On Root Systems and Automaticity of Coxeter Groups

by

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Statement of Authorship

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other University. The material presented here is believed to be original, with the exception of standard general results in Chapter 1, and other cases where due attribution is given.

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Introduction

The aim of this thesis is to examine root systems associated with Coxeter groups. We introduce the notions of dominance and elementary roots, and employ them to show that the stabilizer of a root is the semidirect product of a Coxeter group and a free group, as well as to obtain an automatic structure for finitely generated Coxeter groups.

The structure of this thesis is as follows. We begin in Chapter 1 by reviewing some well known facts about Coxeter groups and root systems. In Chapter 2 a function from the root system to the integers is defined, which is an analogue of the length function on the Coxeter group, and enables us to use inductive proofs on the root system. This is then applied to give an alternative proof of the Theorem, due to Deodhar [4] and Dyer [5], that each reflection subgroup of a Coxeter group is itself a Coxeter group; furthermore, we derive certain properties relating to the coefficients which occur when roots are expressed as linear combinations of simple roots.

In Chapter 3 the concepts of dominance and elementary roots are introduced, and we prove the principal result of this thesis, namely, that the set of elementary roots is finite (provided the Coxeter group has finite rank). We also prove the important technical result that if $r_1 r_2 \cdots r_l$ is a reduced expression for an element of the Coxeter group having the property that $(r_1 r_2 \cdots r_l) \cdot \alpha = \beta$ for some simple roots α and β, then $(r_i r_{i+1} \cdots r_l) \cdot \alpha$ is elementary for all i. This is then used in the subsequent chapters to deal with the stabilizer of a root and the automaticity of Coxeter groups respectively.

Finally, in Chapter 6 we use the properties of elementary roots obtained in Chapter 3 to give an explicit description of the set of elementary roots in every case.

In the interest of keeping this thesis as self-contained as possible, proofs of various well-known properties of Coxeter groups are included in the early chapters.

Chapter 1

Preliminaries

In this chapter some well known properties of Coxeter groups are presented (see [1], [3] or [7]). We begin by introducing some notation. The set of real numbers will be denoted by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, and the set of nonnegative integers by $\mathbb{N}_0$. For any set M, we denote the cardinality of M by $|M|$; where appropriate, $-M$ denotes the set $\{\,-x \mid x \in M\,\}$. For sets M and N, the *set difference* of M and N will be denoted by $M \setminus N$.

Throughout this thesis, W is a *Coxeter group* with *distinguished generating set* R; that is, W has a presentation

$$\big\langle\, r \in R \mid (rs)^{m_{rs}} = 1 \text{ for } r,\ s \in R \,\big\rangle,$$

where $m_{rr} = 1$ for all $r \in R$, and $m_{rs} = m_{sr} \geq 2$ or $m_{rs} = m_{sr} = \infty$ for $r,\ s \in R$ with $r \neq s$. (Here $(rs)^\infty = 1$ is regarded as vacuously true).

The *Coxeter graph* of W has vertex set in one-one correspondence with R, and two distinct vertices corresponding to r and s are joined by an edge or *bond* of weight m_{rs} if $m_{rs} \neq 2$. For convenience of notation we frequently identify $r \in R$ with the vertex corresponding to r. If r and s are joined by an edge, r and s are said to be *adjoined*, and the edge is labelled by m_{rs}; if $m_{rs} = 3$ this label is suppressed. We say that the bond adjoining r and s is *simple*, *non-simple*, *infinite* if $m_{rs} = 3$, $m_{rs} \neq 3$ and $m_{rs} = \infty$ respectively. By abuse of notation, $S \subseteq R$ will denote both a subset of R, and the subgraph of the Coxeter graph consisting of the vertices in S and the bonds adjoining them.

For $w \in W$, define the *length* $l(w)$ of w by

$$l(w) = \min\{\, \mathrm{l} \in \mathbb{N}_0 \mid \mathrm{w} = \mathrm{r}_1 \cdots \mathrm{r}_\mathrm{l} \text{ for some } \mathrm{r}_1, \ldots, \mathrm{r}_\mathrm{l} \in \mathrm{R} \,\}.$$

By definition of W, all elements of R are self inverse, and hence R is closed under taking inverses. Thus $l(w^{-1}) = l(w)$ and $l(w) - 1 \leq l(wr) \leq l(w) + 1$ for all $w \in W$ and $r \in R$; moreover, if w is an element of W with $l(w) \geq 1$, then there exists an $r \in R$ with $l(wr) < l(w)$; that is, $l(wr) = l(w) - 1$.

Let V be an $\mathbb{R}$-vector space with basis Π in one-one correspondence with R, and for $r \in R$ denote the basis element corresponding to r by α_r. The vertex set of the Coxeter graph is by construction in one-one correspondence with Π, and as above for r, we will frequently identify α_r with the vertex corresponding to α_r; moreover, we use $\Pi' \subseteq \Pi$ both to denote a subset of Π, and the subgraph of the Coxeter graph consisting of the vertices in Π' and the bonds adjoining them.

Next, let $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ be a symmetric bilinear form which satisfies $\langle \alpha_r, \alpha_s \rangle = -\cos(\pi/m_{rs})$ for all r, $s \in R$ with m_{rs} finite, and $\langle \alpha_r, \alpha_s \rangle \leq -1$ for r, $s \in R$ with m_{rs} infinite; (in particular, $\langle \alpha_r, \alpha_r \rangle = 1$). Observe that $\langle\ ,\ \rangle$ is uniquely determined by the presentation of W if and only if there are no infinite bonds in the Coxeter graph of W.

For $r \in R$ define $\rho_r : V \to V$ by $\rho_r(v) = v - 2\langle v, \alpha_r \rangle \alpha_r$. Then

$$\begin{aligned}
\rho_r^2(v) &= \rho_r\big(v - 2\langle v, \alpha_r \rangle \alpha_r\big) \\
&= \big(v - 2\langle v, \alpha_r \rangle \alpha_r\big) - 2\big\langle v - 2\langle v, \alpha_r \rangle \alpha_r, \alpha_r \big\rangle \alpha_r \\
&= v - 2\langle v, \alpha_r \rangle \alpha_r - 2\big(\langle v, \alpha_r \rangle - 2\langle v, \alpha_r \rangle \langle \alpha_r, \alpha_r \rangle\big)\alpha_r \\
&= v + \big(-2\langle v, \alpha_r \rangle - 2\langle v, \alpha_r \rangle + 4\langle v, \alpha_r \rangle\big)\alpha_r \\
&= v
\end{aligned}$$

for all $v \in V$; furthermore, for v, $v' \in V$,

$$\begin{aligned}
&\langle \rho_r(v), \rho_r(v') \rangle \\
&\quad = \big\langle v - 2\langle v, \alpha_r \rangle \alpha_r\, ,\, v' - 2\langle v', \alpha_r \rangle \alpha_r \big\rangle \\
&\quad = \langle v, v' \rangle - 2\langle v', \alpha_r \rangle \langle v, \alpha_r \rangle - 2\langle v, \alpha_r \rangle \langle \alpha_r, v' \rangle + 4\langle v, \alpha_r \rangle \langle v', \alpha_r \rangle \langle \alpha_r, \alpha_r \rangle \\
&\quad = \langle v, v' \rangle - 4\langle v', \alpha_r \rangle \langle v, \alpha_r \rangle + 4\langle v, \alpha_r \rangle \langle v', \alpha_r \rangle \\
&\quad = \langle v, v' \rangle.
\end{aligned}$$

So ρ_r is self inverse and preserves the bilinear form, and is thus an element of O(V), the orthogonal group of the bilinear form $\langle\ ,\ \rangle$ on V.

If r, $s \in R$ are distinct, it can be easily seen that ρ_r and ρ_s preserve the space spanned by α_r and α_s. Denote $2\langle \alpha_r, \alpha_s \rangle$ by c; then the matrices of ρ_r, ρ_s and $\rho_r\rho_s$ on this space with respect to the basis α_r, α_s are

$$\begin{pmatrix} -1 & -c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -c & -1 \end{pmatrix} \text{ and } \begin{pmatrix} c^2 - 1 & c \\ -c & -1 \end{pmatrix} \text{ respectively.}$$

The proof of the following assertion is a straightforward induction on n, and will be omitted.

(1.1) Lemma *Let α, $\beta \in V$ be linearly independent, and let A be a linear map on the span of α and β with matrix*

$$\begin{pmatrix} c^2 - 1 & c \\ -c & -1 \end{pmatrix}$$

with respect to the basis α, β. For $n \in \mathbb{N}$, let $A^n(\alpha) = \lambda_n \alpha + \mu_n \beta$ with $\lambda_n, \mu_n \in \mathbb{R}$. Then $A^n(\beta) = -\mu_n \alpha - \lambda_{n-1}\beta$ and

(i) if $c = -2\cos(\theta)$ for some $\theta \in (0, \pi)$,

$$\lambda_n = \frac{\sin\big((2n+1)\theta\big)}{\sin(\theta)} \text{ and } \mu_n = \frac{\sin\big(2n\theta\big)}{\sin(\theta)};$$

(ii) if $c \leq -2$, then $\lambda_n \geq \mu_n + 1$ and $\mu_{n+1} \geq \lambda_n + 1$.

Lemma (1.1) enables us to prove the following proposition.

(1.2) Proposition *There is a representation ρ of W on V with $\rho(r) = \rho_r$ for all $r \in R$.*

Note that (1.1) yields for r, $s \in R$ that $\rho_r \rho_s$ has order at least m_{rs}. So when (1.2) is established, it will follow that rs has order at least m_{rs}. If m_{rs} is finite, rs has order at most m_{rs} by definition of W, therefore m_{rs} is in fact the order of rs for all r, $s \in R$.

Proof of (1.2). Let $F(R)$ be the free group on R; the map $r \mapsto \rho_r$ from R to O(V) extends to a homomorphism $F(R) \to$ O(V). Further, let N be the normal closure in $F(R)$ of the set

$$\{\, (rs)^{m_{rs}} \mid r,\, s \in R \text{ with } m_{rs} < \infty \,\},$$

so that W is isomorphic to the quotient $F(R)/N$. It suffices to show that N is in the kernel of the above homomorphism from $F(R)$ to O(V). This is the case if $(\rho_r\rho_s)^{m_{rs}}$ equals the identity for all r, $s \in R$ with m_{rs} finite. By an earlier remark, ρ_r^2 equals the identity on V for all $r \in R$. It remains to show that $(\rho_r\rho_s)^m$ is the identity for all r, $s \in R$ with $r \neq s$ and $m = m_{rs}$ finite. Since Π is a basis for V, it suffices to show $(\rho_r\rho_s)^m(\alpha_t) = \alpha_t$ for all $\alpha_t \in \Pi$. If $t \in \{r, s\}$, this is true by (1.1)(i) with $\theta = \pi/m$ and $n = m$. So suppose that $\alpha_r, \alpha_s, \alpha_t$ are linearly independent. The space spanned by these vectors is obviously invariant under $\rho_r\rho_s$, and if M is the matrix corresponding to

$\rho_r\rho_s$ on this space with respect to the basis $\alpha_r, \alpha_s, \alpha_t$, it suffices to show that M^m is the 3×3 identity matrix. A short calculation yields that

$$M = \begin{pmatrix} A & b \\ {\scriptstyle 0\,0} & {\scriptstyle 1} \end{pmatrix},$$

where A is the 2×2 matrix corresponding to $\rho_r\rho_s$ on the span of α_r and α_s with respect to the basis α_r, α_s, and b is a 2×1 vector. An easy induction yields that

$$M^n = \begin{pmatrix} A^n & b_n \\ {\scriptstyle 0\,0} & {\scriptstyle 1} \end{pmatrix}$$

for $n \in \mathbb{N}$, where $b_n = (A^0 + \ldots + A^{n-1})b$. As A^m is the matrix of $(\rho_r\rho_s)^m$ on the span of α_r and α_s with respect to the basis α_r, α_s, (1.1)(i) for $\theta = \pi/m$ and $n = m$ gives that A^m equals I_2, the 2×2 identity matrix. Thus

$$(A - I_2)(A^0 + \ldots + A^{m-1}) = A^m - I_2 = \underline{0},$$

the 2×2 zero matrix. The determinant of $A - I_2$ equals $4\sin^2(\pi/m)$, which is nonzero, and so $A - I_2$ is invertible. Hence $A^0 + \ldots + A^{m-1} = \underline{0}$ and b_m is the 2×1 zero vector. This yields that M^m is the 3×3 identity matrix, and we have in fact a homomorphism $\rho\colon W \to \mathrm{O(V)}$. □

The representation ρ of W on V defined in (1.2) will be called a *standard geometric realization* of W. From now on we consider the action of W on V induced by a standard geometric realization of W on V. We have seen that this action preserves $\langle\ ,\ \rangle$. For $w \in W$ and $v \in V$, we denote the image of v under $\rho(w)$ by $w \cdot v$. The set $\Phi = \{\, w \cdot \alpha_r \mid w \in W,\ r \in R \,\}$ is called the *root system* of W in V, and the elements of Π are the *simple roots*. Note that since $\langle \alpha_r, \alpha_r \rangle = 1$ for $r \in R$, we have $\langle \alpha, \alpha \rangle = 1$ for all $\alpha \in \Phi$.

Every element v of V can be uniquely written as $\sum_{\alpha\in\Pi} \lambda_\alpha \alpha$ for some $\lambda_\alpha \in \mathbb{R}$, and λ_α is said to be the *coefficient* of α in v. Define the *support* of v to be the set of $\alpha \in \Pi$ such that the coefficient of α in v is nonzero, and denote this set by $\mathrm{supp}(v)$. Furthermore, let $I(v)$ denote the set of $r \in R$ in one-one correspondence with elements of $\mathrm{supp}(v)$. The set $PLC(\Pi)$ of *positive linear combinations* of Π is the set of vectors in V with all coefficients greater than or equal to 0. The sets of *positive roots* Φ^+ and *negative roots* Φ^- are defined to be $\Phi^+ = \Phi \cap PLC(\Pi)$ and $\Phi^- = -\Phi^+$ respectively.

(1.3) Lemma *Suppose $w \in W$ and $r \in R$ with $l(wr) \geq l(w)$. Then $w \cdot \alpha_r \in \Phi^+$.*

Proof. If $l(w) = 0$, then $w = 1$ and $w \cdot \alpha_r = \alpha_r$ is in Φ^+. Next, suppose that $l(w) \geq 1$, and let $s \in R$ with $l(ws) = l(w) - 1$; then $r \neq s$, since $l(wr) \geq l(w)$. Set $I = \{r, s\}$, and let W_I denote the subgroup of W generated by I. Furthermore, let l_I be the length function on W_I with respect to I. Since $I \subseteq R$, it is clear that $l(z) \leq l_I(z)$ for all $z \in W_I$.

Consider the set

$$A = \{\, u \in W \mid u^{-1}w \in W_I \text{ and } l(u) + l_I(u^{-1}w) = l(w) \,\}.$$

Then $ws \in A$, since $(ws)^{-1}w = s^{-1} = s \in W_I$ and

$$l(ws) + l_I\big((ws)^{-1}w\big) = l(ws) + 1 = l(w).$$

So A is non-empty, and if we let x be an element of A of minimal length, then $l(x) \leq l(ws)$, and thus $l(x) \leq l(w) - 1 < l(w)$. Assume for a contradiction that $l(xr) < l(x)$; that is, $l(xr) = l(x) - 1$. Then $xr \notin A$ by minimality of x. On the other hand, $rx^{-1}w \in W_I$ and

$$\begin{aligned} l(w) &= l\big((xr)(rx^{-1}w)\big) \leq l(xr) + l(rx^{-1}w) \\ &\leq l(xr) + l_I(rx^{-1}w) \leq l(x) - 1 + l_I(x^{-1}w) + 1 \\ &= l(x) + l_I(x^{-1}w) = l(w), \end{aligned}$$

and we must have equality everywhere; hence $l(w) = l(xr) + l_I(rx^{-1}w)$, which implies $xr \in A$, a contradiction. So $l(xr) \geq l(x)$ and similarly $l(xs) \geq l(x)$, and by induction $x \cdot \alpha_r$ and $x \cdot \alpha_s$ are both positive roots.

Let $y = x^{-1}w$; then

$$l(x) + l_I(y) = l(w) \leq l(wr) = l(xyr) \leq l(x) + l(yr) \leq l(x) + l_I(yr),$$

and hence $l_I(yr) \geq l_I(y)$. Therefore y equals $(rs)^n$ or $s(rs)^n$ for some $n \in \mathbb{N}_0$. If rs has infinite order, $(rs)^n \cdot \alpha_r = \lambda\alpha_r + \mu\alpha_s$ for some $\lambda \geq \mu \geq 0$ by (1.1)(ii), and hence $s(rs)^n \cdot \alpha_r = \lambda\alpha_r + \mu'\alpha_s$ for some $\mu' \geq \lambda \geq 0$. If rs has order m, then $l_I(yr) \geq l_I(y)$ yields that $2n+1 \leq m$ if $y = (rs)^n$, and that $2(n+1) \leq m$ if $y = s(rs)^n$, and we deduce from (1.1)(i) that $y \cdot \alpha_r = \lambda\alpha_r + \mu\alpha_s$ for some $\lambda, \mu \geq 0$. So in any case,

$$w \cdot \alpha_r = (xy) \cdot \alpha_r = x \cdot (\lambda\alpha_r + \mu\alpha_s) = \lambda(x \cdot \alpha_r) + \mu(x \cdot \alpha_s)$$

for some λ, $\mu \geq 0$. Since $x \cdot \alpha_r$, $x \cdot \alpha_s \in PLC(\Pi)$, it follows that $w \cdot \alpha_r$ is in $PLC(\Pi)$, and thus in Φ^+, as required. $\square$

It is clear that Φ^+ is a subset of Φ, and since $r \cdot \alpha_r = -\alpha_r$ for $r \in R$, also $\Phi^- \subseteq \Phi$. In fact,

$$\Phi = \Phi^+ \cup \Phi^-.$$

For if $\alpha \in \Phi$, then by definition of Φ there exists a $w \in W$ and an $r \in R$ with $\alpha = w \cdot \alpha_r$. If $l(wr) \geq l(w)$, then $\alpha = w \cdot \alpha_r \in \Phi^+$ by the previous lemma, while if $l(wr) < l(w) = l\big((wr)r\big)$, then $(wr) \cdot \alpha_r \in \Phi^+$ by (1.3) with wr in place of w; hence $w \cdot \alpha_r = -(wr) \cdot \alpha_r \in \Phi^-$.

A trivial, but very useful, consequence of this is the following result.

(1.4) Corollary $\quad r \cdot \big(\Phi^+ \setminus \{\alpha_r\}\big) \subseteq \Phi^+$ *for* $r \in R$.

Next, let $w \in W$ and $r \in R$. By definition of the root system, $w \cdot \alpha_r$ is a root, and since Φ equals the union of Φ^+ and Φ^-, we find that $w \cdot \alpha_r$ is either positive or negative. If $w \cdot \alpha_r \in \Phi^-$, then $l(wr) \not\geq l(w)$ by (1.3); hence $l(wr) < l(w)$, that is, $l(wr) = l(w) - 1$. If $w \cdot \alpha_r \in \Phi^+$, then $(wr) \cdot \alpha_r \in \Phi^-$, and thus

$$l(w) = l\big((wr)r\big) = l(wr) - 1$$

by the previous case with wr in place of w; whence $l(wr) = l(w) + 1$.

(1.5) Lemma $\quad$ *For all* $w \in W$ *and* $r \in R$,

$$l(wr) = \begin{cases} l(w) + 1 & \textit{if } w \cdot \alpha_r \in \Phi^+, \\ l(w) - 1 & \textit{if } w \cdot \alpha_r \in \Phi^-. \end{cases}$$

We can now deduce that the standard geometric realization of W on V induces a faithful action of W on V. For if $w \in W$ such that $w \cdot v = v$ for all $v \in V$, then in particular, $w \cdot \alpha_r = \alpha_r \in \Phi^+$ for all $r \in R$; that is, $l(wr) > l(w)$ for all $r \in R$, and thus $w = 1$.

