \cdot \beta, x_1 \rangle \leq 1 + \left( -\frac{c_m}{2} \right) \mu_1 \leq 1 - c_m^2 \leq 1 - c_4^2 = 1 - \sqrt{2}^2 = -1.
\]

Assume now that \( s = s_j \) for some \( j \in \{1, \ldots, n\} \), and denote the coefficient of \( y_j \) in \( t_j \cdot \alpha \) by \( \mu_j' \). By minimality of \( \beta \) clearly \( \mu_j' < 2 \), and by the connectedness of the support of \( t_j \cdot \beta \) further \( \mu_j' > 0 \); thus \( \mu_j' \in \{1, c_m\} \) by (2.26). Note that \( \mu_{j-1} \geq 2 \) if \( j > 1 \), and \( \lambda \geq c_m \) if \( j = 1 \). Further, \( \mu_{j+1} \geq 2 \) if \( j < n \), and \( \nu_1, \nu_2 \geq 1 \) if \( j = n \). So if \( \mu_j' = 1 \), then

\[
\langle s_j \cdot \beta, y_j \rangle \leq \begin{cases} 
1 + (-\frac{c_m}{2})c_m + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } 1 = j = n, \\
1 + (-\frac{c_m}{2})c_m + (-\frac{1}{2})2 & \text{if } 1 = j < n, \\
1 + (-\frac{1}{2})2 + (-\frac{1}{2})2 & \text{if } 1 < j < n, \\
1 + (-\frac{1}{2})2 + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } 1 < j = n,
\end{cases}
\]

and thus \( \langle s_j \cdot \beta, y_j \rangle \leq -1 \) in any case, as required. Assume now that \( \mu_j' = c_m \). If \( j < n \), then \( \mu_{j+1} \) is an integer multiple of \( c_m \) by (6.97) for \( r = s_j \), and since \( \mu_{j+1} \geq 2 \) we know that \( \mu_{j+1} \geq 2c_m \); if \( j = n \), then \( \nu_1, \nu_2 \geq c_m \) by (6.97) for \( r = s_j \). Therefore

\[
\langle s_j \cdot \beta, y_j \rangle \leq \begin{cases} 
c_m + (-\frac{c_m}{2})c_m + (-\frac{1}{2})c_m + (-\frac{1}{2})c_m & \text{if } 1 = j = n, \\
c_m + (-\frac{c_m}{2})c_m + (-\frac{1}{2})2c_m & \text{if } 1 = j < n, \\
c_m + (-\frac{1}{2})2 + (-\frac{1}{2})2c_m & \text{if } 1 < j < n, \\
c_m + (-\frac{1}{2})2 + (-\frac{1}{2})c_m + (-\frac{1}{2})c_m & \text{if } 1 < j = n,
\end{cases}
\]

and thus \( \langle s_j \cdot \beta, y_j \rangle \leq -1 \), as required.

If \( s = t_1 \), then \( z_1 \notin \text{supp}(t_1 \cdot \beta) \) by minimality of \( \beta \), and

\[
\langle t_1 \cdot \beta, z_1 \rangle \leq 0 + (-\frac{1}{2})\mu_n \leq -1
\]

since \( \mu_n \geq 2 \); symmetrical arguments apply if \( s \) equals \( t_2 \), and this finishes the proof.
Chapter 6  The set of elementary roots 108

Now let \( l \geq 1 \) be maximal such that there exist \( r_2, \ldots, r_l \in Y \) with \( r_i \) adjacent only to \( r_{i-1} \) and \( r_{i+1} \) in \( Y \) for \( i \in \{1, \ldots, l-1\} \) (where \( r_0 = s_1 \)), and denote \( \{r_1, \ldots, r_l\} \) by \( Y'_1 \). Then either \( Y_1 = Y'_1 \), or \( r_l \) has valency greater than or equal to 3. Similarly, let \( n \geq 1 \) be maximal such that there exist \( s_2, \ldots, s_n \in Y_2 \) with \( s_j \) adjacent only to \( s_{j-1} \) and \( s_{j+1} \) in \( Y \) for \( j = 1, \ldots, n-1 \) (where \( s_0 = r_1 \)), and define \( Y'_2 = \{s_1, \ldots, s_n\} \) and \( Y' = Y'_1 \cup Y'_2 \); then \( Y' \) equals

\[
\bullet \quad \bullet \quad \cdots \quad \bullet \quad m \quad \bullet \quad \cdots \quad \bullet \quad r_1 \quad r_{l-1} \quad r_2 \quad r_1 \quad s_1 \quad s_2 \quad \cdots \quad s_{n-1} \quad s_n
\]

We denote the simple roots corresponding to \( r_i, s_j \) by \( x_i \) and \( y_j \) respectively.

The following result enables us to restrict our main focus to the case \( |Y_1| = l \) and \( |Y_2| = n \).

(6.99) Lemma  Suppose \( r_1 \notin X \) and \( |Y_2| > n \) (that is, \( Y_2 \) contains a vertex of valency greater than 2). Then \( \mathcal{E}_Y^X \) is empty unless \( X \subseteq Y_1 \). Moreover, if \( X \subseteq Y_1 \), then \( \mathcal{E}_Y^X \) is the set of

\[
\alpha + c_5 \beta - c_m y_j
\]

with \( j \in \{1, \ldots, n\}, \alpha \in \mathcal{E}_{Y_1 \cup \{s_1, \ldots, s_{j-1}\}} \) with coefficient \( c_m \) for \( y_j \) and coefficient greater than or equal to 2 for \( y_1, \ldots, y_{j-1} \), and \( \beta \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} \) with coefficient 1 for \( y_j \).

Proof. Let \( \gamma \in \mathcal{E}_Y^X \); then (6.98) yields that the coefficients of \( y_1, \ldots, y_n \) in \( \gamma \) cannot all be greater than or equal to 2, and since \( X \) does not contain any of the \( s_j \) by (6.61), there must exist a \( j \in \{1, \ldots, n\} \) such that the coefficient of \( y_j \) in \( \gamma \) equals \( c_m \). If we choose \( j \) minimal with this property, (6.97) yields that

\[
\gamma = \alpha + c_m \beta - c_m y_j
\]

for some \( \alpha \in \mathcal{E}_{Y_1 \cup \{s_1, \ldots, s_{j-1}\}} \) with coefficient \( c_m \) for \( y_j \) and coefficient greater than or equal to 2 for \( y_1, \ldots, y_{j-1} \), and \( \beta \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} \) with coefficient 1 for \( y_j \). Since for \( s \in Y_2 \setminus \{s_1, \ldots, s_n\} \) the coefficients of \( \alpha_s \) in \( \gamma \) are integer multiples of \( c_m \), it follows that \( X \subseteq Y_1 \). Moreover, it is clear that for \( r \in Y_1 \setminus \{s_1, \ldots, s_{j-1}\} \), the coefficient of \( \alpha_r \) in \( \alpha \) equals 1 if and only if \( r \in X \); that is, \( \alpha \in \mathcal{E}_{Y_1 \cup \{s_1, \ldots, s_{j-1}\}} \).
By (6.97), each root of the form described in the assertion is in $\mathcal{E}_Y^X$ if $X \subseteq Y_1$, and this finishes the proof.

If $|Y_1| > l$ and $s_1 \notin X$, then symmetrically $\mathcal{E}_Y^X = \emptyset$ unless $X \subseteq Y_2$, and thus $X = \emptyset$; if $|Y_1| = l$ and $X \subseteq Y_1$, then $X \subseteq \{r_l\}$ since each element of $X$ is adjacent to exactly one element of $Y \setminus X$.

(6.100) Proposition Suppose $r_1, s_1 \notin X$. Then $\mathcal{E}_Y^X$ is empty unless $X$ is empty, or $|Y_1| = l$ and $X = \{r_l\}$, or $|Y_2| = n$ and $\{s_n\}$.

So from now on we only need to determine $\mathcal{E}_Y^\emptyset, \mathcal{E}_Y^{\{r_l\}}$ (for $|Y_1| = l$), and $\mathcal{E}_Y^{\{s_n\}}$ (for $|Y_2| = n$); by symmetry it suffices to investigate only one of the latter two.

Suppose now that $m = 4$. An easy induction shows for $\alpha \in \Phi^+_Y$ that the coefficient of $\alpha_r$ in $\alpha$ is an integer for all $r \in Y_1$, and an integer multiple of $\sqrt{2}$ for all $r \in Y_2$, or vice versa. This together with (6.99) imply:

(6.101) Proposition Suppose that $m = 4$. Then $\mathcal{E}_Y^X$ is empty, unless $|Y_1| = l$ and $X \subseteq Y_1$, or $|Y_2| = n$ and $X \subseteq Y_2$.

If $m = 4$, we assume from now on that $|Y_1| = l$ and $X \subseteq \{r_l\}$. We first determine $\mathcal{E}_Y^X$ for $|Y_2| = n$, and then cope with the case $|Y_2| > n$ using (6.99).

It is clear, that $\mathcal{E}_Y$ is independent of $R$ (as long as $Y \subseteq R$), and so we may assume without loss of generality that $R$ contains $\widehat{Y}_2 = \{\widetilde{s}_1, \ldots, \widetilde{s}_n\}$ and $\overline{Y}_2 = \{\overline{s}_1, \ldots, \overline{s}_n\}$ such that $Y_a = Y_1' \cup \widehat{Y}_2 \cup \overline{Y}_2$ equals.
Denote the simple roots corresponding to $\tilde{s}_j$, $\bar{s}_j$ by $\tilde{y}_j$ and $\bar{y}_j$ respectively, and let $V_Y$, be the subspace of $V$ spanned by $x_1, \ldots, x_l, y_1, \ldots, y_n$, and $V_a$ be the space spanned by $x_1, \ldots, x_l, \tilde{y}_1 + \bar{y}_1, \ldots, \tilde{y}_n + \bar{y}_n$. Further, define $\phi: V_Y \to V_a$ by

$$
\phi\left(\sum_{i=1}^{l} \lambda_i x_i + \sum_{j=1}^{n} \mu_j y_j\right) = \sum_{i=1}^{l} \lambda_i x_i + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j (\tilde{y}_j + \bar{y}_j).
$$

If $v = \sum_{i=1}^{l} \lambda_i x_i + \sum_{j=1}^{n} \mu_j y_j$ for some $\lambda_i, \mu_j \in \mathbb{R}$, then

$$
\langle \phi(v), x_1 \rangle = \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j (\langle \tilde{y}_j, x_1 \rangle + \langle \bar{y}_j, x_1 \rangle)
$$

$$
= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \mu_1 \frac{1}{\sqrt{2}} (\langle \tilde{y}_1, x_1 \rangle + \langle \bar{y}_1, x_1 \rangle)
$$

$$
= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \mu_1 \left( -\frac{1}{\sqrt{2}} \right)
$$

$$
= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \mu_1 \langle y_1, x_1 \rangle
$$

$$
= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \sum_{j=1}^{n} \mu_j \langle y_j, x_1 \rangle = \langle v, x_1 \rangle,
$$

where $\lambda_i, \mu_j \in \mathbb{R}$. 

Chapter 6: The set of elementary roots

Page 110
while for \( i \geq 2 \) clearly \( \langle \phi(v), x_i \rangle = \langle v, x_i \rangle \). It follows that \( \phi(r_i \cdot v) = r_i \cdot \phi(v) \) for all \( i \in \{1, \ldots, l\} \). Furthermore,

\[
\langle \phi(v), \tilde{y}_1 \rangle = \sum_{i=1}^{l} \lambda_i \langle x_i, \tilde{y}_1 \rangle + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j (\langle \tilde{y}_j, \tilde{y}_1 \rangle + \langle \tilde{y}_j, \tilde{y}_1 \rangle)
\]

\[
= \lambda_1 \langle x_1, \tilde{y}_1 \rangle + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j \langle \tilde{y}_j, \tilde{y}_1 \rangle
\]

\[
= \lambda_1 \left( -\frac{1}{2} \right) + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j \langle y_j, y_1 \rangle
\]

\[
= \frac{1}{\sqrt{2}} (\lambda_1 \langle x_1, y_1 \rangle + \sum_{j=1}^{n} \mu_j \langle y_j, y_1 \rangle) = \frac{1}{\sqrt{2}} \langle v, y_1 \rangle
\]

and symmetrically \( \langle \phi(v), \tilde{y}_1 \rangle = \langle v, y_1 \rangle / \sqrt{2} \); also, for \( j \in \{2, \ldots, n\} \) clearly

\[
\langle \phi(v), \tilde{y}_j \rangle = \langle \phi(v), \tilde{y}_j \rangle = \frac{1}{\sqrt{2}} \langle v, y_j \rangle.
\]

A straightforward calculation now yields that \( \phi(s_j \cdot v) = (\tilde{s}_j \tilde{y}_j) \cdot \phi(v) \) for \( j \in \{1, \ldots, n\} \).