It is clear that Φ is finite if W is finite. Faithfulness of the action of W on V yields the converse, and so Φ is finite if and only if W is finite.

Next, let $w \in W$ and $r,\, s \in R$ with $w \cdot \alpha_r = \alpha_s$; then for all $v \in V$,

$$\begin{aligned}
(wrw^{-1}s) \cdot v &= \big(wrw^{-1}\big) \cdot \big(v - 2\langle v, \alpha_s\rangle \alpha_s\big) \\
&= (wr) \cdot \big(w^{-1} \cdot v - 2\langle v, \alpha_s\rangle w^{-1} \cdot \alpha_s\big) \\
&= (wr) \cdot \big(w^{-1} \cdot v - 2\langle v, \alpha_s\rangle \alpha_r\big) \\
&= w \cdot \big(w^{-1} \cdot v - 2\langle w^{-1} \cdot v, \alpha_r\rangle \alpha_r + 2\langle v, \alpha_s\rangle \alpha_r\big) \\
&= w \cdot \big(w^{-1} \cdot v - 2\langle v, w \cdot \alpha_r\rangle \alpha_r + 2\langle v, \alpha_s\rangle \alpha_r\big) \\
&= w \cdot (w^{-1} \cdot v) \\
&= v.
\end{aligned}$$

Since the action of W on V is faithful, this implies $wrw^{-1}s = 1$; that is, $wrw^{-1} = s$.

Now let $\alpha \in \Phi$; by definition of the root system, there exist $w \in W$ and $r \in R$ with $\alpha = w \cdot \alpha_r$. If α also equals $u \cdot \alpha_s$ for some $u \in W$ and $s \in R$, then $(u^{-1}w) \cdot \alpha_r = \alpha_s$, and hence $(u^{-1}w)r(u^{-1}w)^{-1} = s$ by the above; that is, $wrw^{-1} = usu^{-1}$. So without ambiguity we may define the *reflection* r_α to be wrw^{-1}. Then for $v \in V$,

$$\begin{aligned}
r_\alpha \cdot v &= (wrw^{-1}) \cdot v = (wr) \cdot (w^{-1} \cdot v) \\
&= w \cdot \big(w^{-1} \cdot v - 2\langle w^{-1} \cdot v, \alpha_r\rangle \alpha_r\big) \\
&= v - 2\langle w^{-1} \cdot v, \alpha_r\rangle w \cdot \alpha_r \\
&= v - 2\langle v, w \cdot \alpha_r\rangle w \cdot \alpha_r \\
&= v - 2\langle v, \alpha\rangle \alpha.
\end{aligned}$$

Observe that this yields for roots α and β with $r_\alpha = r_\beta$ that $\alpha = \pm\beta$; for then

$$-\alpha = r_\alpha \cdot \alpha = r_\beta \cdot \alpha = \alpha - 2\langle \alpha, \beta\rangle \beta,$$

and thus $\alpha = \langle \alpha, \beta\rangle \beta$, which leaves us with $\alpha = \pm\beta$, since $\langle \alpha, \alpha\rangle = \langle \beta, \beta\rangle = 1$.

(1.6) Proposition *(Strong Exchange Condition)*
Let $r_1,\, r_2, \dots, r_n \in R$ and $\alpha \in \Phi^+$ such that $(r_1 r_2 \cdots r_n) \cdot \alpha \in \Phi^-$. Then there exists an $i \in \{1, \dots, n\}$ with

$$(r_1 r_2 \cdots r_n) r_\alpha = r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_n.$$

Proof. Let $i \in \{1, \ldots, n\}$ be maximal such that $(r_i r_{i+1} \cdots r_n) \cdot \alpha \in \Phi^-$; then $(r_{i+1} \cdots r_n) \cdot \alpha$ must be positive, and thus $(r_{i+1} \cdots r_n) \cdot \alpha = \alpha_{r_i}$ by (1.4). So $\alpha = (r_n \cdots r_{i+1}) \cdot \alpha_{r_i}$ and $r_\alpha = (r_n \cdots r_{i+1}) r_i (r_{i+1} \cdots r_n)$. This implies

$$\begin{aligned}(r_1 r_2 \cdots r_n) r_\alpha &= r_1 r_2 \cdots r_{n-1} r_n (r_n r_{n-1} \cdots r_{i+1} r_i r_{i+1} \cdots r_{n-1} r_n) \\ &= r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_n,\end{aligned}$$

as required. □

Note that (1.6) together with (1.5) imply the *Exchange Condition*: Let $r_1, \ldots, r_n$, $s \in R$ such that $l(r_1 r_2 \cdots r_n) = n$ and $l(r_1 r_2 \cdots r_n s) < n + 1$. Then there exists an $i \in \{1, \ldots, n\}$ such that

$$(r_1 r_2 \cdots r_n) s = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n.$$

For a set of roots Γ, we denote the corresponding set of reflections by S_Γ, and the subspace of V spanned by Γ by V_Γ. The subgroup generated by $S = S_\Gamma$ is denoted by W_S or W_Γ, and Φ_Γ or Φ_S is defined to be the set of roots of the form $w \cdot \gamma$ with $w \in W_\Gamma$ and $\gamma \in \Gamma$. It is clear that $W_{\Gamma'} \subseteq W_\Gamma$ if $\Gamma' \subseteq \Phi_\Gamma$.

Suppose now that Γ is a set of positive roots such that for α, $\beta \in \Gamma$ with $\alpha \neq \beta$, either $\langle \alpha, \beta \rangle \leq -1$ or $\langle \alpha, \beta \rangle = -\cos(\pi/m_{\alpha,\beta})$ for some integer $m_{\alpha,\beta} \geq 2$. Let $\widetilde{S}$ be a set in one-one correspondence with Γ, and for $\gamma \in \Gamma$ denote the element of $\widetilde{S}$ corresponding to γ by σ_γ. Further, let $\widetilde{W}$ denote the Coxeter group with distinguished generating set $\widetilde{S}$ and defining relations

$$(\sigma_\alpha \sigma_\beta)^{m_{\alpha,\beta}} = 1 \text{ for all } \alpha,\ \beta \in \Gamma \text{ such that } \langle \alpha, \beta \rangle > -1,$$

and let $\widetilde{V}$ be an $\mathbb{R}$-vector space with basis $\widetilde{\Pi} = \{\, \widetilde{\gamma} \mid \gamma \in \Gamma \,\}$. If the order of $\sigma_\alpha \sigma_\beta$ equals m, define $(\widetilde{\alpha}, \widetilde{\beta}) = -\cos(\pi/m)$, while if $\sigma_\alpha \sigma_\beta$ is of infinite order, define $(\widetilde{\alpha}, \widetilde{\beta}) = \langle \alpha, \beta \rangle \leq -1$. This determines a bilinear form on $\widetilde{V}$, and we get a standard geometric realization of $\widetilde{W}$ on $\widetilde{V}$ with

$$(\widetilde{\alpha}, \widetilde{\beta}) = \langle \alpha, \beta \rangle \text{ for all } \alpha,\ \beta \in \Gamma.$$

We show now that $\pi\colon \sigma_\gamma \mapsto r_\gamma$ for $\gamma \in \Gamma$ defines a group homomorphism from $\widetilde{W}$ to W_Γ. Since clearly $r_\alpha^2 = 1$ for $\alpha \in \Gamma$, it suffices to show that

$(r_\alpha r_\beta)^m$ equals the identity for $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ such that $\sigma_\alpha \sigma_\beta$ is of order m. So let $\alpha, \beta \in \Gamma$ with $\langle \alpha, \beta \rangle = -\cos(\pi/m)$. Then (1.1)(i) yields that $(r_\alpha r_\beta)^m$ acts as the identity on the space spanned by α and β. In particular, $(r_\alpha r_\beta)^m \cdot \alpha = \alpha$, and thus $(r_\alpha r_\beta)^m r_\alpha (r_\beta r_\alpha)^m = r_\alpha$; that is, $(r_\alpha r_\beta)^{2m} = 1$. Now let $v \in V$; then there exist $\lambda, \mu \in \mathbb{R}$ such that $(r_\alpha r_\beta)^m \cdot v$ equals $v + \lambda\alpha + \mu\beta$, and

$$\begin{aligned} v &= (r_\alpha r_\beta)^{2m} \cdot v \\ &= (r_\alpha r_\beta)^m \cdot (v + \lambda\alpha + \mu\beta) \\ &= v + \lambda\alpha + \mu\beta + \lambda(r_\alpha r_\beta)^m \cdot \alpha + \mu(r_\alpha r_\beta)^m \cdot \beta \\ &= v + 2(\lambda\alpha + \mu\beta). \end{aligned}$$

So $\lambda\alpha + \mu\beta$ equals the zero vector, and thus $(r_\alpha r_\beta)^m \cdot v = v$ for all $v \in V$. It follows by faithfulness of the standard geometric realization that $(r_\alpha r_\beta)^m$ equals 1, as required. Note that π is certainly surjective since $\pi(\widetilde{S}) = S_\Gamma$.

Denote the root system of $\widetilde{W}$ in $\widetilde{V}$ by $\widetilde{\Phi}$, and define $\psi\colon \widetilde{V} \to V_\Gamma$ by linear extension of $\psi(\widetilde{\gamma}) = \gamma$ for $\gamma \in \Gamma$. We show now that ψ maps $\widetilde{\Phi}$ onto Φ_Γ. First, let $\alpha, \beta \in \Gamma$; then

$$\begin{aligned} \psi\big(\sigma_\alpha \cdot \widetilde{\beta}\big) &= \psi\big(\widetilde{\beta} - 2(\widetilde{\alpha}, \widetilde{\beta})\widetilde{\alpha}\big) \\ &= \psi\big(\widetilde{\beta}\big) - 2(\widetilde{\alpha}, \widetilde{\beta})\psi\big(\widetilde{\alpha}\big) \\ &= \beta - 2\langle \alpha, \beta \rangle \alpha \\ &= r_\alpha \cdot \beta. \end{aligned}$$

Since $\widetilde{\Pi}$ forms a basis for $\widetilde{V}$, this yields $\psi(\widetilde{s} \cdot \widetilde{v}) = \pi(\widetilde{s}) \cdot \psi(\widetilde{v})$ for all $\widetilde{s} \in \widetilde{S}$ and $\widetilde{v} \in \widetilde{V}$, and an easy induction on the length of w gives $\psi(w \cdot \widetilde{v}) = \pi(w) \cdot \psi(\widetilde{v})$ for all $w \in \widetilde{W}$ and $v \in \widetilde{V}$. As π is surjective and $\psi(\widetilde{\Pi}) = \Gamma$, it follows that

$$\psi\big(\widetilde{\Phi}\big) = \psi\big(\widetilde{W} \cdot \widetilde{\Pi}\big) = \pi\big(\widetilde{W}\big) \cdot \psi\big(\widetilde{\Pi}\big) = W_\Gamma \cdot \Gamma = \Phi_\Gamma.$$

Denote the set of roots in Φ_Γ which can be written as positive linear combinations of elements of Γ by Φ_Γ^+; since Γ consists of positive roots, it follows easily that Φ_Γ^+ is a subset of Φ^+. Furthermore, $\psi(\widetilde{\Phi}^+) \subseteq \Phi_\Gamma^+$ as $\psi(\widetilde{\Pi}) = \Gamma$, and symmetrically $\psi(\widetilde{\Phi}^-) \subseteq -\Phi_\Gamma^+$. Since $\Phi_\Gamma = \psi(\widetilde{\Phi}) = \psi(\widetilde{\Phi}^+) \cup \psi(\widetilde{\Phi}^-)$, this yields that $\psi(\widetilde{\Phi}^+) = \Phi_\Gamma^+$; therefore $\Phi_\Gamma^+ = \Phi_\Gamma \cap \Phi^+$, and Φ_Γ is the disjoint union of Φ_Γ^+ and $-\Phi_\Gamma^+$.

Now let $w \in \widetilde{W} \setminus \{1\}$. Then there exists an $\widetilde{\alpha} \in \widetilde{\Phi}^+$ such that $w \cdot \widetilde{\alpha}$ is in $\widetilde{\Phi}^-$. Thus $\psi(\widetilde{\alpha}) \in \Phi_\Gamma^+ \subseteq \Phi^+$, and $\pi(w) \cdot \psi(\widetilde{\alpha}) = \psi(w \cdot \widetilde{\alpha}) \in \Phi_\Gamma^- \subseteq \Phi^-$; hence $\pi(w) \neq 1$, and this shows that π is injective. Since π is also surjective, π is a group isomorphism, and thus W_Γ is a Coxeter group with distinguished generating set S_Γ.

We have proved the following theorem, which is due to M.J.Dyer.

(1.7) THEOREM *Let $\Gamma \subseteq \Phi^+$ such that for α, $\beta \in \Gamma$ with $\alpha \neq \beta$ either $\langle \alpha, \beta \rangle \leq -1$ or $\langle \alpha, \beta \rangle = -\cos(\pi/m)$ for some integer m. Then W_Γ is a Coxeter group with distinguished generating set S_Γ.*

Note furthermore that ψ induces a bijection between $\widetilde{\Phi}$ and Φ_Γ. For, as we have already seen, ψ maps $\widetilde{\Phi}$ onto Φ_Γ, and it remains to show that ψ restricted to $\widetilde{\Phi}$ is one-one. Since $\Psi(\widetilde{\Phi}^+) \cap \psi(\widetilde{\Phi}^-) = \emptyset$, and by symmetry of $\widetilde{\Phi}^+$ and $\widetilde{\Phi}^-$, it suffices to show that ψ restricted to $\widetilde{\Phi}^+$ is one-one. So let w, $u \in \widetilde{W}$ and α, $\beta \in \Gamma$ such that $w \cdot \widetilde{\alpha}$ and $u \cdot \widetilde{\beta}$ are positive and $\psi(w \cdot \widetilde{\alpha}) = \psi(u \cdot \widetilde{\beta})$; that is, $\pi(w) \cdot \psi(\widetilde{\alpha}) = \pi(u) \cdot \psi(\widetilde{\beta})$. Then

$$\pi(w\sigma_\alpha w^{-1}) = \pi(w) r_\alpha \pi(w)^{-1} = \pi(u) r_\beta \pi(u)^{-1} = \pi(u\sigma_\beta u^{-1}).$$

Since π is injective, this yields $w\sigma_\alpha w^{-1} = u\sigma_\beta u^{-1}$, and thus $w \cdot \widetilde{\alpha} = \pm u \cdot \widetilde{\beta}$ by an earlier remark. As $w \cdot \widetilde{\alpha}$ and $u \cdot \widetilde{\beta}$ are both positive, we deduce that $w \cdot \widetilde{\alpha} = u \cdot \widetilde{\beta}$, as required.

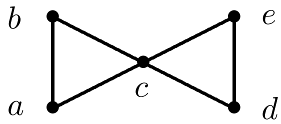
(1.8) PROPOSITION *Let Γ be a set of positive roots as in (1.7), and define Φ_Γ to be the set of roots of the form $w \cdot \gamma$ with $w \in W_\Gamma$ and $\gamma \in \Gamma$. Furthermore, let Φ_Γ^+ denote the set of roots in Φ_Γ which can be written as positive linear combinations of roots in Γ. Then $\Phi_\Gamma^+ = \Phi_\Gamma \cap \Phi^+$, and there exists a standard geometric realization of W_Γ with root system $\widetilde{\Phi}$ and bilinear form $(\ ,\ )$, and a bijection $\psi\colon \widetilde{\Phi} \to \Phi_\Gamma$ such that*

(i) $\psi(\widetilde{\Phi}^+) = \Phi_\Gamma^+$,

(ii) $\psi(w \cdot \widetilde{\alpha}) = w \cdot \psi(\widetilde{\alpha})$ *for all* $w \in W_\Gamma$ *and* $\widetilde{\alpha} \in \widetilde{\Phi}$, *and*

(iii) $\langle \psi(\widetilde{\alpha}), \psi(\widetilde{\beta}) \rangle = (\widetilde{\alpha}, \widetilde{\beta})$ *for all* $\widetilde{\alpha}$, $\widetilde{\beta} \in \widetilde{\Phi}$.

Note that Γ does not necessarily have to be linearly independent. For example, suppose that W has the following Coxeter graph:



Then

$$\Gamma = \{\, \alpha_a + \alpha_b,\ \alpha_a + \alpha_b + 2\alpha_c,\ \alpha_d + \alpha_e,\ 2\alpha_c + \alpha_d + \alpha_e \,\}$$

is certainly not linearly independent, but it can be easily checked that for distinct roots $\alpha, \beta \in \Gamma$, either $\langle \alpha, \beta \rangle = 0$ or $\langle \alpha, \beta \rangle \leq -1$.

If Γ is linearly dependent, the bilinear form $(\ ,\ )$ on $\widetilde{V}$ is not positive definite. For assume that there exist $n > 0$, $m \geq 0$ and pairwise distinct roots $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_m$ in Γ as well as $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_m > 0$ such that $\sum_{i=1}^n \lambda_i \alpha_i - \sum_{j=1}^m \mu_j \beta_j$ equals the zero vector. Define $\widetilde{v} = \sum_{i=1}^n \lambda_i \widetilde{\alpha}_i$; this is a nonzero vector since $\alpha_1, \ldots, \alpha_n$ are linearly independent and $n \geq 1$ with $\lambda_1 > 0$. Then $\psi(\widetilde{v}) = \sum_{i=1}^n \lambda_i \alpha_i = \sum_{j=1}^m \mu_j \beta_j$, and thus

$$(\widetilde{v}, \widetilde{v}) = \langle \psi(\widetilde{v}), \psi(\widetilde{v}) \rangle = \sum_{i,j=1}^{n,m} \lambda_i \mu_j \langle \alpha_i, \beta_j \rangle \leq 0,$$

since $\langle \alpha_i, \beta_j \rangle \leq 0$ for all i and j.

The elements of $R = \{\, r_\alpha \mid \alpha \in \Pi \,\}$ are called *simple reflections.* If J is a set of simple reflections, W_J is called a *parabolic subgroup* of W. This is a Coxeter group with the obvious standard geometric realization on V_J, the space spanned by α_r with $r \in J$.

Denote the length function on W_J with respect to J by l_J; we show now that $l_J(w) = l(w)$ for all $w \in W_J$. For $w \in W$ define

$$N(w) = \{\, \alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^- \,\},$$

and observe that for u and w in W,

$$N(uw) \subseteq N(w) \cup w^{-1} \cdot N(u).$$

For if $\gamma \in N(uw) \setminus N(w)$, then $w \cdot \gamma \in \Phi^+$ and $u \cdot (w \cdot \gamma) = (uw) \cdot \gamma \in \Phi^-$; thus $w \cdot \gamma \in N(u)$, that is $\gamma \in w^{-1} \cdot N(u)$. Note also that

$$\begin{aligned} N(w) \cap w^{-1} \cdot N(u) &= (w^{-1}w) \cdot \big(N(w) \cap w^{-1} \cdot N(u)\big) \\ &= w^{-1} \cdot \big(w \cdot N(w) \cap (ww^{-1}) \cdot N(u)\big) \\ &= w^{-1} \cdot \big(w \cdot N(w) \cap N(u)\big) \\ &\subseteq w^{-1}\big(\Phi^- \cap \Phi^+\big) = \emptyset, \end{aligned}$$

whence the above union is disjoint.