(6.102) **Proposition** Let \( m = 4 \). Then \( \phi \) defines a one-one correspondence between the set of roots in \( \mathcal{E}_{Y'} = \mathcal{E}_{r_1, s_1, \ldots, s_n} \) preceded by \( x_1 \), and \( \mathcal{E}_{Y_a \cap V_a} \), the set of roots in \( \mathcal{E}_{Y_1' \cup Y_2'} \) with coinciding coefficients for \( y_j \) and \( \tilde{y}_j \) for all \( j \in \{1, \ldots, n\} \).

**Proof.** Let \( \alpha \in \Phi_{V'} \) be an elementary root preceded by \( x_1 \); then clearly \( \phi(\alpha) \in V_a \), and we show now that \( \phi(\alpha) \) is an elementary root. If \( \alpha \) has depth 1, then \( \alpha = x_1 \) and \( \phi(\alpha) = x_1 \in \mathcal{E} \). Suppose next that \( \alpha \) is of depth greater than 1, and assume that \( \phi(\beta) \) is an elementary root for all \( \beta \) with \( x_1 \leq \beta < \alpha \). Let \( r \in Y \) such that \( x_1 \leq r \cdot \alpha < \alpha \); then \( \phi(r \cdot \alpha) \) is an elementary root by induction. Further, \( \langle \alpha, \alpha_r \rangle > 0 \), and thus \( \langle \alpha, \alpha_r \rangle \in (0, 1) \), since \( \alpha \) cannot dominate \( \alpha_r \); that is, \( \langle r \cdot \alpha, \alpha_r \rangle \in (-1, 0) \).

If \( r = r_i \) for some \( i \), then \( \phi(\alpha) = r_i \cdot \phi(r_i \cdot \alpha) \) by the above, and since \( \langle \phi(r_i \cdot \alpha), x_i \rangle = \langle r_i \cdot \alpha, x_i \rangle \in (-1, 0) \), it follows by (3.37) that \( \phi(\alpha) \) is an elementary root.
Assume next that \( r = s_j \) for some \( j \). Then \( \phi(\alpha) = (\bar{s}_j \bar{\gamma}_j) \cdot \phi(s_j \cdot \alpha) \) by the above. Since \( \langle \phi(s_j \cdot \alpha), \bar{\gamma}_j \rangle = \langle s_j \cdot \alpha, y_j \rangle / \sqrt{2} \in (-1,0) \), it follows by (3.37) that \( \bar{\gamma}_j \cdot \phi(s_j \cdot \alpha) \) is elementary; furthermore,

\[
\langle \bar{\gamma}_j \cdot \phi(s_j \cdot \alpha), \bar{y}_j \rangle = \langle \phi(s_j \cdot \alpha), \bar{y}_j \rangle = \frac{1}{\sqrt{2}} \langle s_j \cdot \alpha, y_j \rangle \in (-1,0)
\]

and, again by (3.37), we deduce that \( (s_j \bar{\gamma}_j) \cdot \phi(s_j \cdot \alpha) \) is elementary; therefore \( \phi(\alpha) \) is elementary.

Note that this yields that if \( \alpha \in \mathcal{E}_Y \) is preceded by \( x_1 \), then \( \phi(\alpha) \) is an element of \( \mathcal{E}_Y \cap V_a \).

Now let \( \beta \) be an elementary root in \( V_a \) preceded by \( x_1 \). If \( \beta \) is of depth 1, then \( \beta = x_1 = \phi(x_1) \). Suppose next that \( \text{dp}(\beta) > 1 \), and assume that every root \( \gamma \) in \( V_a \) with \( x_1 \prec \gamma \prec \beta \) equals \( \phi(\delta) \) for some elementary root \( \delta \) in \( \Phi_Y \), with \( \delta \succeq x_1 \). Further, let \( t \in Y_a = Y'_a \cup \bar{Y}_2 \cup \bar{Y}_2' \) with \( x_1 \leq t \cdot \beta \prec \beta \). Since \( I(\beta) \subseteq Y_a \) contains only simple bonds, (6.64) implies that \( \langle \beta, \alpha_i \rangle = \frac{1}{2} \).

Suppose first that \( t = r_i \) for some \( i \). Then \( r_i \cdot \beta \in V_a \), and induction yields that \( r_i \cdot \beta \) equals \( \phi(\alpha) \) for some elementary root \( \alpha \in \Phi_Y \), with \( \alpha \succeq x_1 \). Now \( \beta = r_i \cdot \phi(\alpha) = \phi(r_i \cdot \alpha) \); moreover,

\[
\langle \alpha, x_i \rangle = \langle \phi(\alpha), x_i \rangle = \langle r_i \cdot \beta, x_i \rangle = -\langle \beta, x_i \rangle = \frac{1}{2},
\]

and thus \( r_i \cdot \alpha \) is elementary by (3.37), and \( r_i \cdot \alpha \succeq \alpha \succeq x_1 \), as required.

Assume next that \( t = \bar{s}_j \) for some \( j \). By symmetry of \( \beta \), and since \( \bar{s}_j \) and \( \bar{\gamma}_j \) are not adjoined, we know that

\[
\langle \bar{\gamma}_j \cdot \beta, \bar{y}_j \rangle = \langle \beta, \bar{y}_j \rangle = \langle \beta, \bar{\gamma}_j \rangle;
\]

therefore \( (\bar{s}_j \bar{\gamma}_j) \cdot \beta \prec \beta \) since \( \langle \beta, \bar{y}_j \rangle = \frac{1}{2} \). Furthermore, if we denote the coefficient of \( \bar{y}_j \) in \( \beta \) by \( \mu \), the coefficient of \( \bar{\gamma}_j \) in \( (\bar{s}_j \bar{\gamma}_j) \cdot \beta \) equals \( \mu - 2\langle \beta, \bar{\gamma}_j \rangle \), while the coefficient of \( \bar{y}_j \) in \( (\bar{s}_j \bar{\gamma}_j) \cdot \beta \) equals

\[
\mu - 2\langle \bar{s}_j \beta, \bar{y}_j \rangle = \mu - 2\langle \beta, \bar{\gamma}_j \rangle,
\]

and thus \( (\bar{s}_j \bar{\gamma}_j) \cdot \beta \in V_a \). By induction, \( (\bar{s}_j \bar{\gamma}_j) \cdot \beta \) equals \( \phi(\alpha) \) for some elementary root \( \alpha \in \Phi_Y \), preceded by \( x_1 \). So \( \beta = \phi(s_j \cdot \alpha) \), and since

\[
\langle \alpha, y_j \rangle = \sqrt{2} \langle \phi(\alpha), \bar{y}_j \rangle = \sqrt{2} \langle \bar{s}_j \bar{\gamma}_j \cdot \beta, \bar{y}_j \rangle = -\sqrt{2} \langle \bar{\gamma}_j \cdot \beta, \bar{y}_j \rangle = -\frac{1}{\sqrt{2}},
\]
$s_j \cdot \alpha$ is elementary by (3.37); the above also yields that $s_j \cdot \alpha \succeq \alpha \succeq x_1$, as required. Symmetrical arguments apply if $t \in \mathcal{Y}_2$.

We can now deduce that $\phi$ maps the set of roots in $\mathcal{E}_{\mathcal{Y}'}$ preceded by $x_1$ onto the set of roots in $\mathcal{E}_{\mathcal{Y}_a} \cap V_a$. For, if $\beta$ is in $\mathcal{E}_{\mathcal{Y}_a} \cap V_a$, then $I(\beta) = Y_a$ and since $I(\beta)$ contains only simple bonds, it follows by (6.69) that $\beta$ is preceded by $x_1$. Since $\phi$ is clearly one-one this finishes the proof.

Now let $\alpha \in \mathcal{E}_{\mathcal{Y}_a}$ with coefficient 1 for $x_i$. An easy modification of (6.73)(i) yields that the coefficients of $x_{i+1}$ in $\alpha$ must also be equal to 1, and we deduce that

$$\mathcal{E}_{\mathcal{Y}_a} = \bigcup_{i=1}^{l+1} \bigcup_{j=1}^{n+1} \mathcal{E}_{\mathcal{Y}_a}^{\{r_i, \ldots, r_l\} \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\} \cup \{\tilde{s}_{k+1}, \ldots, \tilde{s}_n\}}.$$ 

Since $\mathcal{E}_{\mathcal{Y}_a}^{\{r_i, \ldots, r_l\} \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\} \cup \{\tilde{s}_{k+1}, \ldots, \tilde{s}_n\}} \cap V_a$ is clearly empty if $j \neq k$, this becomes

$$\mathcal{E}_{\mathcal{Y}_a} \cap V_a = \bigcup_{i=1}^{l+1} \bigcup_{j=1}^{n+1} (\mathcal{E}_{\mathcal{Y}_a}^{\{r_i, \ldots, r_l\} \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\} \cup \{\tilde{s}_{j+1}, \ldots, \tilde{s}_n\}} \cap V_a).$$

If $X \subseteq Y_1$, we derive from the definition of $\phi$ that $\phi$ induces a one-one correspondence between the set of roots in $\mathcal{E}_{\mathcal{Y}}^{X}$ preceded by $x_1$, and

$$\bigcup_{j=1}^{n+1} (\mathcal{E}_{\mathcal{Y}_a}^{X \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\} \cup \{\tilde{s}_{j+1}, \ldots, \tilde{s}_n\}} \cap V_a).$$

(6.103) Proposition Suppose that $m = 4$, $|Y_1| = l$ and $|Y_2| = n$. If $l = 2$, the elements of $\mathcal{E}_{\mathcal{Y}}^{\{r_2\}}$ are

$$x_2 + Mx_1 + \sqrt{2}(M(y_1 + \cdots + y_{j}(M-1)-1)$$

$$+ (M-1)(y_{j}(M-1) + \cdots + y_{j}(M-2)-1) + \cdots$$

$$\cdots + 2(y_{j}(2) + \cdots + y_{j}(1)-1) + (y_{j}(1) + \cdots + y_n))$$

with $M \in \{2, \ldots, n+1\}$ and $0 < j(M-1) < j(M-2) < \ldots < j(1) < n+1$. If $l \geq 3$, the elements of $\mathcal{E}_{\mathcal{Y}}^{\{r_l\}}$ are

$$x_l + 2(x_{l-1} + \cdots + x_1) + \sqrt{2}(2(y_1 + \cdots + y_{j-1}) + (y_j + \cdots + y_n))$$
with \( j \in \{1, \ldots, n\} \). Thus

\[
|\mathcal{E}^{\{r_l\}}_Y| = \begin{cases} 
2^n - 1 & \text{if } l = 2 \text{ and } |Y_2| = n, \\
n & \text{if } l \geq 3 \text{ and } |Y_2| = n.
\end{cases}
\]

**Proof.** By (6.59) we know that each root in \( \mathcal{E}^{\{r_l\}}_Y \) is preceded by \( x_l \), and it follows (by a remark at the beginning of this section) that each root in \( \mathcal{E}^{\{r_l\}}_Y \) is preceded by \( x_1 \). Therefore the previous remark yields that

\[
\mathcal{E}^{\{r_l\}}_Y = \phi^{-1}(\bigcup_{j=1}^{n+1} \mathcal{E}^{\{r_l\} \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\} \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\}}_Y \cap V_a).
\]

If \( l = 2 \), then \( Y_1 \cup \tilde{Y}_2 \cup \tilde{Y}_2 \) equals

\[
s_n \quad \tilde{s}_{n-1} \quad \tilde{s}_n \quad r_2 \quad s_1 \quad \tilde{s}_1 \quad \tilde{s}_{n-1} \quad \tilde{s}_n
\]

Proposition (6.74) implies that \( \mathcal{E}^{\{r_2\}}_Y \) is empty, and we deduce from (6.72) (together with (6.57)) that the elements of \( \mathcal{E}^{\{r_l\} \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\} \cup \{\tilde{s}_j, \ldots, \tilde{s}_n\}}_Y \cap V_a \) for \( j \in \{1, \ldots, n\} \) are

\[
x_2 + (\tilde{y}_n + \cdots + \tilde{y}_j) + 2(y_{j-1} + \cdots + y_{k(3)}) + \cdots + (M - 1)(\tilde{y}_{k(M-1)} + \cdots + \tilde{y}_{k(M)}) + M(\tilde{y}_{k(M)-1} + \cdots + \tilde{y}_1) \\
+ Mx_1 + M(\tilde{y}_1 + \cdots + \tilde{y}_{k(M)-1}) + \cdots + (\tilde{y}_j + \cdots + \tilde{y}_m),
\]

with \( M \in \{2, \ldots, j+1\} \) and \( j > k(3) > \cdots > k(M) > 0 \). This yields the assertion, if we set \( j(1) = j \) and \( j(i) = k(i-1) \) for \( i \in \{3, \ldots, M\} \).