If $J \subseteq R$ and $w \in W_J$, then $N_J(w) \subseteq N(w)$, where $N_J(w)$ denotes the set of all roots $\alpha \in \Phi_J^+$ such that $w \cdot \alpha \in \Phi_J^-$. The next lemma yields that $l_J(w) = |N_J(w)| \leq |N(w)| = l(w)$; since $J \subseteq R$, the reverse inequality is certainly true, and so $l_J(w) = l(w)$.

(1.9) LEMMA $|N(w)| = l(w)$ *for all* $w \in W$.

Proof. We use induction on $l(w)$. If $l(w) = 0$, then $w = 1$ and $N(1) = \emptyset$. So suppose that $l(w) \geq 1$, and let $u \in W$ and $r \in R$ such that $w = ur$ and $l(u) = l(w) - 1$; then $|N(u)| = l(u)$ by induction. By the above remark,

$$N(w) = N(ur) \subseteq N(r) \cup r \cdot N(u) = \{\alpha_r\} \cup r \cdot N(u),$$

and this union is disjoint. In order to prove that $|N(w)| = l(w)$, it suffices to show that $\{\alpha_r\}$ and $r \cdot N(u)$ are subsets of $N(w)$. By (1.5) we know that $w \cdot \alpha_r$ is negative and $u \cdot \alpha_r$ is positive, and thus $\{\alpha_r\} \subseteq N(w)$ and $\alpha_r \notin N(u)$. The latter together with (1.4) yield that $r \cdot N(u)$ contains only positive roots, and since $w \cdot \big(r \cdot N(u)\big) = u \cdot N(u) \subseteq \Phi^-$, we conclude that $r \cdot N(u)$ is a subset of $N(w)$, as required. □

Now let $w \in W$ and $\alpha \in \Phi^+$ with $w \cdot \alpha \in \Phi^-$. Then $-w \cdot \alpha \in \Phi^+$ and $w^{-1} \cdot (-w \cdot \alpha) = -\alpha \in \Phi^-$; that is, $-w \cdot \alpha \in N(w^{-1})$. So $-w \cdot N(w) \subseteq N(w^{-1})$, and since $l(w) = l(w^{-1})$, we deduce the following:

(1.10) COROLLARY $-w \cdot N(w) = N(w^{-1})$ *for all* $w \in W$.

Suppose now that we have u, $w \in W$ with $l(uw) = l(u) + l(w)$. Then $N(uw)$ is a subset of $N(w) \cup w^{-1} \cdot N(u)$; since $l(uw) = |N(uw)|$, and this equals $|N(u)| + |N(w)|$, it follows that $N(uw) = N(w) \cup w^{-1} \cdot N(u)$. In particular, $w^{-1} \cdot N(u) \subseteq \Phi^+$, and thus necessarily $N(u) \cap N(w^{-1}) = \emptyset$.

Conversely, suppose that $N(u) \cap N(w^{-1}) = \emptyset$. Then $w^{-1} \cdot N(u) \subseteq \Phi^+$, and

$$(uw) \cdot \big(w^{-1} \cdot N(u)\big) = u \cdot N(u) \subseteq \Phi^-;$$

whence $w^{-1} \cdot N(u) \subseteq N(uw)$. Further, $N(w^{-1}) \subseteq \Phi^+ \setminus N(u)$, and so (1.10) yields that

$$(uw) \cdot N(w) = u \cdot \big(-N(w^{-1})\big) = -u \cdot N(w^{-1}) \subseteq -u \cdot \big(\Phi^+ \setminus N(u)\big) \subseteq \Phi^-;$$

hence also $N(w) \subseteq N(uw)$. Therefore $N(uw) = N(w) \cup w^{-1} \cdot N(u)$, and since this union is disjoint, we have $l(uw) = l(u) + l(w)$.

(1.11) LEMMA *Let u, $w \in W$. Then the following are equivalent:*

(i) $l(uw) = l(u) + l(w)$

(ii) $N(uw) = N(w) \cup w^{-1} \cdot N(u)$

(iii) $N(u) \cap N(w^{-1}) = \emptyset$.

We conclude this chapter with a proposition whose proof is outlined in [1], Ch.V, §4.

(1.12) PROPOSITION *Suppose H is a finite subgroup of W. Then there exist a finite parabolic subgroup W_J of W such that H is conjugate to a subgroup of W_J.*

Proof. Since H is finite, we may assume without loss of generality that R is finite. If $|W| < \infty$, the assertion is true with $J = R$. So suppose from now on that W is infinite; then Φ is infinite and $|R| > 1$, and we proceed by induction on $|R|$.

Let V^* denote the dual space of V, and let $\{ \delta_\alpha \mid \alpha \in \Pi \}$ be the basis dual to Π; that is, $\delta_\alpha(v)$ equals the coefficient of α in v. For $f \in V^*$ and $w \in W$, we define $fw \in V^*$ by $fw\colon v \mapsto f(w \cdot v)$; this determines a right action of W on V^*. For $f \in V^*$ define further $S(f) = \{ \gamma \in \Phi^+ \mid f(\gamma) < 0 \}$.

Let $f = \sum_{\alpha\in\Pi} \delta_\alpha$ and $F = \sum_{h\in H} fh$. Then $f(\alpha) = 1$ for all $\alpha \in \Pi$, and thus

$$f(\gamma) > 0 \text{ for all } \gamma \in \Phi^+. \qquad (*)$$

Now let $A = \bigcup_{h\in H} N(h)$. Since H is finite and $N(h)$ is finite for all $h \in H$, A is finite. Furthermore, for all $\gamma \in \Phi^+ \setminus A$,

$$F(\gamma) = \sum_{h\in H} (fh)(\gamma) = \sum_{h\in H} f(h \cdot \gamma) > 0 \qquad (**)$$

by $(*)$, since $h \cdot \gamma \in \Phi^+$ for all $h \in H$. So $S(F) \cap \big(\Phi^+ \setminus A\big) = \emptyset$; that is, $S(F) \subseteq A$, and in particular, $S(F)$ must be finite.

Let $x \in W$ such that $|S(Fx)|$ is minimal, and assume for a contradiction that $S(Fx) \neq \emptyset$. Then $\big(Fx\big)(\gamma) < 0$ for some $\gamma \in \Phi^+$, and it follows that $(Fx)(\alpha_r) < 0$ for some $r \in R$; that is, $\alpha_r \in S(Fx)$. Now

$$\big(Fxr\big)(\alpha_r) = \big(Fx\big)(r \cdot \alpha_r) = \big(Fx\big)(-\alpha_r) = -\big(Fx\big)(\alpha_r) > 0,$$

and hence $\alpha_r \notin S(Fxr)$. So $\gamma \mapsto r \cdot \gamma$ maps $S(Fxr)$ into $\Phi^+ \setminus \{\alpha_r\}$, and if $\gamma \in S(Fxr)$, then $\big(Fx\big)(r \cdot \gamma) = \big(Fxr\big)(\gamma) < 0$; therefore $\gamma \mapsto r \cdot \gamma$ maps $S(Fxr)$ into $S(Fx) \setminus \{\alpha_r\}$. Since the above map is clearly one-one, this yields

$$|S(Fxr)| \leq |S(Fx) \setminus \{\alpha_r\}| < |S(Fx)|,$$

contrary to the choice of x. Thus $S(Fx) = \emptyset$.

Since $Fh = F$ for $h \in H$, we deduce that

$$Fx(x^{-1}hx) = (Fxx^{-1})hx = Fhx = Fx;$$

so $x^{-1}Hx \subseteq \{\, y \in W \mid (Fx)y = Fx \,\}$, and we show now that

$$\big\{\, y \in W \mid \big(Fx\big)y = Fx \,\big\} \subseteq W_I,$$

where $I = \{\, r \in R \mid \big(Fx\big)(\alpha_r) = 0 \,\}$. Let $y \in W$ with $\big(Fx\big)y = Fx$. If $l(y) = 0$, then $y = 1 \in W_I$. Proceeding by induction, suppose $l(y) \geq 1$, and let $z \in W$ and $s \in R$ such that $y = zs$ and $l(z) = l(y) - 1$. Then $z \cdot \alpha_s \in \Phi^+$ by (1.5), and

$$\big(Fx\big)(\alpha_s) = \big(Fxy\big)(\alpha_s) = \big(Fx\big)(y \cdot \alpha_s) = -\big(Fx\big)((ys) \cdot \alpha_s) = -\big(Fx\big)(z \cdot \alpha_s);$$

since $S(Fx) = \emptyset$, it follows that $\big(Fx\big)(\alpha_s) = 0$, and thus $s \in I$. So

$$\big(Fxs\big)(v) = \big(Fx\big)(v - 2\langle v, \alpha_s\rangle\alpha_s) = \big(Fx\big)(v) - 2\langle v, \alpha_s\rangle\big(Fx\big)(\alpha_s) = \big(Fx\big)(v)$$

for all $v \in V$, which yields $\big(Fx\big)z = \big(Fxy\big)s = \big(Fx\big)s = Fx$; hence $z \in W_I$ by induction, and $y = zs \in W_I$, as required.

Since Φ^+ is infinite by construction, and A is finite, it follows that $\Phi^+ \setminus A$ is nonempty; so $F \neq 0$ by $(**)$, which yields $Fx \neq 0$ and $I \neq R$. By induction there exist $J \subseteq I$ and $u \in W_I$ such that W_J is finite and $u^{-1}(x^{-1}Hx)u \subseteq W_J$, as desired. □

Chapter 2

The Depth of a Root

To facilitate inductive proofs of facts about root systems, it is convenient for us to introduce a concept which, in some sense, measures how far a root is from being simple. For a positive root α, we define the *depth* of α to be

$$\operatorname{dp}(\alpha) = \min\{\, l \in \mathbb{N}_0 \mid w \cdot \alpha \in \Phi^- \text{ for some } w \in W \text{ with } l(w) = l\}.$$

Observe that such an integer always exists, since by definition of the root system every root α has the form $u \cdot \alpha_r$ for some $u \in W$ and $r \in R$, and then $(ru^{-1}) \cdot \alpha = -\alpha_r \in \Phi^-$. Suppose now that $\alpha \in \Phi^+$, and let $w \in W$ with $l(w) = \operatorname{dp}(\alpha)$ such that $w \cdot \alpha \in \Phi^-$. Then $w \neq 1$, and hence there exists an $r \in R$ and a $u \in W$ with $w = ru^{-1}$ and $l(w) = l(u^{-1}) + 1 = l(u) + 1$. Minimality of w yields that $u^{-1} \cdot \alpha$ is positive, and since $r \cdot (u^{-1} \cdot \alpha) = w \cdot \alpha$ is negative, (1.4) gives $u^{-1} \cdot \alpha = \alpha_r$; that is, $\alpha = u \cdot \alpha_r$. Thus every positive root can be written as $u \cdot \alpha_r$ with $l(u)$ equal to the depth of the root minus 1. Moreover, the depth of α equals the minimal integer l such that $w \cdot \alpha \in -\Pi$ for some $w \in W$ of length l. For a negative root β, we define the depth of β to be

$$\operatorname{dp}(\beta) = -\min\{\, l \in \mathbb{N}_0 \mid w \cdot \beta \in -\Pi \text{ for some } w \in W \text{ with } l(w) = l\}.$$

Note that for $r \in R$ and $\alpha \in \Phi$, clearly $\operatorname{dp}(\alpha) - 1 \le \operatorname{dp}(r \cdot \alpha) \le \operatorname{dp}(\alpha) + 1$. Furthermore, if α is a positive root, then there exists an $r \in R$ such that $\operatorname{dp}(r \cdot \alpha) = \operatorname{dp}(\alpha) - 1$. Also, for $w \in W$ and $r \in R$ clearly $\operatorname{dp}(w \cdot \alpha_r) \le l(w) + 1$. Finally, note that $\operatorname{dp}(\alpha) \le l(w)$ if $w \cdot \alpha$ is negative for some root α and $w \in W$; for if α is positive, this follows by definition of the depth of α, and if α is negative, then $\operatorname{dp}(\alpha) \le 0 \le l(w)$.

Now let $\alpha \in \Phi^+$ and $w \in W$, $r \in R$ such that α equals $w \cdot \alpha_r$ and $l(w) = \operatorname{dp}(\alpha) - 1$. Then $-\alpha = -w \cdot \alpha_r$, and so $w^{-1} \cdot (-\alpha) = -\alpha_r$ and $l(w) = l(w^{-1}) \ge -\operatorname{dp}(-\alpha)$. On the other hand, if $u \in W$ and $s \in R$ such that $l(u) = -\operatorname{dp}(-\alpha)$ and $u \cdot (-\alpha) = -\alpha_s$, then $(su) \cdot \alpha = -\alpha_s$; thus

$$\operatorname{dp}(\alpha) \le l(su) \le l(u) + 1 = -\operatorname{dp}(-\alpha) + 1 \le l(w) + 1 = \operatorname{dp}(\alpha),$$

and we must have equality everywhere. In particular, $\operatorname{dp}(\alpha) + \operatorname{dp}(-\alpha) = 1$. Since for $\beta \in \Phi^-$ clearly $\alpha = -\beta \in \Phi^+$, this proves the following result:

(2.13) Lemma $\operatorname{dp}(\alpha) + \operatorname{dp}(-\alpha) = 1$ *for all* $\alpha \in \Phi$.

(2.14) Proposition *Let* $r \in R$ *and* $\alpha \in \Phi$. *Then*

$$\operatorname{dp}(r \cdot \alpha) = \begin{cases} \operatorname{dp}(\alpha) - 1 & \text{if } \langle \alpha, \alpha_r \rangle > 0, \\ \operatorname{dp}(\alpha) & \text{if } \langle \alpha, \alpha_r \rangle = 0, \\ \operatorname{dp}(\alpha) + 1 & \text{if } \langle \alpha, \alpha_r \rangle < 0. \end{cases}$$

Proof. If the assertion is proved for positive roots, (2.13) yields for $\alpha \in \Phi^-$,

$$\begin{aligned}
\mathrm{dp}(r\cdot\alpha) &= -\,\mathrm{dp}\big(-(r\cdot\alpha)\big)+1 = -\,\mathrm{dp}\big(r\cdot(-\alpha)\big)+1\\
&= \begin{cases} -\big(\mathrm{dp}(-\alpha)-1\big)+1 & \text{if } \langle -\alpha,\alpha_r\rangle > 0,\\ -\,\mathrm{dp}(-\alpha)+1 & \text{if } \langle -\alpha,\alpha_r\rangle = 0,\\ -\big(\mathrm{dp}(-\alpha)+1\big)+1 & \text{if } \langle -\alpha,\alpha_r\rangle < 0,\end{cases}\\
&= \begin{cases} \mathrm{dp}(\alpha)+1 & \text{if } \langle \alpha,\alpha_r\rangle < 0,\\ \mathrm{dp}(\alpha) & \text{if } \langle \alpha,\alpha_r\rangle = 0,\\ \mathrm{dp}(\alpha)-1 & \text{if } \langle \alpha,\alpha_r\rangle > 0.\end{cases}
\end{aligned}$$

Hence it suffices to prove the proposition for positive roots. If $\langle \alpha,\alpha_r\rangle = 0$, then $r\cdot\alpha = \alpha - 2\langle\alpha,\alpha_r\rangle\alpha_r = \alpha$, and trivially $\mathrm{dp}(r\cdot\alpha) = \mathrm{dp}(\alpha)$.

Suppose next that $\langle\alpha,\alpha_r\rangle > 0$. It suffices to show that $\mathrm{dp}(r\cdot\alpha) < \mathrm{dp}(\alpha)$; to do so, we construct a $w \in W$ with $w\cdot(r\cdot\alpha) \in \Phi^-$ and $l(w) < \mathrm{dp}(\alpha)$. Choose $u \in W$ such that $u\cdot\alpha \in \Phi^-$ and $l(u) = \mathrm{dp}(\alpha)$. If $u\cdot\alpha_r$ is negative, set $w = ur$; then $l(w) = l(u)-1$ by (1.5), and $w\cdot(r\cdot\alpha) = u\cdot\alpha \in \Phi^-$, as required. Hence we may assume from now on that $u\cdot\alpha_r$ is positive. Clearly $u \neq 1$ (as $u\cdot\alpha$ is negative, while α is positive), and thus there exist $s \in R$ and $w \in W$ with $u = sw$ and $l(u) = l(w)+1$. Now

$$u\cdot(r\cdot\alpha) = u\cdot\big(\alpha - 2\langle\alpha,\alpha_r\rangle\alpha_r\big) = u\cdot\alpha - 2\langle\alpha,\alpha_r\rangle u\cdot\alpha_r$$

is negative, and since $u\cdot\alpha$ and $-2\langle\alpha,\alpha_r\rangle u\cdot\alpha_r$ are both negative linear combinations of simple roots and not scalar multiples of each other (since α and α_r are linearly independent), $u\cdot(r\cdot\alpha)$ cannot be equal to $-\alpha_s$. It follows by (1.4) that $w\cdot(r\cdot\alpha) = s\cdot\big(u\cdot(r\cdot\alpha)\big) \in \Phi^-$.

Finally, suppose that $\langle\alpha,\alpha_r\rangle < 0$. Then $\langle r\cdot\alpha,\,\alpha_r\rangle = -\langle\alpha,\alpha_r\rangle > 0$, and the preceding paragraph shows that $\mathrm{dp}(\alpha) = \mathrm{dp}\big(r\cdot(r\cdot\alpha)\big) = \mathrm{dp}(r\cdot\alpha)-1$. □

(2.15) Lemma *Let $\alpha \in \Phi^+$ and $w \in W$, $r \in R$ such that $\alpha = w\cdot\alpha_r$ and $l(w) = \mathrm{dp}(\alpha)-1$. Then $w \in W_{I(\alpha)}$ and $r \in I(\alpha)$, where $I(\alpha)$ denotes the set of simple reflections in one-one correspondence with the support of α.*

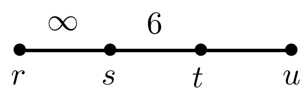
Proof. If $\mathrm{dp}(\alpha) = 1$, the assertion is trivially true. Suppose next that α is of depth greater than 1, and let $u \in W$ and $s \in R$ with $w = su$ and $l(w) = l(u)+1$. Then $u\cdot\alpha_r = s\cdot\alpha$, and thus

$$\mathrm{dp}(s\cdot\alpha) = \mathrm{dp}(u\cdot\alpha_r) \le l(u)+1 = l(w) = \mathrm{dp}(\alpha)-1;$$

this yields that $\mathrm{dp}(s\cdot\alpha)=\mathrm{dp}(\alpha)-1$, and therefore $\langle\alpha,\alpha_s\rangle>0$. Since α is a positive root and $\langle\alpha_s,\alpha_t\rangle\le 0$ for $t\in R\setminus\{s\}$, we deduce that $s\in I(\alpha)$. Further, as $s\cdot\alpha=u\cdot\alpha_r$ and $l(u)=\mathrm{dp}(s\cdot\alpha)-1$, induction yields that $u\in W_{I(s\cdot\alpha)}$ and $r\in I(s\cdot\alpha)$. By definition of the action of s on V, it follows that $I(s\cdot\alpha)\subseteq I(\alpha)\cup\{s\}=I(\alpha)$, and thus $w=su$ is in $W_{I(\alpha)}$ and $r\in I(\alpha)$. □

(2.16) COROLLARY *Let $J\subseteq R$. Then $\Phi_J=\Phi\cap V_J$ and $\mathrm{dp}_J(\alpha)=\mathrm{dp}(\alpha)$ for all $\alpha\in\Phi_J$, where dp_J denotes the depth function on Φ_J with respect to W_J and l_J.*

It is clear that the part of the previous assertion concerning the depth in parabolic subsystems is not in general true for reflection subgroups W_Γ; for if $\Gamma=\{\alpha\}$, then α has depth 1 with respect to Γ, independent of $\mathrm{dp}(\alpha)$. Note furthermore that for $\Gamma\subseteq\Phi$, we do not have in general $\Phi_\Gamma=\Phi\cap V_\Gamma$. For example, suppose that W has the following Coxeter graph



with $\langle\alpha_r,\alpha_s\rangle=-1$. Define $\alpha=tust\cdot\alpha_s$ and $\Gamma=\{\alpha,\alpha_r,\alpha_u\}$; then

$$\begin{aligned}
t\cdot\alpha_s&=\alpha_s+\sqrt{3}\alpha_t,\\
st\cdot\alpha_s&=2\alpha_s+\sqrt{3}\alpha_t,\\
ust\cdot\alpha_s&=2\alpha_s+\sqrt{3}\alpha_t+\sqrt{3}\alpha_u,\\
tust\cdot\alpha_s&=2\alpha_s+2\sqrt{3}\alpha_t+\sqrt{3}\alpha_u,
\end{aligned}$$

and thus $\alpha=2\alpha_s+2\sqrt{3}\alpha_t+\sqrt{3}\alpha_u$. Now $\langle\alpha,\alpha_r\rangle=-2\le-1$, $\langle\alpha,\alpha_u\rangle=0$ and $\langle\alpha_r,\alpha_u\rangle=0$. It follows by (1.7) that W_Γ is a Coxeter group with distinguished generating set S_Γ, and by (1.8) we know further that Φ_Γ is the union of Φ_Γ^+ and $-\Phi_\Gamma^+$, where Φ_Γ^+ denotes the set of the roots in Φ_Γ that can be written as positive linear combinations of elements of Γ. Now let $\beta=ts\cdot\alpha_r$; then

$$\beta=t\cdot(\alpha_r+2\alpha_s)=\alpha_r+2\alpha_s+2\sqrt{3}\alpha_t,$$

and this is equal to $\alpha+\alpha_r-\sqrt{3}\alpha_u$. Hence β is an element of V_Γ; but since Γ is linearly independent, β is neither in Φ_Γ^+ nor in Φ_Γ^-, and thus β cannot be an element of Φ_Γ.

Note that the explicit calculation of α above together with the previous proposition imply that α is of depth 5; for, from each step to the next, application of a simple reflection increases the corresponding coefficient, and this indicates that the depth increases by 1. Similarly, $\gamma = uts \cdot \alpha_r$ is of depth 4, since

$$\begin{aligned} s \cdot \alpha_r &= \alpha_r + 2\alpha_s, \\ ts \cdot \alpha_r &= \alpha_r + 2\alpha_s + 2\sqrt{3}\alpha_t, \\ uts \cdot \alpha_r &= \alpha_r + 2\alpha_s + 2\sqrt{3}\alpha_t + 2\sqrt{3}\alpha_u. \end{aligned}$$

Observe that although α is of depth greater than $\mathrm{dp}(\gamma)$, all coefficients in α are less than or equal to the corresponding coefficients in γ.

In general it is a rather tedious task to calculate the depth of a root using (2.14). In order to find a noninductive formula for the depth of a root, we define for $v \in V$:

$$V_+(v) = \{\, v' \in V \mid \langle v, v' \rangle > 0 \,\}, \;\; V_-(v) = \{\, v' \in V \mid \langle v, v' \rangle < 0 \,\}$$
$$\text{and } V_0(v) = \{\, v' \in V \mid \langle v, v' \rangle = 0 \,\}.$$

Then V is the disjoint union of $V_+(v)$, $V_-(v)$ and $V_0(v)$, and it is clear that $V_+(-v) = -V_+(v) = V_-(v)$ and $V_0(v) = V_0(-v) = -V_0(v)$. Furthermore,

$$\langle w^{-1} \cdot v', v \rangle = \langle w \cdot (w^{-1} \cdot v'), w \cdot v \rangle = \langle v', w \cdot v \rangle$$

for all $v' \in V$, and thus

$$V_+(w \cdot v) = w \cdot V_+(v), \; V_-(w \cdot v) = w \cdot V_-(v) \text{ and } V_0(w \cdot v) = w \cdot V_0(v).$$

Next, define for $v \in V$ and $w \in W$,

$$N_+(w, v) = N(w) \cap V_+(v), \;\; N_-(w, v) = N(w) \cap V_-(v)$$
$$\text{and } N_0(w, v) = N(w) \cap V_0(v).$$

By the above, $N(w)$ is the disjoint union of $N_+(w, v)$, $N_-(w, v)$ and $N_0(w, v)$ and, furthermore, $N_+(w, v) = N_-(w, -v)$ and $N_0(w, v) = N_0(w, -v)$.

Since $N(uw)$ is contained in $N(w) \cup w^{-1} \cdot N(u)$, it follows that

$$\begin{aligned} N_+(uw, v) &= N(uw) \cap V_+(v) \\ &\subseteq \big(N(w) \cup w^{-1} \cdot N(u)\big) \cap V_+(v) \\ &= \big(N(w) \cap V_+(v)\big) \cup \big(w^{-1} \cdot N(u) \cap V_+(v)\big) \\ &= N_+(w, v) \cup w^{-1} \cdot \big(N(u) \cap w \cdot V_+(v)\big). \end{aligned}$$

Now $w \cdot V_+(v) = V_+(w \cdot v)$, and thus

$$\begin{aligned} N_+(uw, v) &\subseteq N_+(w, v) \cup w^{-1} \cdot \big(N(u) \cap V_+(w \cdot v)\big) \\ &= N_+(w, v) \cup w^{-1} \cdot N_+(u, w \cdot v). \end{aligned}$$

Since by (1.11) further $N(uw) = N(w) \cup w^{-1} \cdot N(u)$ if $l(uw) = l(u) + l(w)$, this and similar arguments yield the next lemma, which will enable us to give a formula for the depth of $w \cdot \alpha$.

(2.17) Lemma *Let $v \in V$ and u, $w \in W$. Then*

$$\begin{aligned} N_+(uw, v) &\subseteq N_+(w, v) \cup w^{-1} \cdot N_+(u, w \cdot v), \\ N_-(uw, v) &\subseteq N_-(w, v) \cup w^{-1} \cdot N_-(u, w \cdot v), \\ N_0(uw, v) &\subseteq N_0(w, v) \cup w^{-1} \cdot N_0(u, w \cdot v), \end{aligned}$$

and these unions are disjoint. Moreover, if $l(uw) = l(u) + l(w)$ we have equality; hence in particular, $N_+(w, v) \subseteq N_+(uw, v)$, $N_-(w, v) \subseteq N_-(uw, v)$ and $N_0(w, v) \subseteq N_0(uw, v)$, and furthermore, $N_+(u, w \cdot v) \subseteq w \cdot N_+(uw, v)$, $N_-(u, w \cdot v) \subseteq w \cdot N_-(uw, v)$ and $N_0(u, w \cdot v) \subseteq w \cdot N_0(uw, v)$.

(2.18) Proposition *Let $\alpha \in \Phi$ and $w \in W$. Then*

$$\mathrm{dp}(w \cdot \alpha) = \mathrm{dp}(\alpha) + |N_-(w, \alpha)| - |N_+(w, \alpha)|.$$

Proof. The assertion is trivial if $l(w) = 0$. So suppose $l(w) \geq 1$, and let $u \in W$ and $r \in R$ with $w = ru$ and $l(w) = l(u) + 1$. Then by (2.17),

$$N_+(w, \alpha) = N_+(u, \alpha) \cup u^{-1} \cdot N_+(r, u \cdot \alpha) = N_+(u, \alpha) \cup u^{-1} \cdot \big(\{\alpha_r\} \cap V_+(u \cdot \alpha)\big);$$

and

$$N_-(w, \alpha) = N_-(u, \alpha) \cup u^{-1} \cdot N_-(r, u \cdot \alpha) = N_-(u, \alpha) \cup u^{-1} \cdot \big(\{\alpha_r\} \cap V_-(u \cdot \alpha)\big),$$

and these unions are disjoint. So

$$|N_+(w, \alpha)| - |N_+(u, \alpha)| = \begin{cases} 0 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle \leq 0, \\ 1 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle > 0, \end{cases}$$

and

$$|N_-(w, \alpha)| - |N_-(u, \alpha)| = \begin{cases} 1 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle < 0, \\ 0 & \text{if } \langle u \cdot \alpha, \alpha_r \rangle \geq 0. \end{cases}$$

Using (2.14), we deduce that

$$\mathrm{dp}(w \cdot \alpha) = \mathrm{dp}(u \cdot \alpha) + \big(|N_-(w, \alpha)| - |N_-(u, \alpha)|\big) - \big(|N_+(w, \alpha)| - |N_+(u, \alpha)|\big),$$

and induction finishes the proof. $\square$

If α and β are roots, and $w \in W$ with $w \cdot \alpha = \beta$, Proposition (2.18) yields that $l(w) \geq \mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$, with equality if and only if $N(w) \subseteq V_-(\alpha)$. In particular, if α is positive and β equals $-\alpha$, then

$$l(r_\alpha) \geq \mathrm{dp}(\alpha) - \mathrm{dp}(-\alpha) = 2\,\mathrm{dp}(\alpha) - 1.$$

On the other hand, if $w \in W$ and $r \in R$ with $\alpha = w \cdot \alpha_r$ and $l(w) = \mathrm{dp}(\alpha) - 1$, then $r_\alpha = wrw^{-1}$, and so $l(r_\alpha) \leq l(w) + 1 + l(w) = 2\,\mathrm{dp}(\alpha) - 1$.

(2.19) Corollary *$l(r_\alpha) = 2\,\mathrm{dp}(\alpha) - 1$ for all $\alpha \in \Phi^+$.*

We can now generalize (2.14).

(2.20) Lemma *Let $\alpha \in \Phi$ and $\beta \in \Phi^+$. Then*

$$\mathrm{dp}(r_\beta \cdot \alpha) \begin{cases} < \mathrm{dp}(\alpha) & \text{if } \langle \alpha, \beta \rangle > 0, \\ = \mathrm{dp}(\alpha) & \text{if } \langle \alpha, \beta \rangle = 0, \\ > \mathrm{dp}(\alpha) & \text{if } \langle \alpha, \beta \rangle < 0. \end{cases}$$

Proof. First, suppose that $\langle \alpha, \beta \rangle = 0$. Then $r_\beta \cdot \alpha = \alpha - 2\langle \alpha, \beta \rangle \beta = \alpha$, and hence trivially $\mathrm{dp}(r_\beta \cdot \alpha) = \mathrm{dp}(\alpha)$.

Now let $\langle \alpha, \beta \rangle > 0$. Corollary (1.10) states that $\gamma \mapsto -r_\beta \cdot \gamma$ defines a one-one correspondence on $N(r_\beta)$. If $\gamma \in N_-(r_\beta, \alpha)$, then

$$\langle \alpha, -r_\beta \cdot \gamma \rangle = -\big(\langle \alpha, \gamma \rangle - 2\langle \gamma, \beta \rangle \langle \alpha, \beta \rangle\big) = -\langle \alpha, \gamma \rangle + 2\langle \gamma, \beta \rangle \langle \alpha, \beta \rangle,$$

and this is greater than 0; for $\langle \gamma, \beta \rangle > 0$ as $\gamma \in N(r_\beta)$, $\langle \alpha, \beta \rangle > 0$ by hypothesis and $\langle \alpha, \gamma \rangle < 0$ by choice of γ. So $\gamma \mapsto -r_\beta \cdot \gamma$ embeds $N_-(r_\beta, \alpha)$ into $N_+(r_\beta, \alpha)$. But β is in $N_+(r_\beta, \alpha)$ by hypothesis, and since $\beta = -r_\beta \cdot \beta$ clearly $\beta \notin -r_\beta \cdot N_-(r_\beta, \alpha)$. So $|N_-(r_\beta, \alpha)| < |N_+(r_\beta, \alpha)|$, and hence by (2.18),

$$\mathrm{dp}(r_\beta \cdot \alpha) = \mathrm{dp}(\alpha) + |N_-(r_\beta, \alpha)| - |N_+(r_\beta, \alpha)| < \mathrm{dp}(\alpha).$$

Finally, suppose that $\langle \alpha, \beta \rangle < 0$. Then $\langle r_\beta \cdot \alpha, \beta \rangle = -\langle \alpha, \beta \rangle > 0$, and thus by the preceding paragraph, $\mathrm{dp}(\alpha) = \mathrm{dp}\big(r_\beta \cdot (r_\beta \cdot \alpha)\big) < \mathrm{dp}(r_\beta \cdot \alpha)$. □

We conclude this chapter with some first applications of inductive proofs on the depth. First we give an alternative proof of the next theorem, which was proved independently by V. V. Deodhar [4] and M. J. Dyer [5]. Our proof is closely related to Dyer's proof, translating his ideas into the context of root systems.

(2.21) THEOREM *Let $\Gamma \subseteq \Phi$, and let Ψ be the set of all roots α in $\Phi_\Gamma \cap \Phi^+$ such that*

$$N(r_\alpha) \cap \Phi_\Gamma = \{\alpha\}.$$

Then $W_\Gamma = W_\Psi$, and for $\alpha, \beta \in \Psi$ with $\alpha \neq \beta$, either $\langle \alpha, \beta \rangle \leq -1$ or $\langle \alpha, \beta \rangle = -\cos(\pi/m)$ for some integer m. Thus W_Γ is a Coxeter group with distinguished generating set S_Ψ.

The following technical lemma is a vital tool in our proof of (2.21).

(2.22) LEMMA *Suppose α and β are two positive roots with $\alpha \neq \beta$ and $\mathrm{dp}(\alpha) \leq \mathrm{dp}(\beta)$. Then $\mathrm{dp}(-r_\alpha \cdot \beta) < \mathrm{dp}(\alpha)$.*

Proof. Let $w \in W$ and $r \in R$ with $\alpha = w \cdot \alpha_r$ and $l(w) = \mathrm{dp}(\alpha) - 1$. Then $w^{-1} \cdot \beta \in \Phi^+$, since $l(w^{-1}) = l(w)$ and this is less than the depth of β. Furthermore, $w^{-1} \cdot \beta \neq \alpha_r$ since α and β are distinct, and thus $rw^{-1} \cdot \beta \in \Phi^+$ by (1.4). Now

$$w^{-1} \cdot (-r_\alpha \cdot \beta) = w^{-1} \cdot (-wrw^{-1} \cdot \beta) = -(rw^{-1} \cdot \beta) \in \Phi^-,$$

and so $\mathrm{dp}(-r_\alpha \cdot \beta) \leq l(w^{-1}) < \mathrm{dp}(\alpha)$, as required. □

Proof of (2.9). Clearly $\Psi \subseteq \Phi_\Gamma$, and thus $\Phi_\Psi \subseteq \Phi_\Gamma$. Assume now for a contradiction that $\Phi_\Gamma \neq \Phi_\Psi$; then $\big(\Phi_\Gamma \setminus \Phi_\Psi\big) \cap \Phi^+ \neq \emptyset$, and thus there exists a positive root γ in $\Phi_\Gamma \setminus \Phi_\Psi$ of minimal depth. In particular, $\gamma \notin \Psi$, and hence

$$\big(N(r_\gamma) \cap \Phi_\Gamma\big) \setminus \{\gamma\} \neq \emptyset.$$

Let β be an element of the above set of minimal depth; then $-r_\gamma \cdot \beta$ is also an element of $N(r_\gamma) \setminus \{\gamma\}$ by (1.10), and since β and γ are in Φ_Γ by construction, it follows that $-r_\gamma \cdot \beta$ is in $(N(r_\gamma) \cap \Phi_\Gamma) \setminus \{\gamma\}$. If $\mathrm{dp}(\beta) \geq \mathrm{dp}(\gamma)$, then

$$\mathrm{dp}(-r_\gamma \cdot \beta) < \mathrm{dp}(\gamma) \leq \mathrm{dp}(\beta)$$

by (2.22), contradicting the minimality of β. So $\mathrm{dp}(\gamma) > \mathrm{dp}(\beta)$, and minimality of γ forces $\beta \in \Phi_\Psi$.

If $r_\beta \cdot \gamma$ is positive, then $\mathrm{dp}(\gamma) > \mathrm{dp}(r_\beta \cdot \gamma)$ by (2.20), since $\beta \in N(r_\gamma)$ implies that $\langle \gamma, \beta \rangle > 0$; thus $r_\beta \cdot \gamma \in \Phi_\Psi$ by minimality of γ. If $r_\beta \cdot \gamma$ is negative, then $-r_\beta \cdot \gamma$ is positive, and since $\mathrm{dp}(-r_\beta \cdot \gamma) < \mathrm{dp}(\beta) < \mathrm{dp}(\gamma)$ by (2.22), minimality of γ forces $-r_\beta \cdot \gamma \in \Phi_\Psi$. So in any case $r_\beta \cdot \gamma \in \Phi_\Psi$. But since $r_\beta \in W_\Psi$, this yields $\gamma \in \Phi_\Psi$, a contradiction. Hence $\Phi_\Gamma = \Phi_\Psi$, and thus $W_\Gamma = W_\Psi$.

Now let α, $\beta \in \Psi$ with $\alpha \neq \beta$. It remains to show that $\langle \alpha, \beta \rangle \leq -1$ or $\langle \alpha, \beta \rangle = -\cos(\pi/m)$ for some integer m. If $\langle \alpha, \beta \rangle \leq -1$ or $\langle \alpha, \beta \rangle = 0$ this is certainly true, so assume without loss of generality that $\langle \alpha, \beta \rangle > -1$ and $\langle \alpha, \beta \rangle \neq 0$. Then $r_\alpha \cdot \beta$, $r_\beta \cdot \alpha \in \Phi^+$ since α, $\beta \in \Psi$; furthermore, $r_\beta \cdot (r_\alpha \cdot \beta)$ is also positive, since $r_\alpha \cdot \beta$ is an element of $\Phi_\Psi \cap \Phi^+$ but not equal to β (as $\langle \alpha, \beta \rangle \neq 0$), and $N(r_\beta) \cap \Phi_\Psi = \{\beta\}$. Now $\beta = r_\alpha \cdot \beta + 2\langle \alpha, \beta \rangle \alpha$, and thus

$$-\beta = r_\beta \cdot \beta = r_\beta \cdot (r_\alpha \cdot \beta) + 2\langle \alpha, \beta \rangle r_\beta \cdot \alpha,$$

which forces $\langle \alpha, \beta \rangle$ to be less than 0; that is, $\langle \alpha, \beta \rangle \in (-1, 0)$.

We show now that if $\lambda\alpha + \mu\beta$ is a root in Φ_Γ, then λ and μ are either both nonnegative or both nonpositive. Assume for a contradiction that there exist λ, $\mu > 0$ such that $\lambda\alpha - \mu\beta$ is a root in Φ_Γ, and assume without loss of generality that this is a positive root. Then

$$r_\alpha \cdot (\lambda\alpha - \mu\beta) = \big(-\lambda + 2\langle \alpha, \beta \rangle \mu\big)\alpha - \mu\beta$$

is in Φ^-, since λ, $\mu > 0$ and $\langle \alpha, \beta \rangle < 0$. Thus $\lambda\alpha - \mu\beta \in N(r_\alpha) \cap \Phi_\Gamma$, forcing $\lambda\alpha - \mu\beta = \alpha$, and contradicting $\mu > 0$.

Now let $\theta \in (0, \frac{\pi}{2})$ such that $\langle \alpha, \beta \rangle = -\cos(\theta)$, and let $m \in \mathbb{N}$ be minimal such that $(m+1)\theta > \pi$; then $\sin\big(\theta(m+1)\big) < 0$ and $\sin(\theta m) \geq 0$. If m is even,

$$(r_\alpha r_\beta)^{m/2} \cdot \alpha = \frac{1}{\sin(\theta)}(\sin\big((m+1)\theta\big)\alpha + \sin(m\theta)\beta)$$

by (1.1)(i), while if m is odd,

$$(r_\alpha r_\beta)^{(m+1)/2} \cdot \beta = -\frac{1}{\sin(\theta)}(\sin\big((m+1)\theta\big)\alpha + \sin(m\theta)\beta).$$

Since $\sin\big((m+1)\theta\big) < 0$, the previous paragraph forces $\sin(m\theta) \leq 0$ in both cases, and we deduce that $\sin(m\theta) = 0$; hence $m\theta = \pi$, as required. □

The following corollary is an easy consequence of the previous theorem together with (1.8).