Suppose next that \( l \geq 3 \). If \( n = 1 \), then \( Y_1 \cup \tilde{Y}_2 \cup \tilde{Y}_2 \) equals
and (6.77) yields that

\[ \mathcal{E}^{\{r_i\}} \{ \phi^{-1}(x_l + 2(x_{l-1} + \cdots + x_1) + \tilde{y}_1 + \overline{y}_1) \} = \{ x_l + 2(x_{l-1} + \cdots + x_1) + \sqrt{2}y_1 \}, \]

as required. Suppose now that \( n \geq 2 \). Then \( Y_1 \cup \tilde{Y}_2 \cup \overline{Y}_2 \) equals

\[ \{ s_j, \ldots, s_n \} \]

with \( l \geq 3 \) and \( n \geq 2 \). Now \( \mathcal{E}_\mathcal{Y}^{\{r_i\}} \) is empty by (6.70), and by (6.70) (and (6.57)), \( \mathcal{E}_\mathcal{V}_a^{\{r_i\} \cup \{s_j, \ldots, s_n\} \cup \{\overline{s}_j, \ldots, \overline{s}_n\}} \cap V_a \) has exactly one element for \( j \in \)
Chapter 6  The set of elementary roots

\{1, \ldots, n\}, namely

\[
x_l + 2(x_{l-1} + \cdots + x_2) + (\bar{y}_m + \cdots + \bar{y}_j) + 2(\tilde{y}_{j-1} + \cdots + \tilde{y}_1) + 2x_1 + 2(\bar{y}_1 + \cdots + \bar{y}_{j-1}) + (\bar{y}_j + \cdots + \bar{y}_m),
\]

and the assertion follows trivially.

The previous proposition together with (6.99) allow us to conclude the following:

\begin{equation}
(6.104) \text{ Proposition \ Suppose that } m = 4, |Y_1| = l \text{ and } |Y_2| > n. \text{ If } l = 2, \text{ then } \mathcal{E}_{Y_2}^{\{r_2\}} \text{ is the set of }
\end{equation}

\[
x_2 + Mx_1 + \sqrt{2}(M(y_1 + \cdots + y_{(M-1)}) + (M-1)(y_{(M-1)} + \cdots + y_{(M-2)}) + \cdots + 3(y_{(3)}) + \cdots + y_{(2)}) + 2(y_{(2)} + \cdots + y_{(1)} + \beta)
\]

with \(M \in \{2, \ldots, n+1\}\),

\[
0 < j(M-1) < j(M-2) \ldots < j(2) < j(1) < n + 1
\]

and \(\beta \in \mathcal{E}_{Y_2\setminus\{s_1, \ldots, s_{j(2)}\}}\) with coefficient 1 for \(y_{j(1)}\).

If \(l \geq 3\), then \(\mathcal{E}_{Y_2}^{\{r_l\}}\) is the set of

\[
x_l + 2(x_{l-1} + \cdots + x_1) + \sqrt{2}(2(y_1 + \cdots + y_{j-1}) + \beta)
\]

with \(j \in \{1, \ldots, n\}\) and \(\beta \in \mathcal{E}_{Y_2\setminus\{s_1, \ldots, s_{j-1}\}}\) with coefficient 1 for \(y_{j}\).

Thus

\[
|\mathcal{E}_{Y_2}^{\{r_l\}}| = \begin{cases} 
\sum_{j=1}^{n}(2^j - 1) \sum_{J \subseteq Y_2 \setminus \{s_1, \ldots, s_j\}} |\mathcal{E}_{Y_2\setminus\{s_1, \ldots, s_{j-1}\}}^{\{s_j\}\cup J}| & \text{if } l = 2, \\
\sum_{j=1}^{n} \sum_{J \subseteq Y_2 \setminus \{s_1, \ldots, s_j\}} |\mathcal{E}_{Y_2\setminus\{s_1, \ldots, s_{j-1}\}}^{\{s_j\}\cup J}| & \text{if } l \geq 3.
\end{cases}
\]

Note that the hypothesis \(|Y_2| > n\) is not necessary in (6.104); for if \(|Y_2| = n\), then \(\mathcal{E}_{Y_2\setminus\{s_1, \ldots, s_{j-1}\}} = \{y_j + \cdots + y_n\}\), and the assertion reduces to (6.103).
(6.105) **Proposition**  Suppose that \( m = 4, |Y_1| = l \) and \( |Y_2| = n \). Then \( \mathcal{E}^\emptyset_Y \) is empty unless \( l = n = 2 \), or \( l = 2 \) and \( n \geq 3 \), or \( n = 2 \) and \( l \geq 3 \). If \( l = 2 \) and \( n = 2 \), then

\[
\mathcal{E}^\emptyset_Y = \{2x_2 + 3x_1 + 2\sqrt{2}y_1 + \sqrt{2}y_2, \sqrt{2}x_2 + 2\sqrt{2}x_1 + 3y_1 + 2y_2\},
\]

while if \( l = 2 \) and \( n \geq 3 \), the elements of \( \mathcal{E}^\emptyset_Y \) are

\[
2x_2 + 3x_1 + \sqrt{2}(3(y_1 + \cdots + y_{k-1}) + 2(y_k + \cdots + y_{j-1}) + (y_j + \cdots + y_n))
\]

with \( 1 \leq k < j \leq n \). Hence

\[
|\mathcal{E}^\emptyset_Y| = \begin{cases} 
0 & \text{if } l = 1 \text{ or } n = 1, \\
2 & \text{if } l = n = 2, \\
\binom{n}{2} & \text{if } l = 2, n \geq 3, \\
\binom{l}{2} & \text{if } l \geq 3 \text{ and } n = 2, \\
0 & \text{if } l, n \geq 3.
\end{cases}
\]

**Proof.** As \( \mathcal{E}_Y \) does not depend on \( R \), we may assume without loss of generality that \( R \) contains \( \check{Y}_1 = \{\check{r}_1, \ldots, \check{r}_1\} \) and \( Y_1 = \{r_1, \ldots, r_1\} \) such that \( \check{Y}_1 \cup \check{Y}_1 \cup Y_2 \) equals

![Diagram](image)

Denote the simple roots corresponding to \( \check{r}_1, \check{r}_1 \) by \( \check{x}_i, \check{x}_i \) respectively, and let
V_b the space spanned by \( \tilde{x}_l + \overline{x}_l, \tilde{x}_1 + \overline{x}_1, y_1, \ldots, y_n \). Further, let \( \psi: V_Y \rightarrow V_b \) be defined by

\[
\psi\left(\sum_{i=1}^{l} \lambda_i x_i + \sum_{j=1}^{n} \mu_j y_j\right) = \frac{1}{\sqrt{2}} \sum_{i=1}^{l} \lambda_i (\tilde{x}_i + \overline{x}_i) + \sum_{j=1}^{n} \mu_j y_j.
\]

By a remark at the beginning of this section, each root in \( E_Y \) is preceded by \( x_1 \) or \( y_1 \) (but certainly not by both), and so the remark following (6.102) yields that \( E_Y^{\{r_1\}} \) is equal to the following disjoint union:

\[
\phi^{-1}\left(\bigcup_{j=1}^{n+1} E_{Y_1 \cup \overline{Y}_2 \cup Y_2}^{\{s_j, \ldots, \overline{s}_n\} \cup \{\overline{s}_j, \ldots, \overline{s}_n\}} \cap V_a\right) \cup \psi^{-1}\left(\bigcup_{i=1}^{l+1} E_{Y_2 \cup \overline{Y}_1 \cup Y_1}^{\{\tilde{r}_i, \ldots, \overline{r}_i\} \cup \{\overline{r}_i, \ldots, \overline{r}_l\}} \cap V_b\right).
\]

If \( l = 1 \), then \( Y_1 \cup \overline{Y}_2 \cup Y_2 \) equals

\[
\tilde{s}_n \quad \tilde{s}_{n-1} \quad \tilde{s}_1 \quad \overline{s}_1 \quad \overline{s}_{n-1} \quad \overline{s}_n
\]

and \( E_{Y_1 \cup \overline{Y}_2 \cup Y_2}^{\{s_j, \ldots, \overline{s}_n\} \cup \{\overline{s}_j, \ldots, \overline{s}_n\}} \) is empty for all \( j \in \{1, \ldots, n+1\} \) by (6.66); furthermore, \( \overline{Y}_1 \cup Y_1 \cup Y_2 \) equals

\[
\overline{s}_1 \quad \overline{r}_1 \quad s_1 \quad \overline{s}_1 \quad \overline{s}_2 \quad \overline{s}_1
\]

and \( E_{Y_1 \cup Y_2}^{\{s_1\}} \) and \( E_{Y_1 \cup Y_2}^{\{r_1, \overline{r}_1\}} \) are empty by (6.77). Whence \( E_Y^{\emptyset} = \emptyset \) if \( l = 1 \), and symmetrically \( E_Y^{\emptyset} = \emptyset \) if \( n = 1 \).
Suppose now that $l, n \geq 2$. If both $l$ and $n$ equal 2, $Y_1 \cup \tilde{Y}_2 \cup Y_2$ and $\tilde{Y}_1 \cup \bar{Y}_1 \cup Y_2$ equal

respectively, and (6.85) yields that $E_{\bar{s}}$ has two elements; namely

$$\phi^{-1}\left(2x_2 + 2\bar{y}_2 + 2\tilde{y}_1 + 3x_1 + 2\bar{y}_1 + \bar{y}_2\right) = 2x_2 + 3x_1 + \sqrt{2}(2y_1 + y_2)$$

and

$$\psi^{-1}\left(2y_2 + 3x_2 + 2\tilde{x}_1 + 3y_1 + 2\bar{x}_1 + \bar{x}_2\right) = \sqrt{2}(x_2 + 2x_1) + 3y_1 + 2y_2,$$

as required.

Suppose next that $n \geq 3$. Then $Y_2 \cup \tilde{Y}_1 \cup \bar{Y}_1$ equals

with $n \geq 3$ and $l \geq 2$, and thus $E_{\{\tilde{r}_1, \ldots, \tilde{r}_i\}} \cup \{\bar{r}_1, \ldots, \bar{r}_i\}$ is empty for all $i \in \{2, \ldots, l + 1\}$ by (6.70) (and (6.57)), and furthermore empty by (6.77) for $i = 1$. So

$$E_{\{\tau_1\}} = \phi^{-1}\left(\bigcup_{j=1}^{n+1} E_{Y_1 \cup \tilde{Y}_2 \cup \bar{Y}_2} \cap V_{\alpha}\right).$$
If \( l \) is also greater than or equal to 3, symmetrical arguments yield that \( \mathcal{E}_Y^\theta \) is empty. Assume next that \( l = 2 \); then \( Y_1 \cup Y_2 \cup Y_2 \) equals

![Diagram](image)

with \( n \geq 3 \). Lemma (6.77) yields that \( \mathcal{E}_{Y_1 \cup Y_2 \cup Y_2}^{\{s_1, \ldots, s_n\} \cup \{\bar{s}_1, \ldots, \bar{s}_n\}} = \emptyset \), and (6.80) implies that \( \mathcal{E}_{Y_1 \cup Y_2 \cup Y_2}^\theta \) is also empty. So the above becomes

\[
\mathcal{E}_Y^\{n\} = \phi^{-1}\left( \bigcup_{j=2}^{n} \mathcal{E}_{Y_1 \cup Y_2 \cup Y_2}^{\{s_j, \ldots, s_n\} \cup \{\bar{s}_j, \ldots, \bar{s}_n\}} \cap V_a \right).
\]

Now \( \mathcal{E}_{Y_1 \cup Y_2 \cup Y_2}^{\{s_2, \ldots, s_n\} \cup \{\bar{s}_2, \ldots, \bar{s}_n\}} \) has one element by (6.85), namely

\[
2x_1 + (\bar{y}_n + \cdots + \bar{y}_2) + 2\bar{y}_1 + 3x_0 + +2\bar{y}_1 + (\bar{y}_2 + \cdots + \bar{y}_n).
\]

If \( j \in \{3, \ldots, n\} \), (6.79)(1) yields that \( \mathcal{E}_{Y_1 \cup Y_2 \cup Y_2}^{\{s_j, \ldots, s_n\} \cup \{\bar{s}_j, \ldots, \bar{s}_n\}} \cap V_a \) is the set of

\[
2x_1 + (\bar{y}_n + \cdots + \bar{y}_j) + 2(\bar{y}_j - 1 + \cdots + \bar{y}_k) + 3(\bar{y}_k - 1 + \cdots + \bar{y}_1) + 3x_0 + 3(\bar{y}_1 + \cdots + \bar{y}_k - 1) + 2(\bar{y}_k + \cdots + \bar{y}_j - 1) + (\bar{y}_j + \cdots + \bar{y}_n), \tag{\star}
\]

with \( k \in \{1, \ldots, j - 1\} \). It is clear that \( \mathcal{E}_{Y_1 \cup Y_2 \cup Y_2}^{\{s_j, \ldots, s_n\} \cup \{\bar{s}_j, \ldots, \bar{s}_n\}} \cap V_a \) for \( j = 2 \) is also of the form \( \star \), and one can now easily verify that \( \mathcal{E}_Y^\theta \) is of the required shape.

\[
(6.106) \quad \text{PROPOSITION} \quad \text{Suppose that } m = 4 \text{ and } |Y_1| = l. \text{ Then } \mathcal{E}_Y^\theta \text{ is empty unless } l = 2 \text{ and } n \geq 2, \text{ or } |Y_2| = 2 \text{ (and thus } |Y_2| = n = 2) \text{ and } l \geq 3. \text{ If } l = 2, n \geq 2 \text{ and } |Y_2| \geq 3, \text{ the elements of } \mathcal{E}_Y^\theta \text{ are}
\]

\[
2x_2 + 3x_1 + \sqrt{2}(3(y_1 + \cdots + y_k - 1) + 2(y_k + \cdots + y_{j-1}) + \beta),
\]
where \( 1 \leq k < j \leq n \) and \( \beta \in \mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}} \) with coefficient 1 for \( y_j \).

Hence

\[
|\mathcal{E}_Y^\emptyset| = \sum_{j=2}^{n} (j - 1) \sum_{J \subseteq Y_2 \setminus \{s_1,\ldots,s_j\}} |\mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}^{\{s_j\}\cup J}|.
\]

Proof. If \( |Y_1| = n \),

\[
\bigcup_{J \subseteq Y_2 \setminus \{s_1,\ldots,s_j\}} \mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}^{\{s_j\}\cup J} = \{y_j + \cdots + y_n\},
\]

and the assertion is proved in (6.105). So assume now that \( |Y_2| > n \). By (6.99) we know that every element of \( \mathcal{E}_Y^\emptyset \) can be written as

\[
\alpha + \sqrt{2}\beta - \sqrt{2}y_j
\]

with \( j \in \{1,\ldots,n\} \), \( \alpha \in \mathcal{E}_{Y_1\cup\{s_1,\ldots,s_j\}}^\emptyset \) with coefficient \( \sqrt{2} \) for \( y_j \) and coefficient greater than or equal to 2 for \( y_1,\ldots,y_{j-1} \), and \( \beta \in \mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}} \) with coefficient 1 for \( y_j \). But (6.105) yields that \( \mathcal{E}_{Y_1\cup\{s_1,\ldots,s_j\}}^\emptyset \) is empty unless \( l = 2 \) and \( j \geq 2 \), or \( l \geq 3 \) and \( j = 2 \); if \( l \geq 3 \) and \( j = 2 \), (6.105) yields further that there are no roots in \( \mathcal{E}_{Y_1\cup\{s_1,\ldots,s_j\}}^\emptyset \) with coefficient \( \sqrt{2} \) for \( y_2 \), and thus \( \mathcal{E}_Y^\emptyset = \emptyset \) unless \( l = 2 \) and \( j \geq 2 \) (and thus \( n \geq 2 \)).

If \( l = 2 \) and \( n \geq 2 \), then \( \mathcal{E}_{Y_1\cup\{s_1\}}^\emptyset = \emptyset \) by (6.105), and thus \( \mathcal{E}_Y^\emptyset \) is the set of

\[
2x_2 + 3x_1 + \sqrt{2}(3(y_1 + \cdots + y_{k-1}) + 2(y_k + \cdots + y_{j-1}) + \beta)
\]

with \( 1 \leq k < j \leq n \) and \( \beta \in \mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}} \) with coefficient 1 for \( y_j \), and the assertion follows easily.

This leaves us with \( m = 5 \). Since

\[
\left\{ \frac{\sin(k\pi/n)}{\sin(\pi/n)} \mid n = m_{rs} < \infty \text{ for some } r, s \in Y \text{ and } 1 \leq l \leq \frac{k}{2} \right\}
\]

equals \( \{1, c_5\} \), and \( c_5^2 = c_5 + 1 \), we know by (2.27) that for \( r \in Y \) the coefficient of \( \alpha_r \) in any root in \( \alpha \in \Phi_Y \) equals \( a + bc_5 \) for some \( a, b \in \mathbb{N}_0 \).

We start off by stating two easy results (the former a modification of (6.73)(i), and the latter a corollary of (3.37)). Trivial though they are, these will make life a lot easier for us.
Chapter 6 The set of elementary roots

(6.107) Lemma Suppose that \( \alpha \) is a root in \( \Phi_Y^+ \), with \( x_1 \in \text{supp}(\alpha) \). Then the coefficient of \( x_{i+1} \) in \( \alpha \) is less than or equal to the coefficient of \( x_i \) in \( \alpha \) for all \( i \) in \( \{1, \ldots, l-1\} \).

(6.108) Corollary Suppose that \( \gamma_1, \ldots, \gamma_k \in \Phi^+ \) and \( t_1, \ldots, t_{k-1} \in R \) with \( \gamma_{i+1} = t_i \cdot \gamma_i \) and \( \langle \gamma_i, \alpha_{t_i} \rangle \in (-1, 0) \) for all \( i \) in \( \{1, \ldots, k-1\} \). Suppose furthermore that \( \gamma_1 \in E \). Then \( \gamma_i \in E \) for all \( i \) in \( \{1, \ldots, k\} \).

The next lemma together with (6.99) will enable us to restrict our main attention to the case \( |Y_1| = 1 \) and \( |Y_2| = n \).

(6.109) Lemma Suppose there exist \( t_1, t_2 \in Y \) such that \( \{t_1, r_1, s_1, t_2\} \) equals

\[
\begin{array}{c}
5 \\
t_1 \hspace{1cm} r_1 \hspace{1cm} s_1 \hspace{1cm} t_2
\end{array}
\]

Denote the simple roots corresponding to \( t_1, t_2 \) by \( z_1 \) and \( z_2 \) respectively, and let \( \alpha \in \Phi_Y \) such that \( z_1, z_2 \in \text{supp}(\alpha) \), and the coefficients of \( x_1 \) and \( y_1 \) in \( \alpha \) are greater than or equal to 2. Then \( \alpha \in \Delta \).

Proof. Let \( \beta \preceq \alpha \) be a positive root of minimal depth such that \( z_1 \) and \( z_2 \) are in the support of \( \beta \), and the coefficients of \( x_1 \) and \( y_1 \) in \( \beta \) are greater than or equal to 2. Further, let \( r \in Y \) such that \( \beta > r \cdot \beta \). It suffices to show that \( \langle r \cdot \beta, \alpha_r \rangle \leq -1 \); that is, \( \langle \beta, \alpha_r \rangle \geq 1 \). For then \( \beta \in \Delta \) by (3.32), and thus \( \alpha \in \Delta \) by (3.36).

Denote the coefficients of \( z_1, x_1, y_1, z_2 \) in \( \beta \) by \( \nu_1, \lambda, \mu \) and \( \nu_2 \) respectively. By minimality of \( \beta \) it follows that \( r \in \{t_1, r_1, s_1, t_2\} \), and by symmetry we may assume without loss of generality that \( r \in \{t_1, r_1\} \). Suppose first that \( r = t_1 \). Then minimality of \( \beta \) also yields that \( z_1 \notin \text{supp}(t_1 \cdot \beta) \); that is, the coefficient of \( z_1 \) in \( t_1 \cdot \beta \) equals 0. Since \( \lambda \geq 2 \) we deduce that \( \langle t_1 \cdot \beta, z_1 \rangle \leq 0 + (-\frac{1}{2}) \lambda \leq -1 \), as required.

Suppose next that \( r = r_1 \), and denote the coefficient of \( x_1 \) in \( r_1 \cdot \beta \) by \( \lambda' \). Then \( \lambda' < 2 \) by minimality of \( \beta \), and \( \lambda' > 0 \) by connectedness of the support of \( r_1 \cdot \beta \); hence \( \lambda' \) equals 1 or \( c_5 \). If \( \lambda' = 1 \), Propositions (6.57) and (6.93) together yield that \( \mu = kc_5 \) for some \( k \in \mathbb{N}_0 \), and since \( \mu \geq 2 \) we know that \( k \geq 2 \); since \( \nu_1 \geq 1 \), this implies that

\[
\langle r_1 \cdot \beta, x_1 \rangle \leq 1 + \left( -\frac{1}{2} \right) \nu_1 + \left( -\frac{c_5}{2} \right) \mu \leq 1 - \frac{1}{2} - \frac{kc_5}{2} \leq \frac{1}{2} - c_5^2 \leq -1,
\]
as required.

Finally, suppose that $\lambda' = c_5$. If $\mu = 2$, let $\gamma \succeq r_1 \cdot \beta$ be of minimal depth such that the coefficients of $x_1$ and $y_1$ in $\gamma$ equal $c_5$ and 2 respectively; then

$$\langle \gamma, x_1 \rangle \leq c_5 + \left(-\frac{c_5}{2}\right)2 = 0 \text{ and } \langle \gamma, y_1 \rangle \leq 2 + \left(-\frac{c_5}{2}\right)c_5 = \frac{3}{2} - c_5 \leq 0,$$

and therefore neither $\gamma \succ r_1 \cdot \gamma$ nor $\gamma \succ s_1 \cdot \gamma$, contradicting the minimality of $\gamma$. So $\mu > 2$, and since $\mu$ equals $ac_5 + b$ for some $a, b \in \mathbb{N}_0$ this forces $\mu \geq c_5 + 1$. Hence

$$\langle r_1 \cdot \beta, x_1 \rangle \leq c_5 + \nu_1 \left(-\frac{1}{2}\right) + \mu \left(-\frac{c_5}{2}\right) \leq c_5 - \frac{1}{2} - \frac{c_5^2}{2} = -1,$$

(since $\nu_1 \geq 1$), as required. \(\square\)

(6.110) Lemma. Suppose that $m = 5$, $|Y_1| = 1$ and $|Y_2| = n$, and let $\alpha$ be in $E\{s_n\}$. Then $\alpha$ is preceded by

$$c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n,$$

and this is an elementary root.