(2.23) COROLLARY *Suppose $\Gamma \subseteq \Phi$. Then there exists a standard geometric realization of W_Γ with root system $\widetilde{\Phi}$ and bilinear form $(\,,\,)$, and a bijection $\psi\colon \widetilde{\Phi} \to \Phi_\Gamma$ with*

(i) $\psi\big(\widetilde{\Phi}^+\big) = \Phi_\Gamma \cap \Phi^+$,

(ii) $\psi(w \cdot \widetilde{\alpha}) = w \cdot \psi(\widetilde{\alpha})$ *for* $w \in W_\Gamma$ *and* $\widetilde{\alpha} \in \widetilde{\Phi}$, *and*

(iii) $\big\langle \psi(\widetilde{\alpha}), \psi(\widetilde{\beta})\big\rangle = \big(\widetilde{\alpha}, \widetilde{\beta}\big)$ *for all* $\widetilde{\alpha}$, $\widetilde{\beta} \in \widetilde{\Phi}$.

(2.24) COROLLARY *Let α and β be roots such that $\langle \alpha, \beta\rangle \in (-1, 1)$. Then $W_{\{\alpha,\beta\}}$ is finite.*

Proof. We show first that $\langle\,,\,\rangle$ restricted to $V_{\{\alpha,\beta\}}$ is positive definite. So suppose that $v \in V_{\{\alpha,\beta\}}$ with $\langle v, v\rangle \leq 0$, and let λ, $\mu \in \mathbb{R}$ such that v equals $\lambda\alpha + \mu\beta$. Then

$$\langle v, v\rangle = \lambda^2 + \mu^2 + 2\langle \alpha, \beta\rangle\lambda\mu = \big(\lambda + \langle \alpha, \beta\rangle\mu\big)^2 + \big(1 - \langle \alpha, \beta\rangle^2\big)\mu^2;$$

since $1 - \langle \alpha, \beta\rangle^2 > 0$, this forces $\lambda = \mu = 0$. Thus v equals the zero-vector, as required.

Now let $\Gamma = \{\alpha, \beta\}$; as in (2.22), define Ψ to be the set of all $\gamma \in \Phi_\Gamma \cap \Phi^+$ with $N(r_\gamma) \cap \Phi_\Gamma = \{\gamma\}$. Since $\langle\,,\,\rangle$ restricted to $V_{\{\alpha,\beta\}}$ is positive definite, the remark following (1.8) yields that the elements of Ψ must be linearly independent. So $\Psi = \{\alpha', \beta'\}$ for some roots α' and β', as $V_{\{\alpha,\beta\}} = V_\Psi$ is of dimension 2. Since $\langle\,,\,\rangle$ is positive definite on $V_{\{\alpha',\beta'\}}$, we know further that

$$0 < \langle \alpha' + \beta', \alpha' + \beta'\rangle = 2 + 2\langle \alpha', \beta'\rangle,$$

and thus $\langle \alpha', \beta'\rangle > -1$. So $\langle \alpha', \beta'\rangle = -\cos(\pi/m)$ for some integer m by (2.21), and it follows by (1.1)(i) and the faithfulness of the standard geometric realization of W that $W_\Gamma = W_\Psi$ is finite. □

(2.25) LEMMA *The support of a root is always a connected subgraph of the Coxeter graph.*

Proof. Let α be a root. Since $\operatorname{supp}(\alpha) = \operatorname{supp}(-\alpha)$, we may assume without loss of generality that α is positive. If α has depth 1, then $|\operatorname{supp}(\alpha)| = 1$, and a graph with one vertex is connected. So suppose next that α is of depth greater than 1, and assume that the assertion is true for all positive roots of depth less than $\mathrm{dp}(\alpha)$. Let $r \in R$ such that $\mathrm{dp}(r \cdot \alpha) = \mathrm{dp}(\alpha) - 1$; that is, $\langle \alpha, \alpha_r \rangle > 0$. Since α is a positive root, and $\langle \alpha_s, \alpha_r \rangle \leq 0$ for all $s \in R$ with $s \neq r$, this yields that α_r is an element of $\mathrm{supp}(\alpha)$. By induction, the support of $r \cdot \alpha$ is connected, and the definition of the action of W on V yields that

$$\mathrm{supp}(\alpha) = \mathrm{supp}(r \cdot \alpha) \cup \{\alpha_r\}.$$

If α_r is in the support of $r \cdot \alpha$, then $\mathrm{supp}(\alpha) = \mathrm{supp}(r \cdot \alpha)$, which is connected. If $\alpha_r \notin \mathrm{supp}(r \cdot \alpha)$, then α_r must be adjoined to an element of $\mathrm{supp}(r \cdot \alpha)$, since $\langle r \cdot \alpha, \alpha_r \rangle = -\langle \alpha, \alpha_r \rangle \neq 0$; hence $\mathrm{supp}(\alpha)$ is again connected, and this finishes the proof. □

The next proposition is a generalization of the well known fact that if the coefficient of a simple root in a root is greater than 0, it must be greater than or equal to 1.

(2.26) PROPOSITION *Let α be a positive root, and let $r \in R$. Then the coefficient of α_r in α is either greater than or equal to 2, or equals* 0, 1 *or* $2\cos(\pi/m_{st})$ *for some s, $t \in R$ with $4 \leq m_{st} < \infty$. In particular, since $2\cos(\pi/m) \geq \sqrt{2}$ for $m \geq 4$, this yields that the coefficient of α_r in α equals* 0, 1 *or is greater than or equal to $\sqrt{2}$.*

Proof. Let $\alpha = \sum_{s \in R} \lambda_s \alpha_s$, and assume without loss of generality that λ_r is positive. If $|R| = 1$, then $\alpha = \alpha_r$ and $\lambda_r = 1$, and there is nothing left to show; thus we may assume from now on that $|R| \geq 2$.

Suppose now that $I(\alpha) \subseteq \{r, s\}$ for some $s \in R \setminus \{r\}$; (we call this the rank 2 case). If $\langle \alpha_r, \alpha_s \rangle$ is less than or equal to -1 (that is, rs has infinite order), then the result is an easy consequence of (1.1)(ii). So let $m = m_{rs} < \infty$. We deduce from (1.1)(i) that there exists an $l \in \{0, \ldots, 2m\}$ such that λ_r equals $\sin(l\pi/m)/\sin(\pi/m)$. Since $\lambda_r > 0$, clearly $1 \leq l \leq m$, and by symmetry of sine on the interval $[0, \pi]$, we may assume without loss of generality that $1 \leq l \leq \frac{m}{2}$. Now $\lambda_r = 1$ if $l = 1$, and $\lambda_r = 2\cos(\pi/m)$ if $l = 2$; finally, if $l \geq 3$, in particular $m \geq 2l \geq 6$, and thus

$$\frac{\sin(l\pi/m)}{\sin(\pi/m)} \geq \frac{\sin(3\pi/m)}{\sin(\pi/m)} = 2\cos(2\pi/m) + 1 \geq 2\cos(2\pi/6) + 1 = 2,$$

as required.

The general case is shown by induction on the depth of α. If α has depth 1, then $\alpha = \alpha_r$, and the result follows trivially. So suppose $\mathrm{dp}(\alpha) > 1$, and let $s \in R$ and $w \in W$ with $\alpha = w \cdot \alpha_s$ and $l(w) = \mathrm{dp}(\alpha) - 1$. Further, let $t \in R$ such that $l(wt) < l(w)$; then clearly $t \neq s$, as $w \cdot \alpha_s \in \Phi^+$. Choose $w' \in wW_{\{s,t\}}$ of minimal length. Then $l(w') \leq l(wt) < l(w)$, and also $l(w's) \geq l(w')$ and $l(w't) \geq l(w')$ by minimality of w'. So the roots $w' \cdot \alpha_s$ and $w' \cdot \alpha_t$ are positive by (1.3); moreover, each of these roots is of depth at most $l(w') + 1$, which is less than $\mathrm{dp}(\alpha)$. Hence the inductive hypothesis applies to $w' \cdot \alpha_s$ and $w' \cdot \alpha_t$.

There exists a $u \in W_{\{s,t\}}$ with $w = w'u$, and we have $u \cdot \alpha_s = \mu\alpha_s + \nu\alpha_t$ for some μ, $\nu \in \mathbb{R}$. If $u \cdot \alpha_s \in \Phi^-$, then μ, $\nu \leq 0$, and so

$$\alpha = (w'u) \cdot \alpha_s = \mu(w' \cdot \alpha_s) + \nu(w' \cdot \alpha_t)$$

is negative, contrary to our hypothesis. Hence μ, $\nu \geq 0$; in fact, μ, $\nu > 0$, since otherwise $\alpha = w'u \cdot \alpha_s$ would equal either $w' \cdot \alpha_s$ or $w' \cdot \alpha_t$, contradicting $\mathrm{dp}(\alpha) > l(w') + 1$. Since the assertion is true for $u \cdot \alpha_s$ by the rank 2 case, this implies that μ, $\nu \geq 1$.

Let μ_r, ν_r be the coefficients of α_r in $w' \cdot \alpha_s$, $w' \cdot \alpha_t$ respectively, so that $\lambda_r = \mu\mu_r + \nu\nu_r \geq \mu_r + \nu_r$. If $\mu_r > 0$, then by inductive hypothesis $\mu_r \geq 1$, and the same is true for ν_r. So if both μ_r and ν_r are nonzero, then $\lambda_r \geq 1 + 1 = 2$; since $\lambda_r > 0$ by assumption, it will suffice to consider the case $\nu_r = 0$ and $\mu_r \geq 1$, and hence $\lambda_r = \mu\mu_r \geq 1$. If $\lambda_r = 1$, there is nothing left to show, so suppose $\lambda_r > 1$. We must have either $\mu > 1$ or $\mu_r > 1$. If μ and μ_r are both strictly greater than 1, then $\mu \geq \sqrt{2}$ (by the rank 2 case) and $\mu_r \geq \sqrt{2}$ (by induction), and thus $\lambda_r \geq \sqrt{2}\sqrt{2} = 2$, as required. This leaves us with the case that one of μ and μ_r is 1, and the other one is equal to λ_r. Then by induction or the rank 2 case, $\lambda_r \geq 2$ or $\lambda_r = 2\cos(\pi/m_{xy})$ for some x, $y \in R$ with $4 \leq m_{xy} < \infty$. $\square$

The arguments in the above proof also yield the following scholium.

(2.27) *Suppose $\langle \alpha_s, \alpha_t \rangle \geq -1$ for all s, $t \in R$, and let $\alpha \in \Phi^+$ and $r \in R$. Then the coefficient of α_r in α is a polynomial in C with coefficients in $\mathbb{N}_0$, where C is the set*

$$\left\{ \frac{\sin(l\pi/m)}{\sin(\pi/m)} \;\middle|\; 4 \leq m = m_{st} < \infty \text{ for } s,\, t \in R \text{ and } l \in \mathbb{N} \text{ with } l \leq \frac{m}{2} \right\}.$$

Chapter 3

Dominance and Elementary Roots

The main result of this chapter is the finiteness of the set of elementary roots defined below (given that R is finite). It can be shown that this is equivalent to the Parallel Wall Theorem of the preprint [2], in which the proof given is incomplete.

For α, $\beta \in \Phi^+$, we say that α *dominates* β (we write $\alpha \mathsf{~dom~} \beta$) if and only if $w \cdot \beta$ is negative for all $w \in W$ with $w \cdot \alpha$ negative; equivalently, α dominates β if and only if $w \cdot \alpha \in \Phi^+$ for all $w \in W$ with $w \cdot \beta \in \Phi^+$.

Observe that if α dominates β and $w \cdot \beta$ is positive, it follows trivially that $(w \cdot \alpha) \mathsf{~dom~} (w \cdot \beta)$. Note also that it is not a priori clear that the notion of dominance does not depend on W. That is, if $\Gamma \subseteq \Phi$, and α and β are roots in Φ_Γ such that $w \cdot \beta$ negative whenever $w \cdot \alpha \in \Phi^-$ for $w \in W_\Gamma$, it is not obvious that this means that $\alpha \mathsf{~dom~} \beta$. We will see shortly that it is true, however.

Define Δ to be the set of positive roots α such that α dominates some β in $\Phi^+ \setminus \{\alpha\}$, and define the set of *elementary roots* $\mathcal{E}$ to be $\Phi^+ \setminus \Delta$.

Note that since $r \cdot \alpha_r \in \Phi^-$ and $r \cdot \big(\Phi \setminus \{\alpha_r\}\big) \subseteq \Phi^+$ for $r \in R$, every simple root is elementary. Observe also that if α dominates β and $w \cdot \alpha$ is an elementary root for some $w \in W$, either $\alpha = \beta$ or $w \cdot \beta \in \Phi^-$. If $\alpha \in \Delta$, and $w^{-1} \cdot \alpha$ and $u \cdot \alpha$ are elementary for some u, $w \in W$, then $N(u) \cap N(w^{-1}) \neq \emptyset$; for if α dominates $\beta \in \Phi^+ \setminus \{\alpha\}$, then $w^{-1} \cdot \beta$ and $u \cdot \beta$ must be negative by the above. Thus $l(uw) \neq l(u) + l(w)$ by (1.11).

(3.28) Lemma *Let $\alpha \in \Delta$ and u, $w \in W$ with $u \cdot \alpha$, $w^{-1} \cdot \alpha \in \Pi$. Then $l(uw) \neq l(u) + l(w)$.*

By using $r_1 r_2 \cdots r_{j-1} r_j$ in place of u and $r_{j+1} r_{j+2} \cdots r_{l-1} r_l$ in place of w (with $\alpha = (r_{j+1} \cdots r_{l-1} r_l) \cdot \alpha_s$), we obtain

(3.29) Corollary *Let $r_1, \ldots, r_l \in R$ such that $l(r_1 \cdots r_l) = l$, and suppose furthermore that $(r_1 \cdots r_l) \cdot \alpha_s = \alpha_t$ for some s, $t \in R$. Then*

$$(r_j r_{j-1} \cdots r_2 r_1) \cdot \alpha_t = (r_{j+1} r_{j+2} \cdots r_{l-1} r_l) \cdot \alpha_s \in \mathcal{E} \text{ for all } j \in \{1, \ldots, l\}.$$

(3.30) LEMMA *Let α and β be distinct positive roots. Then α dom β if and only if* $\mathrm{dp}(w \cdot \alpha) > \mathrm{dp}(w \cdot \beta)$ *for all $w \in W$.*

Proof. Suppose first that $w \cdot \alpha$ is of depth strictly greater than $\mathrm{dp}(w \cdot \beta)$ for all $w \in W$. If $u \cdot \alpha$ is negative for some $u \in W$, then $\mathrm{dp}(u \cdot \beta) < \mathrm{dp}(u \cdot \alpha)$, and this is less than or equal to 0; whence $u \cdot \beta$ is negative, and we conclude that α dom β.

For the converse suppose that α dom β, and let $w \in W$. Suppose first that $w \cdot \alpha$ is positive, and let $u \in W$ and $r \in R$ such that $w \cdot \alpha = u \cdot \alpha_r$ and $l(u) = \mathrm{dp}(w \cdot \alpha) - 1$. Since $r \cdot (u^{-1}w) \cdot \alpha = -\alpha_r \in \Phi^-$ and α dom β, we know that $(ru^{-1}w) \cdot \beta$ is negative. So by (1.4), either $(u^{-1}w) \cdot \beta = \alpha_r$ or $(u^{-1}w) \cdot \beta$ is negative. But since α and β are distinct, the former is impossible, and thus $(u^{-1}w) \cdot \beta \in \Phi^-$. Hence $\mathrm{dp}(w \cdot \beta) \le l(u) < \mathrm{dp}(w \cdot \alpha)$, as required.

Finally, suppose that $w \cdot \alpha$ is negative, and thus $w \cdot \beta \in \Phi^-$ since α dom β. Take $u \in W$ and $r \in R$ with $l(u) = -\mathrm{dp}(w \cdot \beta)$ and $u \cdot (w \cdot \beta) = -\alpha_r$; then $(ruw) \cdot \beta = \alpha_r$ is positive, and thus $(ruw) \cdot \alpha$ must also be positive. Since $\alpha \ne \beta$ further $(ruw) \cdot \alpha \ne \alpha_r$, so $(uw) \cdot \alpha$ is still positive; that is, $u \cdot (-w \cdot \alpha) = -uw \cdot \alpha \in \Phi^-$. Hence

$$-\mathrm{dp}(w \cdot \beta) = l(u) \ge \mathrm{dp}(-w \cdot \alpha) = 1 - \mathrm{dp}(w \cdot \alpha),$$

and thus $\mathrm{dp}(w \cdot \beta) < \mathrm{dp}(w \cdot \alpha)$, as required.

□

Note that if α dom β, the previous lemma implies that $\mathrm{dp}(\alpha) \ge \mathrm{dp}(\beta)$, with equality only if $\alpha = \beta$, and so dom is antisymmetric. It is clear that dom is also transitive, and so dom is a partial order on Φ^+. The elementary roots are the minimal elements in this partial order, and for each $\alpha \in \Phi^+$ there exists a $\beta \in \mathcal{E}$ such that α dom β. So if u, $w \in W$, then $N(u) \cap N(w^{-1}) = \emptyset$ if and only if $N(u) \cap N(w^{-1}) \cap \mathcal{E} = \emptyset$; hence (1.11) yields the following result.

(3.31) LEMMA *Let $u, w \in W$. Then $l(uw) = l(u) + l(w)$ if and only if $N(u) \cap N(w^{-1}) \cap \mathcal{E} = \emptyset$.*

Suppose next that W is finite, and let w_R denote an element of W of maximal length. Then $l(w_R r) \le l(w_R)$ for all $r \in R$, and thus $w_R \cdot \alpha_r \in \Phi^-$. Hence $N(w_R) = \Phi^+$ (note that this also yields the uniqueness of w_R). We show now that $\mathrm{dp}(w_R \cdot \alpha) = \mathrm{dp}(-\alpha)$ for all $\alpha \in \Phi$. By (2.13) it suffices to show this for $\alpha \in \Phi^+$. Then $\langle \alpha, \beta \rangle > 0$ for $\beta \in N(r_\alpha)$; furthermore, r_α permutes $\Phi^+ \setminus N(r_\alpha)$ with $\langle \alpha, \beta \rangle > 0$ if and only if $\langle \alpha, r_\alpha \cdot \beta \rangle < 0$ for $\beta \in \Phi^+ \setminus N(r_\alpha)$. Therefore

$$|\Phi^+ \cap V_+(\alpha)| - |\Phi^+ \cap V_-(\alpha)| = |N(r_\alpha)|;$$

that is, $|N_+(w_R, \alpha)| - |N_-(w_R, \alpha)| = |N(r_\alpha)| = l(r_\alpha)$. Now (2.18) and (2.19) yield that

$$\mathrm{dp}(w_R \cdot \alpha) = \mathrm{dp}(\alpha) - l(r_\alpha) = \mathrm{dp}(\alpha) - \big(2\,\mathrm{dp}(\alpha) - 1\big) = 1 - \mathrm{dp}(\alpha),$$

as required. So if $\mathrm{dp}(\alpha) > \mathrm{dp}(\beta)$ for some roots α and β, then

$$\mathrm{dp}(w_R \cdot \alpha) = 1 - \mathrm{dp}(\alpha) < 1 - \mathrm{dp}(\beta) = \mathrm{dp}(w_R \cdot \beta).$$

This together with (3.30) yield that there is no non-trivial dominance in finite root systems.

The next proposition provides us with an alternative characterization of dominance, which shows that dominance is independent of W.

(3.32) PROPOSITION *Let α and β be positive roots. Then α dom β if and only if $\langle \alpha, \beta \rangle \geq 1$ and $\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta)$.*

Proof. Suppose first that α dom β. By (3.30) we need only show that $\langle \alpha, \beta \rangle$ is greater than or equal to 1. Since $r_\alpha \cdot \alpha$ is negative, $r_\alpha \cdot \beta$ must be negative, and this forces $\langle \alpha, \beta \rangle > 0$.

Assume for a contradiction that $\langle \alpha, \beta \rangle \in (0, 1)$; then (2.24) yields that $W_{\{\alpha, \beta\}}$ is a finite Coxeter group, and by (1.12) there exists a finite parabolic subgroup W_J of W and a $w \in W$ such that $wW_{\{\alpha,\beta\}}w^{-1} \subseteq W_J$. Then $w \cdot \alpha$ and $w \cdot \beta$ are in Φ_J, and by (3.30) and (2.16),

$$\mathrm{dp}_J(w \cdot \alpha) = \mathrm{dp}(w \cdot \alpha) > \mathrm{dp}(w \cdot \beta) = \mathrm{dp}_J(w \cdot \beta).$$

But now the remark preceding this proposition together with (2.16) imply

$$\mathrm{dp}(w_J w \cdot \alpha) = \mathrm{dp}_J\big(w_J \cdot (w \cdot \alpha)\big) < \mathrm{dp}_J\big(w_J \cdot (w \cdot \beta)\big) = \mathrm{dp}(w_J w \cdot \beta),$$

where w_J denotes the element of W_J of maximal length. This contradicts (3.30) with $w_J w$ in place of w, and hence $\langle \alpha, \beta \rangle \geq 1$ after all.

For the converse, assume that $\langle \alpha, \beta \rangle \geq 1$ and $\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta)$. First consider the case $\beta \in \Pi$, say $\beta = \alpha_r$. If $\alpha = \alpha_r$ there is nothing left to show, so suppose $r \cdot \alpha \in \Phi^+$. Then

$$\langle \alpha, r \cdot \alpha \rangle = \langle \alpha, \alpha \rangle - 2\langle \alpha, \alpha_r \rangle^2 = 1 - 2\langle \alpha, \alpha_r \rangle^2 \leq -1.$$

By (1.1)(ii) there are infinitely many roots of the form $\lambda\alpha + \mu(r \cdot \alpha)$ with $\lambda, \mu > 0$. Assume for a contradiction that α does not dominate β, and choose $w \in W$ such that $w \cdot \alpha \in \Phi^-$ and $w \cdot \alpha_r \in \Phi^+$. Then

$$w \cdot (r \cdot \alpha) = w \cdot \alpha + 2\langle \alpha, \alpha_r \rangle (-w \cdot \alpha_r)$$

is a positive linear combination of negative roots, and must therefore be negative. So $N(w)$ contains α and $r \cdot \alpha$, and hence also contains all roots of the form $\lambda\alpha + \mu(r \cdot \alpha)$ with $\lambda, \mu > 0$. This contradicts the finiteness of $N(w)$ (see (1.9)).

Proceeding by induction on $\mathrm{dp}(\beta)$, suppose now that $\mathrm{dp}(\beta) > 1$, and choose $r \in R$ such that $\mathrm{dp}(r \cdot \beta) = \mathrm{dp}(\beta) - 1$. Since $\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta) > 1$, clearly $r \cdot \alpha \in \Phi^+$. Further $\langle r \cdot \alpha, r \cdot \beta \rangle \geq 1$, and

$$\mathrm{dp}(r \cdot \alpha) \geq \mathrm{dp}(\alpha) - 1 \geq \mathrm{dp}(\beta) - 1 = \mathrm{dp}(r \cdot \beta).$$

Now $(r \cdot \alpha)$ dom $(r \cdot \beta)$ by induction, and therefore α dom β. □

Observe that if $\langle \alpha, \beta \rangle \geq 1$ for positive roots α and β, then by (3.32), α dom β or β dom α.

Next, let $\Gamma \subseteq \Phi$ and let α and β be positive roots in Φ_Γ such that $w \cdot \beta$ is negative for all $w \in W_\Gamma$ with $w \cdot \alpha$ negative. We show now that this yields that α dom β. If $\alpha = \beta$ this is certainly true, so suppose without loss of generality that $\alpha \neq \beta$. By (2.21) we know that W_Γ is a Coxeter group, and by (2.23) there exists a standard geometric realization of W_Γ with bilinear form $(\ ,\ )$ and root system $\widetilde{\Phi}$, and a bijection $\psi\colon \widetilde{\Phi} \to \Phi_\Gamma$ such that

(i) $\psi(\widetilde{\Phi}^+) = \Phi_\Gamma \cap \Phi^+$,

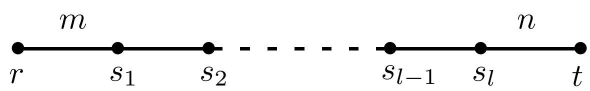
(ii) $\psi(w \cdot \widetilde{\alpha}) = w \cdot \psi(\widetilde{\alpha})$ for all $w \in W$ and $\widetilde{\alpha} \in \widetilde{\Phi}$, and

(iii) $\langle \psi(\widetilde{\alpha}), \psi(\widetilde{\beta}) \rangle = (\widetilde{\alpha}, \widetilde{\beta})$ for all $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\Phi}$.

Now $\psi^{-1}(\alpha)$ and $\psi^{-1}(\beta)$ are in $\widetilde{\Phi}^+$ by (i). If $w \cdot \psi^{-1}(\alpha) \in \widetilde{\Phi}^-$ for some $w \in W_\Gamma$, then $\psi\big(w \cdot \psi^{-1}(\alpha)\big) \in \Phi^-$ by (i), and thus $w \cdot \alpha \in \Phi^-$ by (ii);

hence $w \cdot \beta \in \Phi^-$ by hypothesis, and (again by (i) and (ii)), we find that $w \cdot \psi^{-1}(\beta) = \psi^{-1}(w \cdot \beta) \in \widetilde{\Phi}^-$. So $\psi^{-1}(\alpha)$ dominates $\psi^{-1}(\beta)$ with respect to W_Γ. Proposition (3.32) now yields that $\big(\psi^{-1}(\alpha), \psi^{-1}(\beta)\big) \geq 1$, and thus $\langle \alpha, \beta \rangle \geq 1$ by (iii). Hence α dom β or β dom α. Since $\alpha \neq \beta$, and thus $\psi^{-1}(\alpha) \neq \psi^{-1}(\beta)$, Lemma (3.30) implies that the depth of $\psi^{-1}(\alpha)$ with respect to the distinguished generating set of W_Γ is strictly greater than the depth of $\psi^{-1}(\beta)$. Let $w \in W_\Gamma$ such that $w \cdot \psi^{-1}(\beta) \in \widetilde{\Phi}^-$ with the length of w (with respect to the distinguished generating set of W_Γ) equal to the depth of $\psi^{-1}(\beta)$. Then $w \cdot \psi^{-1}(\alpha) \in \widetilde{\Phi}^+$, since $\psi^{-1}(\alpha)$ is of depth greater than the length of w. Therefore $w \cdot \alpha \in \Phi^+$ and $w \cdot \beta \in \Phi^-$ by (i) and (ii), and thus β cannot dominate α; whence α dom β.

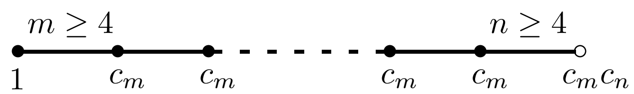
We now make use of (3.32) to give an alternative derivation of the well known classification of finite Coxeter groups. Suppose that W_J is a parabolic subgroup of W with Coxeter diagram



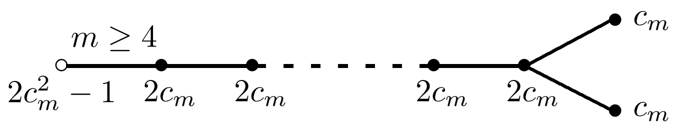
with m, $n \geq 4$. Denote the simple roots corresponding to r, s_j, t by x, y_j and z respectively, and define γ to be $t(s_l \cdots s_1) \cdot x$. Then

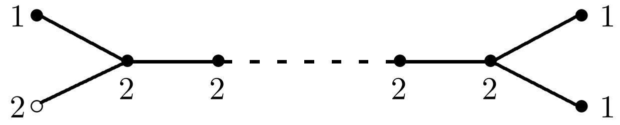
$$\gamma = x + c_m(y_1 + \cdots + y_l) + c_m c_n z,$$

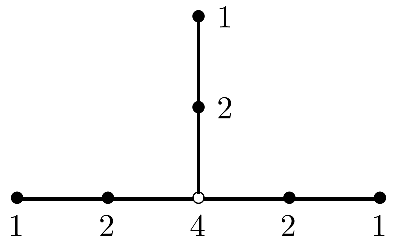
where $c_m = 2\cos(\pi/m)$ and $c_n = 2\cos(\pi/n)$, and an easy calculation yields that$\langle \gamma, z \rangle \geq 1$; since γ is certainly of depth greater than 1, we conclude that γ dom z and $\gamma \in \Delta$. So W_J must be infinite, and hence W must be infinite. The root γ appearing above can be conveniently described by means of the following diagram:

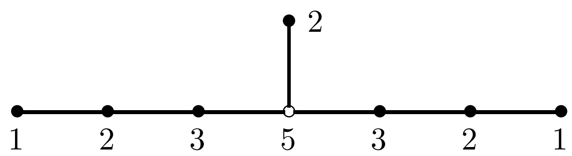


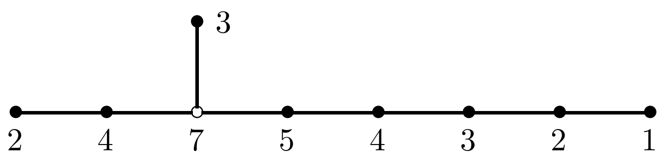
Note that the vertex in the above diagram corresponding to z (which is dominated by γ) is denoted by a circuit rather than a dot. Similarly, roots described by the following diagrams are necessarily in Δ.

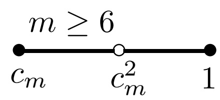

$m \geq 4$
$2c_m^2 - 1$
$2c_m$
$2c_m$
$2c_m$
$2c_m$
c_m
c_m

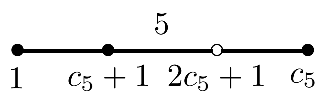

1
2
2
2
2
2
1
1

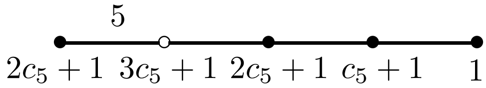

1
2
1 2 4 2 1

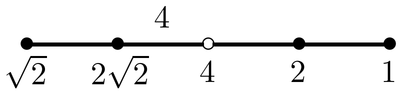

2
1 2 3 5 3 2 1


3
2 4 7 5 4 3 2 1


$m \geq 6$
c_m c_m^2 1

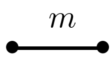

5
1 $c_5 + 1$ $2c_5 + 1$ c_5





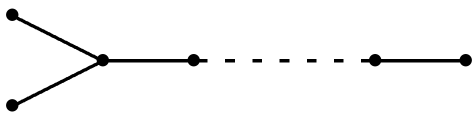
Consequently a finite Coxeter group cannot have a parabolic subgroup of type corresponding to any of the above diagrams. In (3.12) and (3.14) we show that Δ is also non-empty if R contains a circuit or an infinite bond, and this yields the following well-known theorem:

(3.33) THEOREM *Suppose W is finite. Then the Coxeter graph of W has finitely many connected components, and each of these is one of the following shapes:*

$I_2(m)$ 

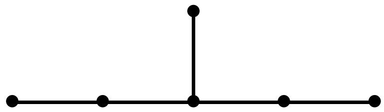
A_n

B_n 

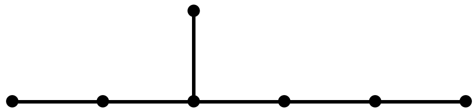
D_n 

F_4 

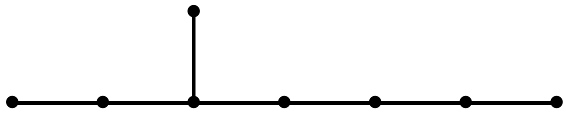
E_6



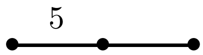
E_7



E_8



H_3



H_4

It is a straightforward matter to show that the form $\langle\,,\,\rangle$ is positive definite for each of the diagrams in this list. Consequently there can be no nontrivial dominance in these root systems, since if α dominates β,

$$\langle \alpha-\beta, \alpha-\beta\rangle = \langle \alpha,\alpha\rangle + \langle \beta,\beta\rangle - 2\langle \alpha,\beta\rangle = 2(1-\langle\alpha,\beta\rangle) \le 0.$$

From Theorem (3.17) below it follows that the root systems (and hence the groups) are finite in these cases.

We now define a second partial order $\preceq$ on Φ, which will enable us to stop our search for elementary roots in an ascending chain with respect to $\preceq$, as soon as we find a non-elementary root (see (3.9)). This fact is an important tool in the proof of the finiteness of the set of elementary roots.

For roots α and β we say that α *precedes* β (and write $\alpha \preceq \beta$) if there exists a $w \in W$ with $\beta = w \cdot \alpha$ and $N(w) \subseteq V_-(\alpha)$; that is, $N(w) = N_-(w,\alpha)$ and $N_+(w,\alpha) = N_0(w,\alpha) = \emptyset$. If $\alpha \preceq \beta$, we also write $\beta \succeq \alpha$ and say that β is a *successor* of α. We write $\alpha \prec \beta$ or $\beta \succ \alpha$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$.

Note that if α is a root and $r \in R$, then

$$r \cdot \alpha \begin{cases} \prec \alpha & \text{if } \langle \alpha, \alpha_r \rangle > 0\text{; that is, } \mathrm{dp}(r \cdot \alpha) = \mathrm{dp}(\alpha) - 1, \\ = \alpha & \text{if } \langle \alpha, \alpha_r \rangle = 0, \\ \succ \alpha & \text{if } \langle \alpha, \alpha_r \rangle < 0\text{; that is, } \mathrm{dp}(r \cdot \alpha) = \mathrm{dp}(\alpha) + 1. \end{cases}$$

In particular, if α is a positive root, then there exists an $r \in R$ with $\alpha \succ r \cdot \alpha$. Since $\mathrm{dp}(r \cdot \alpha) = \mathrm{dp}(\alpha) - 1$ in this case, an iteration yields that each positive root is preceded by a simple one.

If $\beta = w \cdot \alpha$ for some $w \in W$, then

$$\mathrm{dp}(\beta) - \mathrm{dp}(\alpha) = \mathrm{dp}(w \cdot \alpha) - \mathrm{dp}(\alpha) = |N_-(w, \alpha)| - |N_+(w, \alpha)|$$

by (2.18), and it is clear that $\alpha \preceq \beta$ if and only if there exists a $w \in W$ of length equal to $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$ such that $\beta = w \cdot \alpha$. Therefore $\preceq$ is antisymmetric, and we show now that $\preceq$ is also transitive, and thus a partial order. So let α, β and γ be roots with $\alpha \preceq \beta$ and $\beta \preceq \gamma$. Then there exist u, $w \in W$ such that $\beta = w \cdot \alpha$ and $\gamma = u \cdot \beta$ with $N(w) = N_-(w, \alpha)$ and $N(u) = N_-(u, \beta)$. Thus $\gamma = uw \cdot \alpha$, and (2.17) gives

$$N_+(uw, \alpha) \subseteq N_+(w, \alpha) \cup w^{-1} \cdot N_+(u, w \cdot \alpha) = N_+(w, \alpha) \cup w^{-1} \cdot N_+(u, \beta) = \emptyset,$$

and similarly $N_0(uw, \alpha) = \emptyset$; hence $N(uw) = N_-(uw, \alpha)$ and $\alpha \preceq \gamma$, as required.

We can now state a slightly weaker criterion for precedence.

(3.34) LEMMA *Let α, $\beta \in \Phi$ such that $\beta = w \cdot \alpha$ for some $w \in W$ with $N_+(w, \alpha) = \emptyset$. Then $\alpha \preceq \beta$.*

Proof. If $w = 1$, this is certainly true; so suppose $l(w) > 0$, and proceed by induction. Let $r \in R$ and $u \in W$ such that $w = ur$ and $l(w) = l(u) + 1$. Then $N_+(u, r \cdot \alpha) \subseteq r \cdot N_+(w, \alpha) = \emptyset$ by (2.17), and as $\beta = u \cdot (r \cdot \alpha)$, it follows by induction that $\beta \succeq r \cdot \alpha$. Lemma (2.17) implies furthermore that $N_+(r, \alpha) \subseteq N_+(w, \alpha) = \emptyset$, and thus $\langle \alpha, \alpha_r \rangle \leq 0$; we deduce that $r \cdot \alpha \succeq \alpha$, and hence $\beta \succeq \alpha$ by transitivity of $\succeq$, as required. □

We show now that if $\beta \succeq \alpha$, then there exist roots $\alpha_1, \ldots, \alpha_{d-1}$ with $\mathrm{dp}(\alpha_i) = \mathrm{dp}(\alpha) + i$ and $d = \mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$ such that

$$\alpha \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{d-1} \prec \beta.$$

If $\alpha = \beta$ this is trivially true; so suppose $\alpha \prec \beta$, and let $w \in W$ with $l(w) = \mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$ such that $\beta = w \cdot \alpha$. Then $w \neq 1$ as $\alpha \neq \beta$, and thus there exist $u \in W$, $r \in R$ such that $w = ru$ and $l(w) = l(u) + 1$. Now

$$\begin{aligned} \mathrm{dp}(\beta) - 1 \leq \mathrm{dp}(r \cdot \beta) &= \mathrm{dp}(u \cdot \alpha) \\ &= \mathrm{dp}(\alpha) + |N_-(u, \alpha)| - |N_+(u, \alpha)| \\ &\leq \mathrm{dp}(\alpha) + l(u) = \mathrm{dp}(\alpha) + l(w) - 1 = \mathrm{dp}(\beta) - 1, \end{aligned}$$

and we must have equality everywhere; in particular, $\mathrm{dp}(r \cdot \beta) = \mathrm{dp}(\beta) - 1$ and $l(u) = |N_-(u, \alpha)|$. So $\beta \succ r \cdot \beta = u \cdot \alpha$ and $N(u) \subseteq V_-(\alpha)$, and thus $r \cdot \beta \succeq \alpha$. Since $\mathrm{dp}(r \cdot \beta) - \mathrm{dp}(\alpha) = \mathrm{dp}(\beta) - \mathrm{dp}(\alpha) - 1$, induction yields that there exist roots $\alpha_1, \ldots, \alpha_{d-2}$ with $\mathrm{dp}(\alpha_i) = \mathrm{dp}(\alpha) + i$ such that

$$\alpha \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{d-2} \prec r \cdot \beta\,(= \alpha_{d-1} \prec \beta),$$

as required. In particular, if $\alpha \prec \beta$, then there exist $r, s \in R$ such that $\alpha \preceq r \cdot \beta \prec \beta$ and $\alpha \prec s \cdot \alpha \preceq \beta$.