Proof. By (6.59) we know that $\alpha$ is preceded by $y_n$, and we let $\beta$ be of maximal depth with $\alpha \succeq \beta \succeq y_n$ such that $I(\beta) \subseteq Y_2$. Maximality of $\beta$ yields that $\alpha \succeq r_1 \cdot \beta \succeq \beta$, and thus $\langle \beta, x_1 \rangle \in (-1, 0)$ by (3.38); if we denote the coefficient of $y_1$ in $\beta$ by $\mu$, we find that $\langle \beta, x_1 \rangle = -\frac{c_5}{2} \mu$, and (2.26) yields that $\mu = 1$. As $s_1, s_n \in I(\beta)$, we deduce that $I(\beta) = \{s_1, \ldots, s_n\}$. Now $\beta$ is elementary, and $I(\beta)$ contains only simple bonds, therefore (6.69) implies that $r_1 \cdot \beta \succeq y_1 + \cdots + y_n$; furthermore, the coefficients of $y_1$ in $r_1 \cdot \beta$ and $y_1 + \cdots + y_n$ coincide, and thus

$$r_1 \cdot \beta \succeq r_1 \cdot (y_1 + \cdots + y_n) = c_5 x_1 + y_1 + \cdots + y_n$$

by (6.58). It follows by transitivity of $\succeq$ that $\alpha$ is preceded by

$$\gamma_1 = c_5 x_1 + y_1 + \cdots + y_n;$$
Chapter 6 The set of elementary roots

since $y_1 + \cdots + y_n$ is clearly elementary, and $(y_1 + \cdots + y_n, x_1) = -\frac{c_5}{2} \in (-1, 0)$, (3.37) yields that $\gamma_1$ is elementary. For $j \in \{1, \ldots, n-1\}$, let $\gamma_{j+1}$ be equal to $s_j \cdot \gamma_j$. Then a straightforward calculation yields that

$$\gamma_j = c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + y_j + \cdots + y_n$$

and $(\gamma_j, y_j) = -\frac{1}{2} \in (-1, 0)$; since $\gamma_1$ is elementary, (6.108) implies that $\gamma_j$ is elementary for all $j \in \{1, \ldots, n\}$. In particular $\gamma_n \in \mathcal{E}$, and it remains to show that $\alpha$ is preceded by $\gamma_n$.

By the above $\alpha \succeq \gamma_1$, and the proof is finished if $n = 1$; so suppose that $n \geq 2$. The coefficient of $y_1$ in $\gamma_1$ equals 1, while the coefficient of $y_1$ in $\alpha$ is greater than 1, and we let $\delta$ be a root with $\gamma_1 \preceq \delta \preceq \alpha$ such that the coefficient of $y_1$ in $\delta$ equals 1, and $\delta \prec s_1 \cdot \delta \preceq \alpha$. As $\alpha$ is elementary, clearly $s_1 \cdot \delta \in \mathcal{E}$, and thus $(\delta, y_1) > -1$ by (3.38). The coefficient of $y_2$ in $\delta$ equals 1 by (6.107), and we denote the coefficient of $y_1$ in $\delta$ by $\lambda$. Then

$$\langle \delta, y_1 \rangle \leq 1 + \left( -\frac{1}{2} \right) 1 + \left( -\frac{c_5}{2} \right) \lambda,$$

and thus $\lambda < \frac{3}{c_5} \leq 2$; since $\lambda \geq c_5$ as $\delta \succeq \gamma_1$, this yields $\lambda = c_5$. Hence the coefficients of $x_1$ in $s_1 \cdot \delta$ and $\gamma$ coincide; as $|Y_2| = n$, and the coefficients of $y_2$ in $s_1 \cdot \delta$ and $\gamma$ both equal 1, (6.58) yields that $s_1 \cdot \delta$ is a successor of $s_1 \cdot \gamma_1$, which equals $\gamma_2$.

If $n = 2$, this finishes the proof, so suppose $n > 2$, and assume that $\alpha \succeq \gamma_j$ for some $j \in \{2, \ldots, n-1\}$. The coefficient of $y_j$ in $\gamma_j$ equals 1, while the coefficient of $y_j$ in $\alpha$ is greater than 1, and (again) we let $\delta$ be a root with $\gamma_j \preceq \delta \preceq \alpha$ such that the coefficient of $y_j$ in $\delta$ equals 1, and $\delta \prec r_j \cdot \delta \preceq \alpha$. As $\alpha$ is elementary, clearly $s_j \cdot \delta \in \mathcal{E}$, and thus $(\delta, y_j) > -1$ by (3.38). The coefficient of $y_{j+1}$ in $\delta$ equals 1 by (6.107), and we denote the coefficient of $y_j$ in $\delta$ again by $\lambda$; then

$$\langle \delta, y_j \rangle \leq 1 + \left( -\frac{1}{2} \right) 1 + \left( -\frac{1}{2} \right) \lambda = \frac{1}{2} (1 - \lambda),$$

and we deduce that $\lambda < 3$. But $\lambda \geq c_5 + 1$ as $\delta \succeq \gamma_j$, and this only leaves us with $\lambda = c_5 + 1$ (since $\lambda$ equals $ac_5 + b$ for some $a, b \in \mathbb{N}_0$). Hence the coefficients of $y_j$ in $s_j \cdot \delta$ and $\gamma$ coincide again. As $|Y_2| = n$, and the coefficients of $y_{j+1}$ in $s_j \cdot \delta$ and $\gamma$ both equal 1, (6.58) yields that $s_j \cdot \delta$ is a successor of $s_j \cdot \gamma_j$, which equals $\gamma_{j+1}$. So by induction $\alpha \succeq \gamma_n$, as required.
(6.111) **Proposition** Suppose \( m = 5 \), \(|Y_1| = 1\) and \(|Y_2| = n\).

(i) If \( n = 2 \), then

\[
\mathcal{E}_Y\{s_2\} = \{c_5x_1 + (c_5 + 1)y_1 + y_2, (c_5 + 1)x_1 + (c_5 + 1)y_1 + y_2\}.
\]

(ii) If \( n = 3 \), then

\[
\mathcal{E}_Y\{s_3\} = \{c_5x_1 + (c_5 + 1)y_1 + (c_5 + 1)y_2 + y_3,
(c_5 + 1)x_1 + (c_5 + 1)y_1 + (c_5 + 1)y_2 + y_3,
(c_5 + 1)x_1 + (2c_5 + 1)y_1 + (c_5 + 1)y_2 + y_3,
(2c_5 + 1)x_1 + (c_5 + 1)y_1 + (c_5 + 1)y_2 + y_3,
(2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 1)y_2 + y_3,
(2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + y_3\}.
\]

(iii) If \( n \geq 4 \), the elements of \( \mathcal{E}_Y\{s_n\} \) are exactly the following:

\[
\alpha^{(1)} = c_5x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n,
\]

\[
\alpha_j^{(2)} = (c_5+1)x_1 + (2c_5+1)(y_1 + \cdots + y_{j-1}) + (c_5+1)(y_j + \cdots + y_{n-1}) + y_n,
\]

with \( j \in \{1, \ldots, n-1\} \),

\[
\alpha_j^{(3)} = (2c_5+1)x_1 + (2c_5+1)(y_1 + \cdots + y_{j-1}) + (c_5+1)(y_j + \cdots + y_{n-1}) + y_n,
\]

with \( j \in \{2, \ldots, n-1\} \), and

\[
\alpha^{(4)} = (2c_5+1)x_1 + (2c_5+2)y_1 + (c_5+1)(y_2 + \cdots + y_{n-1}) + y_n.
\]

Hence

\[
|\mathcal{E}_Y\{r_i\}| = \begin{cases} 
2 & \text{if } |Y_1| = 1 \text{ and } n = 2, \\
6 & \text{if } |Y_1| = 1 \text{ and } n = 3, \\
2n - 1 & \text{if } |Y_1| = 1 \text{ and } n \geq 4.
\end{cases}
\]

**Proof.** If \( n = 2, 3 \) it can be easily verified that we have in fact enumerated all roots preceded by \( c_5x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n \) with coefficient 1 for \( y_n \), and that these are elementary. It remains to show (iii).
By (6.110) we know that \( \alpha^{(1)} \) is elementary. Since \( \langle \alpha^{(1)}, x_1 \rangle = -\frac{1}{2} \), it follows by (3.37) that \( r_1 \cdot \alpha^{(1)} \) is also elementary; that is, \( \alpha_1^{(2)} \in \mathcal{E} \). Now \( \alpha_{j+1}^{(2)} = s_j \cdot \alpha_j^{(2)} \) for \( j \in \{1, \ldots, n-1\} \) and further \( \langle \alpha_j^{(2)}, y_j \rangle = -\frac{c_2}{2} \in (-1, 0) \); hence \( \alpha_j^{(2)} \in \mathcal{E} \) for all \( j \in \{1, \ldots, n\} \) by (6.108). For \( j \in \{2, \ldots, n\} \) clearly \( \alpha_j^{(3)} = r_1 \cdot \alpha_j^{(2)} \) and \( \langle \alpha_j^{(2)}, x_1 \rangle = -\frac{c_2}{2} \); thus it follows by (3.37) that \( \alpha_j^{(3)} \) is elementary for all \( j \in \{2, \ldots, n\} \). Finally, \( \langle \alpha_2^{(3)}, y_1 \rangle = -\frac{1}{2} \) and therefore \( \alpha^{(4)} = s_1 \cdot \alpha_2^{(3)} \in \mathcal{E} \) by (3.37). So the above listed vectors are in \( \mathcal{E} \), and thus certainly in \( \mathcal{E}_Y^{(s_n)} \).

We prove now that all elements of \( \mathcal{E}_Y^{(s_n)} \) have been accounted for. So let \( \beta \in \mathcal{E}_Y^{(s_n)} \), and assume for a contradiction that \( \beta \) is not equal to any of the roots listed above. By (6.110) we know that \( \beta \) is preceded by \( \alpha^{(1)} \). As \( \beta \neq \alpha^{(1)} \), and the coefficients of \( y_n \) in \( \alpha^{(1)} \) and \( \beta \) coincide, there exists an \( r \in \{r_1, s_1, \ldots, s_{n-1}\} \) with \( \beta \geq r \cdot \alpha^{(1)} \); that is, \( \langle \alpha^{(1)}, \alpha_r \rangle < 0 \). This forces \( r = r_1 \), and thus \( \beta \geq r_1 \cdot \alpha^{(1)} = \alpha_1^{(2)} \). Now let \( j \in \{1, \ldots, n-1\} \) be maximal such that \( \beta \geq \alpha_j^{(2)} \), that is, \( \beta \succ \alpha_j^{(2)} \) by assumption. Then there exists an \( s \in \{r_1, s_1, \ldots, s_{n-1}\} \) with \( \beta \geq s \cdot \alpha_j^{(2)} \), and maximality of \( j \) forces \( s = r_1 \) and \( j \geq 2 \). Whence \( \beta \geq r_1 \cdot \alpha_j^{(2)} = \alpha_j^{(3)} \), and hence \( \beta \succ \alpha_j^{(3)} \) by assumption. Now let \( t \in \{r_1, s_1, \ldots, s_{n-1}\} \) with \( \beta \geq t \cdot \alpha_j^{(3)} \). Then \( \langle \alpha_j^{(3)}, \alpha_t \rangle < 0 \), and thus \( \langle \alpha_j^{(3)}, \alpha_t \rangle \in (-1, 0) \), as \( \beta \) is elementary. We find that \( t = s_1 \) and \( j = 2 \); so \( \beta \) is preceded by \( s_1 \cdot \alpha_2^{(3)} = \alpha^{(4)} \). Since \( \langle \alpha^{(4)}, x_1 \rangle, \langle \alpha^{(4)}, y_2 \rangle \leq -1 \) and \( \langle \alpha^{(4)}, \alpha_r \rangle \geq 0 \) for \( r \in Y \setminus \{r_1, s_1\} \), we deduce from (3.38) that \( \beta = \alpha^{(4)} \), contrary to our assumption, and this finishes the proof. \( \square \)

(6.112) Proposition Suppose \( m = 5 \), \( |Y_2| = n \geq 2 \) and \( |Y_1| > 1 \). Then \( \mathcal{E}_Y^{(s_n)} \) equals

\[
\{ c_5 \alpha + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n \mid \alpha \in \mathcal{E}_{Y_1} \text{ with coefficient 1 for } x_1 \}
\]

\[
= \{ c_5 \alpha + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n \mid \alpha \in \bigcup_{J \subseteq Y_1 \setminus \{r_1\}} \mathcal{E}_{Y_1}^{J \cup \{r_1\}} \}.
\]

Hence

\[
|\mathcal{E}_Y^{(s_n)}| = \sum_{J \subseteq Y_1 \setminus \{r_1\}} |\mathcal{E}_{Y_1}^{J \cup \{r_1\}}|.
\]
In particular, if $|Y_1| = l$, 

$$\mathcal{E}_{Y_n}^{(s_n)} = \{ c_5(x_1 + \cdots + x_1) + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n \}.$$ 

Proof. Since $c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n$ is elementary by (6.110), Proposition (6.97) yields that 

$$c_5 \alpha + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n$$ 

is an elementary root for $\alpha \in \mathcal{E}_{Y_1}$ with coefficient 1 for $x_1$, and thus certainly in $\mathcal{E}_{Y_1}^{(s_n)}$. It remains to show that all elements of $\mathcal{E}_{Y_1}^{(s_n)}$ are of this form.

Let $\gamma \in \mathcal{E}_{Y_1}^{(s_n)}$. Since the coefficient of $y_n$ in $\gamma$ equals 1, we deduce from (6.97) that the coefficients of $y_1, \ldots, y_{n-1}$ in $\gamma$ cannot be equal to $c_5$, and thus must be greater than or equal to 2. In particular, the coefficient of $y_1$ in $\gamma$ is greater than or equal to 2; since $n \geq 2$ and $|Y_1| \geq 2$, Lemma (6.109) now forces the coefficient of $x_1$ in $\gamma$ to be less than 2, and thus equal to $c_5$. So by (6.97),

$$\gamma = c_5 \alpha + \beta - c_5 x_1$$

for some $\alpha \in \mathcal{E}_{Y_1}$ with coefficient 1 for $x_1$, and $\beta \in \mathcal{E}_{(r_1) \cup Y_2}$ with coefficient $c_5$ for $x_1$. It is clear that $\beta$ must be an element of $\mathcal{E}_{(r_1) \cup Y_2}$, and since the coefficient of $x_1$ in $\beta$ equals $c_5$, we deduce from (6.111) that

$$\beta = c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n;$$

whence $\gamma$ is of the desired shape.

This leaves us with $X = \emptyset$.

(6.113) Lemma Suppose that $m = 5$, $|Y_1| = l$ and $|Y_2| = n$, and let $\alpha \in \mathcal{E}_{Y_1}^0$. Then

$$\alpha \geq c_5 x_1 + c_5(y_1 + \cdots + y_n)$$

or

$$\alpha = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + (c_5 + 1)y_3.$$ 

Proof. If $n = 1$, the assertion is certainly true. Now suppose that $n \geq 2$ and assume that the assertion is true for $n - 1$. Let $\beta$ be of minimal depth with
β ≤ α such that β ∈ ℰ_Y^0, and denote the coefficients of x_1, y_j in β by λ and μ_j respectively. Let r ∈ Y such that r · β < β; then ⟨r · β, α_r⟩ ∈ (−1, 0) by (3.38). Moreover, r · β is elementary, and since r · β ∉ ℰ_Y^0 by minimality of β, the coefficient of α_r in r · β must be less than or equal to 1, and thus equal to 0 or 1 by (2.26).

Suppose first that r = r_1. If the coefficient of x_1 in r_1 · β equals 0, we find that ⟨r_1 · β, x_1⟩ = −\frac{c_5}{2}μ_1 ≤ −1; for μ_1 > 1, and thus μ_1 ≥ √2 by (2.26). But this contradicts our conclusion that ⟨r · β, α_r⟩ ∈ (−1, 0), and so the coefficient of x_1 in r_1 · β equals 1. By (6.93) (together with (6.57)), μ_1 equals k5c for some k ∈ N_0, and

\[ ⟨r_1 · β, x_1⟩ = 1 + \left(−\frac{c_5}{2}\right)k = 1 − \left(c_5 + \frac{1}{2}\right)k \]

forces k = 1 and μ_1 = c_5. So by (6.97), r_1 · β equals x_1 + c_5γ for some γ ∈ ℰ_{Y_2} with coefficient 1 for y_1. Since I(γ) contains only simple bonds, (6.69) yields that γ ≥ y_1 + ⋯ + y_n, and by definition of ≥ there exists a w ∈ W_{Y_2} with γ = w · (y_1 + ⋯ + y_n) and N(w) = N_−(w, y_1 + ⋯ + y_n). Since the coefficients of y_1 in γ and y_1 + ⋯ + y_n coincide, we conclude that w ∈ W_{Y_2 \setminus \{s_1\}}, and thus N(w) ⊆ ℨ_{Y_2 \setminus \{s_1\}}. So if δ ∈ N(w), then

\[ ⟨c_5x_1 + c_5(y_1 + ⋯ + y_n), δ⟩ = c_5⟨y_1 + ⋯ + y_n, δ⟩ < 0, \]

as δ ∈ N_−(w, y_1 + ⋯ + y_n); thus N_−(w, c_5x_1 + c_5(y_1 + ⋯ + y_n)) = N(w). Now

\[ β = c_5x_1 + c_5γ = c_5x_1 + c_5(w · (y_1 + ⋯ + y_n)) \]
\[ = w · (c_5x_1 + c_5(y_1 + ⋯ + y_n)) \]

and by definition of ≥ this is preceded by c_5x_1 + c_5(y_1 + ⋯ + y_n), as required.

Suppose next that r = s_j for some j ∈ \{1, ⋯, n\}. Since the coefficient of y_j in s_j · β equals 0 or 1, while the coefficient of α_s in s_j · β is greater than 1 for s ∈ Y \ {s_j}, Lemma (6.107) yields that j = n. Suppose first that the coefficient of y_n in s_n · β equals 1, and thus s_n · β ∈ ℰ_{Y_2}^{(n)}; then

\[ s_n · β ≥ c_5x_1 + (c_5 + 1)(y_1 + ⋯ + y_{n-1}) + y_n \]

by (6.111). If μ_{n-1} = c_5 + 1, (6.58) implies that

\[ β ≥ s_n · (c_5x_1 + (c_5 + 1)(y_1 + ⋯ + y_{n-1}) + y_n) \]
\[ = c_5x_1 + (c_5 + 1)(y_1 + ⋯ + y_{n-1}) + c_5y_n \]
\[ = (s_{n-1} ⋅ s_1) · (c_5x_1 + c_5(y_1 + ⋯ + y_n)), \]
Chapter 6  The set of elementary roots

and it can be easily verified that this is preceded by \( c_5 x_1 + c_5 (y_1 + \cdots + y_n) \).

Assume now that \( \mu_{n-1} \neq c_5 + 1 \); then (6.111) gives that \( n = 3 \) and

\[
\langle s_3 \cdot \beta \rangle = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + y_3.
\]

If \( \alpha = \beta = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + (c_5 + 1)y_3 \) there is nothing left to show, so suppose that \( \alpha \neq \beta \). Since \( \langle \beta, x_1 \rangle = 0 \) and \( \langle \beta, y_1 \rangle \) as well as \( \langle \beta, y_3 \rangle \) are positive, we deduce that \( \alpha \) must be a successor of \( s_2 \cdot \beta \). Now

\[
\langle s_2 \cdot \beta \rangle = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (2c_5 + 1)y_2 + (c_5 + 1)y_3
\]

and it can be easily verified that this is preceded by \( c_5 (x_3 + x_2 + x_1) + c_5 y_1 \).

Finally, suppose that the coefficient of \( y_n \) in \( s_n \cdot \beta \) equals 0. Then \( \mu_n \geq c_5 \) as \( \beta \in \mathcal{E}_\emptyset^n \); furthermore \( 0 = \mu_n - 2\langle \beta, y_n \rangle \), and as \( \beta \) cannot dominate \( y_n \), this yields that \( 1 > \langle \beta, y_n \rangle \geq \frac{c_5}{2} \), and thus \( \mu_n = c_5 \) by (2.26). Since \( s_n \cdot \beta \) is certainly an element of \( \mathcal{E}_\emptyset^n \setminus \langle s_n \rangle \), induction yields that either

\[
s_n \cdot \alpha \preceq c_5 x_1 + c_5 (y_1 + \cdots + y_{n-1}),
\]

or \( n = 4 \) and \( s_4 \cdot \beta = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + (c_5 + 1)y_3 \). The coefficient of \( y_{n-1} \) in \( s_n \cdot \beta \) equals \( c_5 \), and so the latter is impossible, while the former case together with (6.58) yield that

\[
\beta \preceq s_n \cdot (c_5 x_1 + c_5 (y_1 + \cdots + y_{n-1})) = c_5 x_1 + c_5 (y_1 + \cdots + y_n),
\]

and this finishes the proof.

Observe that (6.108) implies that \( c_5 x_1 + c_5 (y_1 + \cdots + y_n) \) is an elementary root. For \( \delta_1 = c_5 x_1 + c_1 y_1 \) is clearly elementary, and if we define \( \delta_j = s_j \cdot \delta_{j-1} \) for \( j \in \{2, \ldots, n\} \), then an easy calculation yields that

\[
\delta_j = c_5 x_1 + c_5 (y_1 + \cdots + y_j)
\]

and \( \langle \delta_{j-1}, y_j \rangle = -\frac{c_5}{2} \in (-1, 0) \), and thus \( \delta_n = c_5 x_1 + c_5 (y_1 + \cdots + y_n) \in \mathcal{E} \) by (6.108).

Next let \( \beta_1^{(1)} = \delta_n \), and for \( j \in \{1, \ldots, n-1\} \) define \( \beta_{j+1}^{(1)} = s_j \cdot \beta_j^{(1)} \). Then a straightforward calculation yields that

\[
\beta_j^{(1)} = c_5 x_1 + (c_5 + 1)(y_j + \cdots + y_{j-1}) + c_5 (y_j + \cdots + y_n),
\]

and \( \langle \beta_j^{(1)}, y_j \rangle = -\frac{c_5}{2} \in (-1, 0) \); hence (6.108) implies that \( \beta_j^{(1)} \in \mathcal{E} \) for all \( j \in \{1, \ldots, n\} \).