(3.35) LEMMA *Let $\alpha = \sum_{r \in R} \lambda_r \alpha_r$ and $\beta = \sum_{r \in R} \mu_r \alpha_r$ be such that $\alpha \preceq \beta$. Then $\lambda_r \leq \mu_r$ for all $r \in R$.*

Proof. If $\alpha = \beta$, the assertion is trivially true, so suppose $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha) > 0$, and let $s \in R$ with $\beta \succeq s \cdot \alpha \succ \alpha$; then $\mathrm{dp}(s \cdot \alpha) = \mathrm{dp}(\alpha) + 1$, and thus $\mathrm{dp}(\beta) - \mathrm{dp}(s \cdot \alpha) < \mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$. Further, $\langle \alpha, \alpha_s \rangle < 0$ and

$$s \cdot \alpha = \sum_{r \in R \setminus \{s\}} \lambda_r \alpha_r + \big(\lambda_s - 2\langle \alpha, \alpha_s \rangle\big) \alpha_s;$$

so by induction, $\mu_r \geq \lambda_r$ for all $r \in R \setminus \{s\}$ and $\mu_s \geq \lambda_s - 2\langle \alpha, \alpha_s \rangle > \lambda_s$, as required. $\square$

Note that this also yields that if $\beta = w \cdot \alpha$ and $N(w) \subseteq V_-(\alpha)$ for $w \in W$, then $w \in W_I$, where I consists of all $r \in R$ such that the coefficient of α_r in α is strictly less than the coefficient of α_r in β.

(3.36) LEMMA *Let $\alpha, \beta \in \Phi^+$ such that $\alpha \preceq \beta$ and $\alpha \in \Delta$. Then $\beta \in \Delta$.*

Proof. If $\alpha = \beta$, the assertion is again trivially true. Assume next that $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha) > 0$, and let $s \in R$ with $\beta \succeq s \cdot \alpha \succ \alpha$. Then $s \cdot \alpha$ is of depth $\mathrm{dp}(\alpha) + 1$, and thus $\mathrm{dp}(\beta) - \mathrm{dp}(s \cdot \alpha) < \mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$; furthermore, $\langle \alpha, \alpha_s \rangle < 0$. Next, let $\gamma \in \Phi^+ \setminus \{\alpha\}$ such that α dominates γ; then $\langle \alpha, \gamma \rangle \geq 1$ by (3.32), and thus clearly $\gamma \neq \alpha_s$. So $s \cdot \gamma \in \Phi^+$ by (1.4), and it follows easily that $(s \cdot \alpha)$ dom $(s \cdot \gamma)$; since obviously $s \cdot \alpha \neq s \cdot \gamma$, we deduce that $s \cdot \alpha$ is in Δ, and thus $\beta \in \Delta$ by induction. □

The next lemma gives an algorithm that has as its input the set of elementary roots of depth n, and computes the set of elementary roots of depth $n + 1$.

(3.37) LEMMA *For all $n \in \mathbb{N}$, define $\mathcal{E}_n = \{\, \alpha \in \mathcal{E} \mid \mathrm{dp}(\alpha) = n \,\}$. Then*

$$\mathcal{E}_{n+1} = \big\{\, r \cdot \alpha \mid \alpha \in \mathcal{E}_n \text{ and } r \in R \text{ with } \langle \alpha, \alpha_r \rangle \in (-1, 0) \,\big\}.$$

Proof. First, let $\alpha \in \mathcal{E}_n$ and $r \in R$ with $\langle \alpha, \alpha_r \rangle \in (-1, 0)$, and suppose that $r \cdot \alpha$ dominates some $\beta \in \Phi^+$; then $\langle r \cdot \alpha, \beta \rangle \geq 1$, and thus $\beta \neq \alpha_r$, since $\langle r \cdot \alpha, \alpha_r \rangle = -\langle \alpha, \alpha_r \rangle \in (0, 1)$. So $r \cdot \beta \in \Phi^+$ by (1.4), and it follows that α dom $(r \cdot \beta)$. As α is elementary, this implies that $r \cdot \beta$ equals α; that is, $\beta = r \cdot \alpha$. Therefore $r \cdot \alpha$ is elementary, and since $\mathrm{dp}(r \cdot \alpha) = \mathrm{dp}(\alpha) + 1$ by (2.14) we have $r \cdot \alpha \in \mathcal{E}_{n+1}$.

For the converse, suppose that α is an elementary root of depth $n + 1$ (which is greater than 1), and let $r \in R$ with $r \cdot \alpha \prec \alpha$. Then $r \cdot \alpha \in \mathcal{E}_n$, since $r \cdot \alpha$ is of depth n by (2.14), and elementary by (3.36). Furthermore, $\langle \alpha, \alpha_r \rangle > 0$ since $r \cdot \alpha \prec \alpha$; on the other hand, $\langle \alpha, \alpha_r \rangle < 1$ by (3.32), as α is of depth greater than 1 and cannot dominate α_r. Thus $\langle \alpha, \alpha_r \rangle \in (0, 1)$, and hence $\langle r \cdot \alpha, \alpha_r \rangle \in (-1, 0)$, as required. □

(3.38) LEMMA *Let α, $\beta \in \Phi^+$ with $\beta \preceq \alpha$, and let $r \in R$ such that $\langle \beta, \alpha_r \rangle$ is less than or equal to -1. Then $\alpha \in \Delta$, or the coefficients of α_r in α and β coincide.*

Proof. Suppose that the coefficients of α_r in α and β do not coincide, and let γ be a root of maximal depth with $\beta \preceq \gamma \preceq \alpha$ such that the coefficients of α_r in β and γ coincide; then $\gamma \prec r \cdot \gamma \preceq \alpha$ by maximality of γ. Now $\gamma - \beta = \sum_{s \in R \setminus \{r\}} \lambda_s \alpha_s$ for some $\lambda_s \geq 0$, and hence

$$\langle r \cdot \gamma, \alpha_r \rangle = -\langle \gamma, \alpha_r \rangle = -\langle \beta, \alpha_r \rangle - \sum_{s \in R \setminus \{r\}} \lambda_s \langle \alpha_s, \alpha_r \rangle \geq 1,$$

since $\langle \alpha_s, \alpha_r \rangle \leq 0$ for $s \neq r$. As $\mathrm{dp}(r \cdot \gamma) > 1 = \mathrm{dp}(\alpha_r)$, this implies that $r \cdot \gamma \in \Delta$, and thus $\alpha \in \Delta$ by (3.36). □

(3.39) COROLLARY *Let $\alpha \in \Phi^+$ such that* $\mathrm{supp}(\alpha)$ *contains a circuit. Then $\alpha \in \Delta$.*

Proof. Let β be a positive root of minimal depth preceding α such that $\mathrm{supp}(\beta)$ contains a circuit, and let $r \in R$ with $r \cdot \beta \prec \beta$. By minimality of β we find that $\mathrm{supp}(r \cdot \beta)$ does not contain a circuit. Now let $\beta = \sum_{x \in R} \lambda_x \alpha_x$ and $r \cdot \beta = \sum_{x \in R} \mu_x \alpha_x$. Since $\lambda_x = \mu_x$ for all $x \neq r$, and the support of β contains a circuit, while the support of $r \cdot \beta$ does not, it follows that $\mu_r = 0$, and that α_r is part of a circuit in $\mathrm{supp}(\beta)$. Hence there exist at least two elements α_s, α_t of $\mathrm{supp}(\beta) \setminus \{\alpha_r\}$ such that α_r is adjoined to α_s as well as α_t. By definition of $\langle\, ,\, \rangle$, it follows that $\langle \alpha_r, \alpha_s \rangle$ and $\langle \alpha_r, \alpha_t \rangle$ are both at most $-\cos(\pi/3) = -\frac{1}{2}$, while $\langle \alpha_r, \alpha_x \rangle \leq 0$ for all other $x \in R \setminus \{r\}$. Furthermore, μ_s, $\mu_t \geq 1$ by (2.26), and thus

$$\langle r \cdot \beta, \alpha_r \rangle = \sum_{x \in R} \mu_x \langle \alpha_x, \alpha_r \rangle = \sum_{x \neq r} \mu_x \langle \alpha_x, \alpha_r \rangle \leq \mu_s \langle \alpha_s, \alpha_r \rangle + \mu_t \langle \alpha_t, \alpha_r \rangle \leq -1.$$

Since the coefficient of α_r in β is not equal to the coefficient of α_r in $r \cdot \beta$, Lemma (3.38) implies that $\beta \in \Delta$; hence $\alpha \in \Delta$ by (3.36), as required. □

(3.40) Corollary *Let r, $s \in R$ be adjoined. Further, suppose that α and β are positive roots with $\alpha \succeq \beta$ such that $r \in I(\alpha) \setminus I(\beta)$, and the coefficient of α_s in β is greater than or equal to $|\langle \alpha_r, \alpha_s \rangle|^{-1}$. Then $\alpha \in \Delta$.*

Proof. Let $\beta = \sum_{t \in R \setminus \{r\}} \lambda_t \alpha_t$. Then

$$\langle \beta, \alpha_r \rangle = \sum_{t \in R \setminus \{r\}} \lambda_t \langle \alpha_t, \alpha_r \rangle = \sum_{t \neq r,s} \lambda_t \langle \alpha_t, \alpha_r \rangle + \lambda_s \langle \alpha_s, \alpha_r \rangle \leq \lambda_s \langle \alpha_s, \alpha_r \rangle \leq -1.$$

Since the coefficient of α_r in α is not equal to the coefficient of α_r in β, Lemma (3.38) implies that $\alpha \in \Delta$. □

(3.41) Corollary *Let r, $s \in R$ such that r and s are adjoined by an infinite bond, and suppose that α is a positive root with both α_r and α_s in its support. Then $\alpha \in \Delta$.*

Proof. Interchanging r and s if necessary, we may choose $\beta \preceq \alpha$ such that $r \in I(\beta)$ and $s \notin I(\beta)$. Since the coefficient of α_r in β is greater than or equal to 1 by (2.26), and thus greater than or equal to $|\langle \alpha_r, \alpha_s \rangle|^{-1}$, (3.40) implies that $\alpha \in \Delta$. □

Our proof of the fact that $\mathcal{E}$ is finite (if R is finite) depends on the finiteness of the set of real numbers $\{ \langle \alpha, \alpha_r \rangle \mid \alpha \in \mathcal{E} \text{ and } r \in R \}$. The next definition facilitates the statement of the relevant facts.

Define $\mathcal{C}(R)$ to be the set of all real numbers of the form $\cos(n\pi/m)$ with $n \in \{1, \ldots, m-1\}$ and $m = m_{rs} < \infty$ for some r, $s \in R$. If R is finite, then $|\mathcal{C}(R)|$ is less than or equal to the sum of all $m - 1$ with $m = m_{rs} < \infty$ for some r, $s \in R$.

The next proposition is a slight variation of (1.12). It yields that if $\langle \alpha, \beta \rangle \in (-1, 1)$ for some roots α and β, it follows that $\langle \alpha, \beta \rangle \in \mathcal{C}(R)$. For (2.24) implies that $W_{\{\alpha,\beta\}}$ is finite if $\langle \alpha, \beta \rangle \in (-1, 1)$, and by the next assertion there exist r, $s \in R$ and $w \in W$ such that $w \cdot \alpha$, $w \cdot \beta \in \Phi_{\{r,s\}}$; therefore we can deduce from (1.1)(i) that

$$\langle \alpha, \beta \rangle = \langle w \cdot \alpha, w \cdot \beta \rangle = \cos(n\pi/m_{rs})$$

for some $n \in \mathbb{N}$, as required.

(3.42) PROPOSITION *Let $\Gamma \subseteq \Phi$ such that W_Γ is finite. Then there exists a finite parabolic subgroup W_I of W with $|I| \leq |\Gamma|$ such that W_Γ is conjugate to a subgroup of W_I.*

Proof. Since W_Γ is finite, we know by (1.12) that W_Γ is conjugate to a subgroup of a finite parabolic subgroup W_J, and we may assume without loss of generality that $W_\Gamma \subseteq W_J$. If $|J| = |\Gamma|$ the assertion is true, so suppose now that $|J| > |\Gamma|$. We show that there exists an $I \subseteq J$ with $I \neq J$ such that W_Γ is conjugate to a subgroup of W_I, and the assertion will follow by induction.

As in the proof of (1.12), let V_J^* denote the dual space of V_J, acted upon from the right by W_J, and for $f \in V_J^*$ define $S(f) = \{\, \gamma \in \Phi_J^+ \mid f(\gamma) < 0 \,\}$. Since the space spanned by Γ is a subspace of V_J, and has dimension less than $|J|$ (the dimension of V_J), there exists a nonzero vector $v_0 \in V_J$ such that $\langle v_0, \gamma \rangle = 0$ for all $\gamma \in \Gamma$. Define $F \in V_J^*$ by $F\colon v \mapsto \langle v, v_0 \rangle$; then $S(F)$ is finite, since Φ_J^+ is finite. As in the proof of (1.12) there exists an $x \in W$ such that $S(Fx) = \emptyset$ and $xW_\Gamma x^{-1} \subseteq W_I$, where $I = \{\, r \in J \mid (Fx)(\alpha_r) = 0 \,\}$. Theorem (3.33) states that the Coxeter graph of J consists of finitely many connected components, each of which is of one of the shapes described in (3.33), and thus it can be easily verified that $\langle\ ,\ \rangle$ restricted to V_J is positive definite; therefore $F \neq 0$, and thus $I \neq J$, as required. □

The following technical lemma, though trivial, provides the key for our proof of the main theorem.

(3.43) LEMMA *Let $\alpha = \Sigma_{r \in R} \lambda_r \alpha_r$ and $\beta = \Sigma_{r \in R} \mu_r \alpha_r$ be positive roots. Furthermore, suppose that there exist R_1, $R_2 \subseteq R$ with $R = R_1 \cup R_2$ such that $\langle \alpha, \alpha_r \rangle = \langle \beta, \alpha_r \rangle$ for all $r \in R_1$, and $\lambda_r = \mu_r$ for all $r \in R_2$. Then $\langle \alpha, \beta \rangle = 1$.*

Proof. Since $\alpha - \beta = \sum_{r \in R_1} (\lambda_r - \mu_r)\alpha_r$, we have

$$\langle \alpha, \alpha - \beta \rangle = \sum_{r \in R_1} (\lambda_r - \mu_r)\langle \alpha, \alpha_r \rangle = \sum_{r \in R_1} (\lambda_r - \mu_r)\langle \beta, \alpha_r \rangle = \langle \beta, \alpha - \beta \rangle,$$

and as $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$, this becomes $1 - \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle - 1$, and the result follows. □

(3.44) Theorem *$\mathcal{E}$ is finite, provided that R is finite.*

Proof. If R is finite, it is clear that $\mathcal{C}(R)$ is finite, and we set $c = |\mathcal{C}(R)|$. Since every root of depth d can be expressed as $(r_d r_{d-1} \cdots r_2) \cdot \alpha_{r_1}$ with each $r_i \in R$, there are no more than $|R|^d$ roots of depth d. If we can show that no root in $\mathcal{E}$ can have depth exceeding $c^{|R|}(|R|+1)+1$, the proof will be complete. So let $\beta \in \mathcal{E}$ have depth d, and let $\beta_1 \prec \cdots \prec \beta_d = \beta$ be a sequence of positive roots such that $\mathrm{dp}(\beta_i) = i$. Note that $\beta_i \in \mathcal{E}$ for each $i \in \{1, \ldots, d\}$. For $i \in \{1, \ldots, d\}$, define $J_i = \{\, r \in R \mid \langle \beta_i, \alpha_r \rangle > -1 \,\}$. If $r \notin J_i$, the coefficient of α_r in β_j is constant for all $j \geq i$ by (3.38); since $\langle \alpha_s, \alpha_r \rangle \leq 0$ for all $s \in R \setminus \{r\}$, it follows from (3.35) that $\langle \beta_j, \alpha_r \rangle \leq \langle \beta_i, \alpha_r \rangle$ for $j \geq i$, and hence $r \notin J_j$ for all $j \geq i$. Thus the sets J_i form a decreasing chain.

Suppose $J_i = \cdots = J_j = J$ for some $2 \leq i \leq j$. If $k \in \{i, \ldots, j\}$ and $r \in R$, then $\langle \beta_k, \alpha_r \rangle < 1$ by (3.32); (since β_k is of depth greater than 1 and cannot dominate α_r). Hence $\langle \beta_k, \alpha_r \rangle \in (-1, 1)$ for $r \in J$, and thus $\langle \beta_k, \alpha_r \rangle \in \mathcal{C}(R)$ by the remark preceding (3.42). So if $j - i \geq c^{|R|}$, then there exist $m, n \in \{i, i+1, \ldots, j\}$ with $n > m$ and $\langle \beta_n, \alpha_r \rangle = \langle \beta_m, \alpha_r \rangle$ for all $r \in J$. But if $r \notin J$, then α_r has the same coefficient in β_m as in β_n, and it follows by (3.43) that $\langle \beta_n, \beta_m \rangle = 1$. This contradicts (3.32), since $\beta_n \notin \Delta$. We conclude that if $j - i \geq c^{|R|}$, then J_j is strictly smaller than J_i. Since $J_2 \subseteq R$, it follows that the chain $J_2 \supseteq J_3 \supseteq \cdots \supseteq J_d$ can have length at most $c^{|R|}(|R|+1)$, and this finishes the proof. □

(3.45) Lemma *Let $\alpha \in \Phi^+$. Then $\alpha \in \mathcal{E}$ or there exists $\beta \in \Phi^+$ such that $\alpha \operatorname{\mathsf{dom}} \beta$ and $r_\beta \cdot \alpha \in \Phi^+$.*

Proof. If $\mathrm{dp}(\alpha) = 1$ then $\alpha \in \mathcal{E}$ and there is nothing left to show., So suppose $\mathrm{dp}(\alpha) > 1$ and $\alpha \notin \mathcal{E}$. Now let γ be a positive root different from α which is dominated by α. and let $r \in R$ such that $r \cdot \alpha \prec \alpha$. If $\gamma = \alpha_r$ choose $\beta = \alpha_r$. Then $r_\beta \cdot \alpha = r \cdot \alpha$ has depth greater or equal to 1 and hence is in Φ^+.

Suppose next $\gamma \neq \alpha_r$. Then $r \cdot \gamma \in \Phi^+$ by (1.6) and hence $r \cdot \alpha$ dominates $r \cdot \gamma$, whence $r \cdot \alpha \in \Delta_W$. By induction there exists a root $\beta' \in \Phi^+$ which is dominated by $r \cdot \alpha$ such that $r_{\beta'} \cdot (r \cdot \alpha) \in \Phi^+$. Since $\langle r \cdot \alpha, \alpha_r \rangle < 0$ as $r \cdot \alpha \prec \alpha$ while on the other hand $\langle r \cdot \alpha, \beta' \rangle \geq 1$ by (3.32) , we know $\beta' \neq \alpha_r$. Hence $r \cdot \beta' \in \Phi^+$ by (1.6) and thus $\alpha \operatorname{\mathsf{dom}} r \cdot \beta'$ as $\beta' \in \Phi^+$.

If $r_{r \cdot \beta'} \cdot \alpha \in \Phi^+$ choose $\beta = r \cdot \beta'$. Then clearly $\alpha \operatorname{\mathsf{dom}} \beta$ and $r_\beta \cdot \alpha \in \Phi^+$. This leaves us with the case $r_{r \cdot \beta'} \cdot \alpha \in \Phi^-$. Since $r_{r \cdot \beta'} = r r_{\beta'} r$ by (1.9) we find

$$r_{r\cdot\beta'} \cdot \alpha = r r_{\beta'} r \cdot \alpha = r \cdot (r_{\beta'} \cdot (r \cdot \alpha)).$$

As $r_{\beta'} \cdot (r \cdot \alpha) \in \Phi^+$ by choice of β', this forces $r_{\beta'} \cdot r \cdot \alpha$ to be equal to α_r. That is,

$$r \cdot \alpha = r_{\beta'} \cdot \alpha_r = \alpha_r - 2\langle \alpha_r, \beta' \rangle \beta'.$$

Now $1 \leq \langle r \cdot \alpha, \beta' \rangle$ by (3.32) and this equals

$$\langle \alpha_r, \beta' \rangle - 2\langle \alpha_r, \beta' \rangle \langle \beta', \beta' \rangle = -\langle \alpha_r, \beta' \rangle.$$

Thus $\langle r \cdot \beta', \alpha_r \rangle = -\langle \alpha_r, \beta' \rangle \geq 1$ and clearly $\mathrm{dp}(r \cdot \beta') \geq 1 = \mathrm{dp}(\alpha_r)$; by (3.32) this yields $r \cdot \beta'$ dom α_r, and by transitivity of dominance we find α dom α_r. So if we choose $\beta = \alpha_r$ then certainly α dom β and $r_\beta \cdot \alpha = r \cdot \alpha \in \Phi^+$. □

Chapter 4

The Stabilizer of a Root

We now show that the stabilizer of a root is the semidirect product of a Coxeter group and a free group.