Note also that \( \beta_1^{(1)}, \ldots, \beta_n^{(1)} \) are the only elements of \( \mathcal{E}_{\{r_1\} \cup \{s_1, \ldots, s_n\}} \) with coefficient \( c_5 \) for \( x_1 \), and this enables us to prove the next result.
Suppose that \( m = 5 \) and \(|Y_1|, |Y_2| \geq 2\). Then \( \mathcal{E}_Y^\emptyset \) is the set of
\[
c_5 \alpha + (c_5 + 1)(x_{i-1} + \cdots + x_1) + c_5 \beta
\]
with \( i \in \{1, \ldots, l\} \), \( \alpha \in \mathcal{E}_{Y_1 \setminus \{r_1, \ldots, r_{i-1}\}} \) with coefficient 1 for \( x_i \), and \( \beta \in \mathcal{E}_{Y_2} \) with coefficient 1 for \( y_1 \), and
\[
c_5 \alpha + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5 \beta
\]
with \( j \in \{1, \ldots, n\} \), \( \alpha \in \mathcal{E}_{Y_1} \) with coefficient 1 for \( x_1 \) and \( \beta \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} \) with coefficient 1 for \( y_j \). Hence
\[
|\mathcal{E}_Y^\emptyset| = \left( \sum_{J \subseteq Y_1 \setminus \{r_1\}} |\mathcal{E}_Y^{J \cup \{r_1\}}| \right) \times \left( \sum_{K \subseteq Y_2 \setminus \{s_1\}} |\mathcal{E}_Y^{\{s_1\} \cup K}| \right)
\]
\[
\quad + \left( \sum_{i=2}^{l} \sum_{J \subseteq Y_1 \setminus \{r_1, \ldots, r_i\}} |\mathcal{E}_Y^{J \cup \{r_i\}}| \right) \times \left( \sum_{K \subseteq Y_2 \setminus \{s_1\}} |\mathcal{E}_Y^{\{s_1\} \cup K}| \right)
\]
\[
\quad + \left( \sum_{J \subseteq Y_1 \setminus \{r_1\}} |\mathcal{E}_Y^{J \cup \{r_1\}}| \right) \times \left( \sum_{j=2}^{n} \sum_{K \subseteq Y_2 \setminus \{s_1, \ldots, s_j\}} |\mathcal{E}_Y^{\{s_j\} \cup K}| \right).
\]

**Proof.** First, let \( \alpha \in \mathcal{E}_{Y_1} \) with coefficient 1 for \( x_1 \), \( j \in \{1, \ldots, n\} \) and \( \beta \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} \) with coefficient 1 for \( y_j \). Since
\[
c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5 y_j
\]
is in \( \mathcal{E}_{\{r_1, s_1, \ldots, s_j\}} \) by the remark preceding this proposition, (6.97) yields that
\[
c_5 \alpha + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5 y_j
\]
is an element of \( \mathcal{E}_{Y_1 \cup \{s_1, \ldots, s_j\}} \), and a repeated application of (6.97) implies that
\[
c_5 \alpha + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5 \beta
\]
is an element of \( \mathcal{E}_Y \), and hence clearly in \( \mathcal{E}_Y^\emptyset \). Symmetrical arguments apply for \( i \in \{1, \ldots, l\} \), \( \alpha \in \mathcal{E}_{Y_1 \setminus \{r_1, \ldots, r_{i-1}\}} \) with coefficient 1 for \( x_i \), and \( \beta \in \mathcal{E}_{Y_2} \) with coefficient 1 for \( y_1 \); hence it remains to show that all elements of \( \mathcal{E}_Y^\emptyset \) can be obtained in this way.
Let $\gamma \in \mathcal{E}_Y^\emptyset$. Since $|Y_1|, |Y_2| \geq 2$, Lemma (6.109) gives that the coefficient of $x_1$ or $y_1$ in $\gamma$ equals $c_5$, and by symmetry we may assume without loss of generality that the coefficient of $x_1$ in $\gamma$ equals $c_5$. Then

$$\gamma = c_5\alpha + \beta - c_5x_1$$

for some $\alpha \in \mathcal{E}_Y$ with coefficient 1 for $x_1$, and $\beta \in \mathcal{E}_{\{r_1\} \cup Y_2}$ with coefficient $c_5$ for $x_1$.

If $|Y_2| = n$, the remark preceding this proposition yields that $\beta$ equals $\beta_j^{(1)}$ for some $j \in \{1, \ldots, n\}$, and thus

$$\gamma = c_5\alpha + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n);$$

as $y_j + \cdots + y_n$ is certainly an element of $\mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_j-1\}}$ with coefficient 1 for $y_j$, it follows that $\gamma$ is of the required form.

Suppose next that $|Y_1| > n$. Then (6.99) yields that

$$\beta = \beta_1 + x_5\beta_2 - c_5y_j$$

for some $\beta_1 \in \mathcal{E}_{\{r_1, s_1, \ldots, s_j\}}$ with coefficient $c_5$ for $y_j$, and coefficient greater than or equal to 2 for $y_1, \ldots, y_{j-1}$, and $\beta_2 \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}$ with coefficient 1 for $y_j$. Since the coefficient of $x_1$ in $\beta_1$ must equal $c_5$, and the coefficients of $y_1, \ldots, y_{j-1}$ in $\beta$ have to be greater than or equal to 2, we find that

$$\beta_1 = c_5x_1 + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5y_j.$$

So

$$\gamma = c_5\alpha + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5\beta_2,$$

as required.

(6.115) **Corollary** Suppose that $m = 5$, $|Y_1| = l$ and $|Y_2| = n$. Then $\mathcal{E}_Y^\emptyset$ is the set of

$$c_5(x_i + \cdots + x_1) + (c_5 + 1)(x_{i-1} + \cdots + x_1) + c_5(y_1 + \cdots + y_n)$$

with $i \in \{1, \ldots, l\}$ and

$$c_5(x_i + \cdots + x_1) + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$$
with \( j \in \{1, \ldots, n\} \). Hence \(|E^\emptyset Y| = l + n - 1\).

(6.116) **Corollary** Suppose \( m = 5, |Y_1| = l \) and \(|Y_2| > n\). Then \( E^\emptyset Y \) is the set of

\[
c_5(x_1 + \cdots + x_i) + (c_5 + 1)(x_{i-1} + \cdots + x_1) + c_5\beta
\]

with \( i \in \{1, \ldots, l\} \) and \( \beta \in E_{Y_2} \) with coefficient 1 for \( y_1 \), and

\[
c_5(x_1 + \cdots + x_j) + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5\beta
\]

with \( j \in \{2, \ldots, n\} \) and \( \beta \in E_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} \) with coefficient 1 for \( y_j \). Hence

\[
|E^\emptyset Y| = l \times \sum_{K \subseteq Y_2 \setminus \{s_1\}} |E_{Y_2}^{\{s_1\} \cup K}| + \sum_{j=2}^{n} \sum_{K \subseteq Y_2 \setminus \{s_1, \ldots, s_j\}} |E_{Y_2}^{\{s_j\} \cup K}|.
\]

Assume from now on that \(|Y_1| = 1\). It is a tedious but finite task to verify that the next lemma lists all the roots in \( \Phi_Y \) preceded by

\[
c_5x_1 + c_5(y_1 + \cdots + y_n),
\]

or equal to \((2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + (c_5 + 1)y_3\) for \(|Y_2| = n \leq 3\), and that these are elementary roots.

(6.117) **Lemma** Suppose that \( m = 5, |Y_1| = l \) and \(|Y_2| = n\).

(i) If \( n = 1 \), then \( E^\emptyset Y = \{c_5x_1 + c_5y_1\} \).

(ii) If \(|Y_2| = n = 2\), then

\[
E^\emptyset Y = \{c_5x_1 + c_5(y_1 + y_2), c_5x_1 + (c_5 + 1)y_1 + c_5y_2, (c_5 + 1)x_1 + (c_5 + 1)y_1 + c_5y_2, (c_5 + 1)x_1 + 2c_5y_1 + c_5y_2\}.
\]

(iii) If \(|Y_2| = n = 3\),

\[
(2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + (c_5 + 1)y_3
\]

is an element of \( E^\emptyset Y \), and we denote this by means of the following diagram:
Chapter 6  The set of elementary roots

The remaining elements of \( \mathcal{E}_Y^0 \) are represented by the following diagrams:

\[
\begin{align*}
&\begin{array}{c}
\bullet \bullet \bullet \bullet \\
2c + 1 & 2c + 2 & c + 2 & c + 1
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \bullet \\
c & c & c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
c & c + 1 & c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
c + 1 & c + 1 & c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
c + 1 & 2c & c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
c + 1 & 2c & 2c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
c + 1 & 2c + 1 & 2c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
2c + 1 & 2c + 1 & 2c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
2c + 1 & 3c + 1 & 2c & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
2c + 1 & 2c + 2 & c + 1 & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
2c + 1 & 2c + 2 & c + 1 & c
\end{array}
\\
&\begin{array}{c}
\bullet \bullet \bullet \\
2c + 1 & 2c + 2 & c + 1 & c
\end{array}
\end{align*}
\]
Chapter 6  

The set of elementary roots

\[
\begin{array}{cccc}
2c + 2 & 3c + 1 & 2c & c \\
2c + 1 & 3c + 1 & 2c + 1 & c + 1 \\
5 & 5 \\
\end{array}
\]
Chapter 6  The set of elementary roots

135

Hence

\[ |E^\emptyset_Y| = \begin{cases} 
1 & \text{if } n = 1, \\
4 & \text{if } n = 2, \\
32 & \text{if } n = 3.
\end{cases} \] (6.118)

**Proposition** Suppose \( m = 5, |Y_1| = 1 \) and \(|Y_2| = n \geq 4\). Then \( E^\emptyset_Y \) consists of the following roots

\[ \beta^{(1)}_j = c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n), \]
with \( 1 \leq j \leq n \),

\[ \beta^{(2)}_{j,k} = (c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + (c_5 + 1)(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n), \]
with \( 1 \leq k < j \leq n \),

\[ \beta^{(3)}_{j,k} = (2c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + (c_5 + 1)(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n), \]
with \( 2 \leq k < j \leq n \),

\[ \beta^{(4)}_j = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 1)(y_2 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n), \]
with \( 3 \leq j \leq n \),

\[ \beta^{(5)}_{j,k} = (c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + 2c_5(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n), \]
with \( 1 \leq k < j \leq n \),

\[ \beta^{(6)}_{j,k} = (2c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + 2c_5(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n), \]
with \( 1 \leq k < j \leq n \).
Chapter 6  The set of elementary roots

with \(2 \leq k < j \leq n\),

\[
\beta_j^{(7)} = (2c_5 + 1)x_1 + (3c_5 + 1)y_1 + 2c_5(y_2 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n),
\]

with \(3 \leq j \leq n\),

\[
\beta_j^{(8)} = (2c_5 + 2)x_1 + (3c_5 + 1)y_1 + 2c_5(y_2 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n),
\]

for \(3 \leq j \leq n\), and

\[
\beta_i^{(9)} = \alpha_i + c_5(y_4 + \cdots + y_n),
\]

where \(i \in \{1, \ldots, 5\}\) and

\[
\begin{align*}
\alpha_1 &= (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (2c_5 + 1)y_2 + c_5y_3, \\
\alpha_2 &= (2c_5 + 1)x_1 + (3c_5 + 1)y_1 + (2c_5 + 1)y_2 + c_5y_3, \\
\alpha_3 &= (2c_5 + 2)x_1 + (3c_5 + 1)y_1 + (2c_5 + 1)y_2 + c_5y_3, \\
\alpha_4 &= (2c_5 + 1)x_1 + (3c_5 + 2)y_1 + (2c_5 + 1)y_2 + c_5y_3, \\
\alpha_5 &= (3c_5 + 1)x_1 + (3c_5 + 2)y_1 + (2c_5 + 1)y_2 + c_5y_3.
\end{align*}
\]

Hence \(|E_Y^0| = 2n^2 + 1\).

**Proof.** We show first that the above listed vectors are elementary roots, and it follows trivially that they are in \(E_Y^0\). We saw before that \(\beta_j^{(1)}\) is elementary for \(j \in \{1, \ldots, n\}\). Since \(\beta_j^{(2)} = r_j \cdot \beta_j^{(1)}\) and \(\langle \beta_j^{(1)}, x_1 \rangle = -\frac{1}{2} \in (-1, 0)\) for \(j \in \{2, \ldots, n\}\), it follows by (3.37) that \(\beta_j^{(2)}\) is elementary. Furthermore, \(\beta_{j,k+1}^{(2)} = s_k \cdot \beta_{j,k}^{(2)}\) and \(\langle \beta_{j,k}^{(2)}, y_k \rangle = -\frac{c_5}{2}\) for \(k \in \{1, \ldots, j-2\}\), and thus (6.108) implies that \(\beta_{j,k}^{(2)}\) is elementary for \(k \in \{1, \ldots, j-1\}\), as required.

If \(j \in \{3, \ldots, n\}\) and \(k \in \{2, \ldots, j-1\}\), then \(\beta_{j,k}^{(3)} = r_j \cdot \beta_{j,k}^{(2)}\) and \(\langle \beta_{j,k}^{(2)}, x_1 \rangle = -\frac{c_5}{2}\), and it follows by (3.37) that \(\beta_{j,k}^{(3)} \in \mathcal{E}\). Next, \(\beta_j^{(4)} = s_1 \cdot \beta_{2,j}^{(3)}\) and \(\langle \beta_{2,j}^{(3)}, y_1 \rangle = -\frac{1}{2}\) for \(j \in \{3, \ldots, n\}\); therefore \(\beta_j^{(4)}\) is elementary for all \(j \in \{3, \ldots, n\}\) by (3.37).
Clearly $\beta_{2,1}^{(5)} = s_1 \cdot \beta_{2,1}^{(2)}$ and $\langle \beta_{2,1}^{(2)}, y_1 \rangle = \frac{1-c_5}{2}$, and (3.37) yields that $\beta_{2,1}^{(5)}$ is elementary. For $j \in \{2, \ldots, n-1\}$ we find that $\beta_{j+1,1}^{(5)} = s_j \cdot \beta_{j,1}^{(5)}$ and $\langle \beta_{j,1}^{(5)}, y_j \rangle = \frac{1-c_5}{2}$; hence $\beta_{j,1}^{(5)}$ is elementary for all $j \in \{2, \ldots, n\}$ by (6.108).

Further, $\beta_{j,k+1}^{(5)} = s_k \cdot \beta_{j,k}^{(2)}$ and $\langle \beta_{j,k}^{(2)}, y_k \rangle = -\frac{1}{2}$ for $k \in \{1, \ldots, j-2\}$, therefore $\beta_{j,k}^{(5)}$ is elementary for all $k \in \{1, \ldots, j-1\}$ by (6.108).

If $j \in \{3, \ldots, n\}$ and $k \in \{2, \ldots, j-1\}$, then $\beta_{j,k}^{(6)} = r_1 \cdot \beta_{j,k}^{(5)}$ and $\langle \beta_{j,k}^{(5)}, x_1 \rangle = -\frac{c_5}{2}$, and it follows by (3.37) that $\beta_{j,k}^{(6)}$ is elementary.

Next, $\beta_{j}^{(7)} = s_1 \cdot \beta_{j,2}^{(6)}$ and $\langle \beta_{j,2}^{(6)}, y_1 \rangle = -\frac{c_5}{2}$ for $j \in \{3, \ldots, n\}$, and thus $\beta_{j}^{(7)} \in \mathcal{E}$ by (3.37). For $j \in \{3, \ldots, n\}$ also $\beta_{j}^{(8)} = r_1 \cdot \beta_{j}^{(7)}$, and $\langle \beta_{j}^{(7)}, y_1 \rangle = -\frac{1}{2}$, and hence $\beta_{j}^{(8)}$ is clearly elementary by (3.37).

Finally, since $y_3 + \cdots + y_n$ is in $\mathcal{E}_{\{s_3, \ldots, s_n\}}$ with coefficient 1 for $y_3$, and $\alpha_i \in \mathcal{E}_{\{r_1, s_1, s_2, s_3\}}$ by (6.117)(iii) with coefficient $c_5$ for $y_3$, Proposition (6.97) yields furthermore that $\beta_{i}^{(9)}$ is elementary for $i \in \{1, \ldots, 5\}$.

It remains to show that all elements of $\mathcal{E}_{Y}^\emptyset$ have been listed. Suppose $\alpha \in \mathcal{E}_{Y}^\emptyset$; then $\alpha \succeq \beta_{1}^{(1)}$ by (6.113) as $n \geq 4$. If $\alpha = \beta_{1}^{(1)}$, there is nothing left to show. So assume next that $\alpha \succ \beta_{1}^{(1)}$, and proceed by induction. Let $r \in Y$ with $\alpha \succ r \cdot \alpha \succeq \beta_{1}^{(1)}$; then $\langle r \cdot \alpha, \alpha \rangle \in (-1, 0)$, as $\alpha$ is elementary. Furthermore, $r \cdot \alpha$ is elementary, and since $r \cdot \alpha \succeq \beta_{1}^{(1)}$ clearly $r \cdot \alpha \in \mathcal{E}_{Y}^\emptyset$. By induction this gives rise to the following cases:

Case 1: $r \cdot \alpha = c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$

  for some $j \in \{1, \ldots, n\}$; then

  (i) $r = r_1$ and $j \geq 2$, and thus $\alpha = \beta_{j,2}^{(2)}$, or

  (ii) $r = s_j$ and $j \leq n-1$, hence $\alpha = \beta_{j+1}^{(1)}$.

Case 2: $r \cdot \alpha = (c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1})$

  $+ (c_5 + 1)(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$

  for some $1 \leq k < j \leq n$; then

  (i) $r = r_1$ and $k \geq 2$, and hence $\alpha = \beta_{j,k}^{(3)}$, or

  (ii) $r = s_1$ with $j = 2$ and $k = 1$, and $\alpha = \beta_{2,1}^{(5)}$, or

  (iii) $r = s_k$ with $k \leq j-2$, and $\alpha = \beta_{j,k+1}^{(2)}$, or
(iv) \( r = s_j \) with \( j < n \), and thus \( \alpha = \beta_{j+1,k}^{(2)} \).

Case 3: \( r \cdot \alpha = (2c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + (c_5 + 1)(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n) \)
for some \( 2 \leq k < j \leq n \); then

(i) \( r = s_1 \) and \( k = 2 \), and thus \( \alpha = \beta_j^{(4)} \), or

(ii) \( r = s_k \) and \( k \leq j - 2 \), and \( \alpha = \beta_{j,k+1}^{(3)} \), or

(iii) \( r = s_j \) and \( j < n \), and thus \( \alpha = \beta_{j+1,k}^{(3)} \).

Case 4: \( r \cdot \alpha = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 1)(y_2 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n) \)
for some \( j \in \{3, \ldots, n\} \); then \( r = s_2 \) with \( j = 3 \), and \( \alpha = \beta_4^{(9)} \).

Case 5: \( r \cdot \alpha = (c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + 2c_5(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n) \)
for some \( 1 \leq k < j \leq n \); then

(i) \( r = r_1 \) and \( k \geq 2 \), and hence \( \alpha = \beta_{j,k}^{(6)} \), or

(ii) \( r = s_k \) with \( k \leq j - 2 \), and \( \alpha = \beta_{j,k+1}^{(5)} \), or

(iii) \( r = s_j \) and \( j < n \), and thus \( \alpha = \beta_{j+1,k}^{(5)} \).

Case 6: \( r \cdot \alpha = (2c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + 2c_5(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n) \)
for some \( 1 \leq k < j \leq n \); then

(i) \( r = s_1 \) and \( k = 1 \), and \( \alpha = \beta_j^{(7)} \), or

(ii) \( r = s_k \) with \( k \leq j - 2 \), and \( \alpha = \beta_{j,k+1}^{(6)} \), or

(iii) \( r = s_j \) and \( j < n \), and thus \( \alpha = \beta_{j+1,k}^{(6)} \).

Case 7: \( r \cdot \alpha = (2c_5 + 1)x_1 + (3c_5 + 1)y_1 + 2c_5(y_2 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n) \),
for some \( j \in \{3, \ldots, n\} \); then

(i) \( r = r_1 \) and \( \alpha = \beta_j^{(8)} \), or

(ii) \( r = s_j \) and \( j < n \), and hence \( \alpha = \beta_{j+1}^{(7)} \).
Chapter 6  The set of elementary roots

139

Case 8: \( r \cdot \alpha = (2c_5 + 2)x_1 + (3c_5 + 1)y_1 + 2c_5(y_2 + \cdots + y_j) + c_5(y_j + \cdots + y_n) \)
for some \( j \in \{3, \ldots, n\} \); then \( r = s_2 \) and \( j = 3 \), and \( \alpha = \beta_3^{(9)} \).

Case 9: \( r \cdot \alpha = \alpha_i + c_5(y_4 + \cdots + y_n) \) for some \( i \in \{1, \ldots, 5\} \); then

(i) \( i = 1 \), \( r = s_1 \) and \( \alpha = \beta_2^{(9)} \), or
(ii) \( i = 2 \), \( r = r_1 \) and \( \alpha = \beta_3^{(9)} \), or
(iii) \( i = 3 \), \( r = s_1 \) and \( \alpha = \beta_4^{(9)} \), or
(iv) \( i = 4 \), \( r = r_1 \) and \( \alpha = \beta_5^{(9)} \),
and this completes the proof. □

Lemma (6.99) now yields the following.

(6.119) PROPOSITION Suppose \( m = 5 \), \( |Y_1| = 1 \) and \( |Y_2| > n \).

(i) If \( n = 1 \), the elements of \( \mathcal{E}_Y^0 \) are

\[ c_5x_1 + c_5\alpha \]

with \( \alpha \in \mathcal{E}_{Y_2} \) with coefficient 1 for \( y_1 \).

(ii) If \( n = 2 \), the elements of \( \mathcal{E}_Y^0 \) are the roots in (i) plus, additionally,

\[ \beta - c_5y_2 + c_5\alpha, \]

with \( \alpha \in \mathcal{E}_{Y_2 \setminus \{s_1\}} \) with coefficient 1 for \( y_2 \), and \( \beta \) equal to one of the 3 roots in \( (6.117)(ii) \) with coefficient \( c_5 \) for \( y_2 \) and coefficient greater than or equal to 2 for \( y_1 \).

(iii) If \( n = 3 \), the elements of \( \mathcal{E}_Y^0 \) are the roots named in (ii), plus, additionally, the roots

\[ \beta + c_5\alpha - c_5y_3 \]

with \( \alpha \in \mathcal{E}_{Y_2 \setminus \{s_1,s_2\}} \) with coefficient 1 for \( y_3 \), and \( \beta \) equal to one of the 15 roots in \( (6.117)(iii) \) with coefficient \( c_5 \) for \( y_3 \) and coefficient greater than or equal to 2 for \( y_1 \) and \( y_2 \).

(iv) If \( n \geq 4 \), the elements of \( \mathcal{E}_Y^0 \) are the roots named in (iii), plus, additionally,

\[ \beta - c_5(y_j + \cdots + y_n) + c_5\alpha, \]
with $j \in \{4, \ldots, n\}$, $\alpha \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}$ with coefficient 1 for $y_j$, and $\beta$ equal to $\beta_j^{(1)}$, $\beta_j^{(4)}$, $\beta_j^{(7)}$, $\beta_j^{(8)}$, or $\beta_k^{(2)}$, $\beta_{k,j}^{(5)}$ with $k \in \{1, \ldots, j-1\}$, or $\beta_k^{(3)}$, $\beta_k^{(6)}$ with $k \in \{2, \ldots, j-1\}$ as defined in (6.118).

Hence

$$|\mathcal{E}_{Y}^\emptyset| = \sum_{j=1}^{n} M(j) \sum_{J \subseteq Y_2 \setminus \{s_1, \ldots, s_{j-1}\}} |\mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}^{\{s_j\} \cup J}|,$$

with

$$M(j) = \begin{cases} 
1 & \text{if } j = 1, \\
3 & \text{if } j = 2, \\
15 & \text{if } j = 3, \\
4j - 2 & \text{if } j \geq 4. 
\end{cases}$$
Chapter 6  The set of elementary roots  141

Bibliography

[1] N. Bourbaki
*Groupes et Algebres de Lie, Chapitres 4, 5 et 6*

‘Coxeter groups are automatic’
*preprint, Ohio State University* (1991)

[3] V. V. Deodhar
‘Some Characterizations of Coxeter Groups’

[4] V. V. Deodhar
‘A Note on Subgroups Generated by Reflections in Coxeter Groups’

‘Reflection Subgroups of Coxeter Systems’

M. S. Paterson and W. P. Thurston
*Word Processing in Groups*

*Reflection Groups and Coxeter Groups*

[8] W. S. Massey
*Algebraic Topology: An Introduction*
Springer-Verlag (1967).
Contents

Statement of Authorship iii

Acknowledgements iv

Introduction v

Chapter 1: Preliminaries . . . . . . . . . . . . . 1

Chapter 2: The Depth of a Root . . . . . . . 14

Chapter 3: Dominance and Elementary Roots . . . . 26

Chapter 4: The Stabilizer of a Root . . . . . . . 41

Chapter 5: An Automatic Structure . . . . . . . 48

Chapter 6: The Set of Elementary Roots . . . . . 59

§6a Simple bonds only 66

§6b One non-simple bond 94

Bibliography 141