For a root α, we denote the stabilizer of α in W by $W(\alpha)$. Any root can be written as $w \cdot \alpha_r$ for some $w \in W$ and $r \in R$, and an easy calculation yields that $W(w \cdot \alpha_r) = wW(\alpha_r)w^{-1}$; therefore we can restrict our attention to $W(\alpha_r)$ for $r \in R$.

Let $\Gamma(r)$ be the set of roots γ with $\langle \alpha_r, \gamma \rangle = 0$; that is, $\Gamma(r)$ equals $V_0(\alpha_r) \cap \Phi$. The group $W_{\Gamma(r)}$ generated by the reflections corresponding to the roots in $\Gamma(r)$ is a normal subgroup of $W(\alpha_r)$; moreover, Theorem (2.21) states that $W_{\Gamma(r)}$ is a Coxeter group. We will show that $W_{\Gamma(r)}$ has a complement Y_r in $W(\alpha_r)$, and that Y_r is isomorphic to the fundamental group of a certain graph. Well known arguments then show that Y_r is a free group.

Next, let X_r be a set of coset representatives of $W_{\Gamma(r)}$ in $W(\alpha_r)$ of minimal length. For $w \in X_r$, the minimality of $l(w)$ yields that $l(wr_\gamma) \geq l(w)$ for all roots $\gamma \in \Gamma(r)$, and it follows by the Strong Exchange Condition that

$N(w) \cap \Gamma(r) = \emptyset$; that is, $N_0(w, \alpha_r) = \emptyset$. So X_r is a subset of Y_r, the set of all $w \in W(\alpha_r)$ with $N_0(w, \alpha_r) = \emptyset$. We will see shortly that $X_r = Y_r$, and then use the concepts developed in the preceding chapter to prove that Y_r has the properties we have described.

(4.46) LEMMA *Let $\alpha \in \Phi$ and $w \in W$ such that $N_0(w, \alpha) = \emptyset$. Then $N_0(w^{-1}, w \cdot \alpha) = \emptyset$.*

Proof. Since $V_0(w \cdot \alpha) = w \cdot V_0(\alpha)$, we know that

$$\begin{aligned} N_0(w^{-1}, w \cdot \alpha) &= N(w^{-1}) \cap V_0(w \cdot \alpha) \\ &= N(w^{-1}) \cap w \cdot V_0(\alpha) \\ &= w \cdot \big(w^{-1} \cdot N(w^{-1}) \cap V_0(\alpha)\big). \end{aligned}$$

Further, $w^{-1} \cdot N(w^{-1}) = -N(w)$ by (1.10), while $-V_0(\alpha) = V_0(\alpha)$; therefore

$$N_0(w^{-1}, w \cdot \alpha) = -w \cdot \big(N(w) \cap V_0(\alpha)\big) = -w \cdot N_0(w, \alpha) = \emptyset.$$

□

Now let $w \in Y_r$. Then $N_0(w^{-1}, \alpha_r) = N_0(w^{-1}, w \cdot \alpha_r) = \emptyset$ by (4.46), and thus $w^{-1} \in Y_r$. If $u, w \in Y_r$, (2.17) yields that

$$\begin{aligned} N_-(uw, \alpha_r) &\subseteq N_0(w, \alpha_r) \cup w^{-1} \cdot N_0(u, w \cdot \alpha_r) \\ &= N_0(w, \alpha_r) \cup w^{-1} \cdot N_0(u, \alpha_r) \\ &= \emptyset, \end{aligned}$$

and therefore also $uw \in Y_r$. Since clearly $1 \in Y_r$, this proves that Y_r is a group.

We show next that $Y_r = X_r$. Let $w \in Y_r$, and let $u \in X_r$ be the representative of the coset $wW_{\Gamma(r)}$; then $u \in Y_r$, and thus $w^{-1}u \in Y_r$ and

$$N(w^{-1}u) \cap \Gamma(r) = N_0(w^{-1}u, \alpha_r) = \emptyset.$$

It is clear that $\Phi_{\Gamma(r)} = \Gamma(r)$, and so the above becomes:

$$(w^{-1}u) \cdot \big(\Phi_{\Gamma(r)} \cap \Phi^+\big) \subseteq \Phi_{\Gamma(r)} \cap \Phi^+.$$

As $w^{-1}u \in W_{\Gamma(r)}$, faithfulness of the standard geometric realization together with (2.23) now imply that $w^{-1}u$ equals 1; that is, $w = u \in X_r$. Whence $X_r = Y_r$, and it remains to show that Y_r is a free group.

We obtain the *odd Coxeter graph* of W by deleting all edges of even weight as well as all edges of infinite weight from the Coxeter graph. Let the set $\mathcal{F}$ consist of all $(l+1)$-tuples $(r_0, \dots, r_l)$ with $r_0, \dots, r_l \in R$ such that r_{i-1} and r_i are adjoined in the odd Coxeter graph for all $i \in \{1, \dots, l\}$. If $s, t \in R$ are adjoined by a bond of weight $2n+1$, we define $\pi(s,t) = (ts)^n$; then $s\pi(s,t)$ is the uniquely determined word in $W_{\{s,t\}}$ of maximal length, and thus

$$N\bigl(\pi(s,t)\bigr) = \Phi^+_{\{\alpha_s,\alpha_t\}} \setminus \{\alpha_t\}.$$

Define $\pi\colon \mathcal{F} \to W$ by $\pi(r_0) = 1$ and for $l \ge 1$,

$$\pi(r_0, \dots, r_l) = \pi(r_0, r_1)\pi(r_1, r_2)\pi(r_2, r_3)\cdots\pi(r_{l-2}, r_{l-1})\pi(r_{l-1}, r_l).$$

Now let $(r_0, \dots, r_l) \in \mathcal{F}$, and set $w = \pi(r_0, \dots, r_l)$. It follows from (1.1)(i) that $\pi(r_{i-1}, r_i)\cdot\alpha_{r_i} = \alpha_{r_{i-1}}$ for all i, and thus $\pi(r_i, \dots, r_l)\cdot\alpha_{r_l} = \alpha_{r_i}$; in particular, $w \cdot \alpha_{r_l} = \alpha_{r_0}$. An iteration of (2.17) yields that

$$\begin{aligned} N_0(w, \alpha_{r_l}) &\subseteq \bigcup_{i=1}^{l} \pi(r_l, \cdots r_i)\cdot N_0\bigl(\pi(r_{i-1}, r_i), \pi(r_i, \dots, r_l)\cdot\alpha_{r_l}\bigr) \\ &= \bigcup_{i=1}^{l} \pi(r_l, \cdots r_i)\cdot N_0\bigl(\pi(r_{i-1}, r_i), \alpha_{r_i}\bigr). \end{aligned}$$

Since the order of $r_{i-1}r_i$ is odd, we deduce from (1.1)(i) that $\Phi_{\{r_{i-1},r_i\}}$ contains no roots perpendicular to α_{r_i}, and thus $N_0\bigl(\pi(r_{i-1}, r_i), \alpha_{r_i}\bigr)$ is empty for all i; hence $N_0(w, \alpha_{r_l}) = \emptyset$. In particular, if $r_0 = r_l = r$ then $\pi(r_0, \dots, r_l)$ is an element of Y_r.

Next, define $L\colon \mathcal{F} \to \mathbb{N}_0$ by $L(r_0, \dots, r_l) = \sum_{i=1}^{l} l\bigl(\pi(r_{i-1}, r_i)\bigr)$. It is clear that $L(\underline{s}) \ge l\bigl(\pi(\underline{s})\bigr)$ for all $\underline{s} \in \mathcal{F}$, and we define $\widetilde{\mathcal{F}} \subseteq \mathcal{F}$ to be the set of all $\underline{s} \in \mathcal{F}$ with $L(\underline{s}) = l\bigl(\pi(\underline{s})\bigr)$. Note that $(s,t) \in \widetilde{\mathcal{F}}$ for all $(s,t) \in \mathcal{F}$. If $(r_0, \dots, r_l) \in \widetilde{\mathcal{F}}$, it is certainly necessary that $r_{i-1} \ne r_{i+1}$ for all i, since

$$\pi(s,t,s) = \pi(s,t)\pi(t,s) = 1.$$

We will soon see that this condition is also sufficient.

(4.47) Proposition *Let $w \in W$ and $r, s \in R$. Then $w \cdot \alpha_r = \alpha_s$ with $N_0(w, \alpha_r) = \emptyset$ if and only if there exists some $(r_0, \ldots, r_l) \in \widetilde{\mathcal{F}}$ with $r_0 = s$ and $r_l = r$ such that $w = \pi(r_0, \ldots, r_l)$.*

Proof. We have seen above that $w \cdot \alpha_r = \alpha_s$ and that $N_0(w, \alpha_r)$ is empty for $w = \pi(r_0, \ldots, r_l)$ with $r_l = r$, $r_0 = s$ and $(r_0, \ldots, r_l) \in \widetilde{\mathcal{F}} \subseteq \mathcal{F}$; so it suffices to show the converse.

Suppose that $w \cdot \alpha_r = \alpha_s$ and $N_0(w, \alpha_r) = \emptyset$. If $w = 1$, the assertion is true with $l = 0$. So assume that $w \neq 1$, and proceed by induction. Let $t \in R$ with $l(wt) < l(w)$; then $t \neq r$ by (1.5), as $w \cdot \alpha_r = \alpha_s$ is positive. Set $I = \{r, t\}$, and let $u \in W_I$ be of maximal length such that $w = du$ for some $d \in W$ with $l(w) = l(d) + l(u)$. Then by (2.17),

$$N_0(u, \alpha_r) = \emptyset \text{ and } N_0(d, u \cdot \alpha_r) = \emptyset. \qquad (*)$$

Now let $\lambda, \mu \in \mathbb{R}$ such that $u \cdot \alpha_r = \lambda\alpha_r + \mu\alpha_t$; then

$$\alpha_s = w \cdot \alpha_r = d \cdot (u \cdot \alpha_r) = \lambda(d \cdot \alpha_r) + \mu(d \cdot \alpha_t)$$

and λ, $\mu \geq 0$ or λ, $\mu \leq 0$. Maximality of u together with (1.5) force $d \cdot \alpha_r$ and $d \cdot \alpha_t$ to be positive, and as α_t is positive, we deduce that $\lambda, \mu \geq 0$; moreover, $\lambda = 0$ or $\mu = 0$ since α_s is simple. That is, $u \cdot \alpha_r = \alpha_r$ or $u \cdot \alpha_r = \alpha_t$, and we denote $u \cdot \alpha_r$ by α_z.

Since $u \cdot \alpha_r$ is positive, we know by (1.5) that $l_I(ur) = l_I(u) + 1$, and as u is an element of W_I and $u \neq 1$, this forces $l_I(ut) < l_I(u)$; that is, $u \cdot \alpha_t \in \Phi^-$. Furthermore, $u \cdot \alpha_r$ and $u \cdot \alpha_t$ are linearly independent since α_r and α_t are linearly independent, and thus in particular $u \cdot \alpha_t \neq -u \cdot \alpha_r = -\alpha_z$. So $(zu) \cdot \alpha_t \in \Phi^-$ by (1.4) and clearly $(zu) \cdot \alpha_r = -\alpha_z \in \Phi^-$. Now $N(zu)$ includes all positive roots which are linear combinations of α_r and α_t, and thus rt must have finite order, and zu is the uniquely determined word in W_I of maximal length.

Assume for a contradiction that the order of rt is even and equals $2m$. Then $u = t(rt)^{m-1}$, and we can deduce from (1.1)(i) that $(tr)^{\frac{m-1}{2}} \cdot \alpha_t$ is in $N_0(u, \alpha_r)$ if m is odd, and $t(rt)^{\frac{m}{2}-1} \cdot \alpha_r$ is in $N_0(u, \alpha_r)$ if m is even. Both of these contradict $(*)$, so rt must have odd order. Then $u = \pi(t, r)$; moreover, $d \cdot \alpha_t = d \cdot (u \cdot \alpha_r) = w \cdot \alpha_r = \alpha_s$ with $N_0(d, \alpha_t) = N_0(d, u \cdot \alpha_r) = \emptyset$ by $(*)$. Since $l(u) \geq l(t) = 1$, we know that $l(d) < l(w)$, and by induction there exists an $(r_0, \ldots, r_l) \in \widetilde{\mathcal{F}}$ with $r_0 = s$ and $r_l = t$ such that $d = \pi(r_0, \ldots, r_l)$. Now $(r_0, \ldots, r_l, r) \in \mathcal{F}$, and

$$w = du = \pi(r_0, \ldots, r_l)\pi(r_l, r) = \pi(r_0, \ldots, r_l, r);$$

furthermore, $l(w) = l(d) + l(u) = L(r_0, \ldots, r_l) + L(r_l, r) = L(r_0, \ldots, r_l, r)$, and thus $(r_0, \ldots, r_l, r)$ is in $\widetilde{\mathcal{F}}$, as required. □

It is clear that the elements of the fundamental group of a connected graph Ξ can be identified with paths in Ξ which start and end at a fixed vertex v, and which never back-track upon themselves; that is, at no stage does the path traverse an edge and then immediately traverse it again in the opposite direction. It is well known (see, for example, [8], Chapter 6, Theorem (5.2), p.198) that this fundamental group is free of rank $\epsilon - \nu + 1$, where ϵ is the number of edges, and ν is the number of vertices of Ξ. (This follows from the fact that Ξ is homotopy equivalent to a graph on one vertex with $\epsilon - \nu + 1$ edges - topologically, a bouquet of circles - as can be seen by shrinking a spanning tree of Γ to a single vertex.)

Specifically, if the graph Ξ is the connected component of the odd Coxeter graph containing the vertex $r \in R$, then the elements of the fundamental group of Ξ can be identified with the set

$$\mathcal{F}_r = \big\{(r_0, \ldots, r_l) \in \mathcal{F} \mid r_0 = r_l = r \text{ and } r_{i-1} \neq r_{i+1} \text{ for all } i \in \{1, \ldots, l\}\big\},$$

multiplication being defined by the rule

$$(r_0, \ldots, r_l) * (s_0, \ldots, s_m) = (r_0, \ldots, r_{l-i-1}, s_i, \ldots, s_m),$$

where i is the maximal integer such that $s_j = r_{l-j}$ for all $j \in \{0, \ldots, i\}$. Note that the identity of this group is (r). At the beginning of this chapter we have seen that π maps $\mathcal{F}_r$ into Y_r, and since $\pi(s, t, s) = 1$ for $(s, t, s) \in \mathcal{F}$, we conclude that π induces a homomorphism π_r from $\mathcal{F}_r$ to Y_r. If $(r_0, \ldots, r_l)$ is in $\widetilde{\mathcal{F}}$ with $r_0 = r_l = r$, clearly $(r_0, \ldots, r_l) \in \mathcal{F}_r$, and so (4.47) implies that π_r is surjective. The next proposition yields that $\mathcal{F}_r \subseteq \widetilde{\mathcal{F}}$. So if $\underline{s} \in \mathcal{F}_r$ is in the kernel of π_r, then

$$L(\underline{s}) = l\big(\pi(\underline{s})\big) = l\big(\pi_r(\underline{s})\big) = l(1) = 0,$$

and thus $\underline{s} = (r)$. Hence π_r is also injective, and therefore a group isomorphism.

(4.48) PROPOSITION *Let $(r_0, \ldots, r_l) \in \mathcal{F} \setminus \widetilde{\mathcal{F}}$. Then $r_{i-1} = r_{i+1}$ for some $i \in \{1, \ldots, l-1\}$.*

Proof. In order to avoid double indices, we denote the simple root corresponding to r_i by α_i for all i. Now let $m \in \{0, \ldots, l\}$ be minimal such that $(r_0, \ldots, r_m) \notin \widetilde{\mathcal{F}}$, and let $n \in \{0, \ldots, m\}$ be maximal such that $(r_n, \ldots, r_m)$ is not in $\widetilde{\mathcal{F}}$. Since $(r_n, \ldots, r_m)$ is also in $\mathcal{F} \setminus \widetilde{\mathcal{F}}$, and as it suffices to find one i such that $r_{i-1} = r_{i+1}$, we may assume without loss of generality that $n = 0$ and $m = l$. Then $(r_0, \ldots, r_{l-1})$, $(r_1, \ldots, r_l) \in \widetilde{\mathcal{F}}$ by minimality of m and maximality of n respectively. Further $l \geq 2$, since $(s, t) \in \widetilde{\mathcal{F}}$ for all $(s, t) \in \mathcal{F}$. Now define $u_1 = \pi(r_0, r_1)$, $w = \pi(r_1, \ldots, r_{l-1})$ and $u_2 = \pi(r_{l-1}, r_l)$. The above yields that $l(u_1 w) = l(u_1) + l(w)$ as well as $l(w u_2) = l(w) + l(u_2)$, and thus by (1.11),

$$N(u_1 w) = w^{-1} \cdot N(u_1) \cup N(w) \text{ and } N(w) \cap N(u_2^{-1}) = \emptyset.$$

On the other hand, $(r_0, \ldots, r_l) \notin \widetilde{\mathcal{F}}$ implies that $l(u_1 w u_2) < l(u_1 w) + l(u_2)$, and therefore $N(u_1 w) \cap N(u_2^{-1}) \neq \emptyset$ by (1.11); hence $w^{-1} \cdot N(u_1) \cap N(u_2^{-1}) \neq \emptyset$ by the above. Now

$$N(u_1) = \Phi^+_{\{\alpha_0, \alpha_1\}} \setminus \{\alpha_1\} \text{ and } N(u_2^{-1}) = \Phi^+_{\{\alpha_{l-1}, \alpha_l\}} \setminus \{\alpha_{l-1}\};$$

thus there exist $\lambda > 0$, $\mu \geq 0$ and $x \geq 0, y > 0$ such that

$$w^{-1} \cdot (\lambda \alpha_0 + \mu \alpha_1) = x \alpha_{l-1} + y \alpha_l.$$

Since $w^{-1} \cdot \alpha_1 = \alpha_{l-1}$, this yields that $\lambda(w^{-1} \cdot \alpha_0) = (x - \mu)\alpha_{l-1} + y \alpha_l$; as $y > 0$ and $w^{-1} \cdot \alpha$ is either positive or negative, this forces $x \geq \mu$. Symmetrically, $y(w \cdot \alpha_l) = (\mu - x)\alpha_1 + \lambda \alpha_0$ and $\lambda > 0$, and thus $\mu = x$ and $w \cdot \alpha_l = \alpha_0$. If $l = 2$, then $w = 1$ and $\alpha_0 = 1 \cdot \alpha_2 = \alpha_2$, and thus $r_0 = r_2$, as required.

Assume for a contradiction that $l > 2$; then (r_0, r_1, r_2) is in $\widetilde{\mathcal{F}}$ by minimality of $m = l$, and this yields that $r_0 \neq r_2$, and thus $\alpha_0 \neq \alpha_2$. Let $u = \pi(r_2, \ldots, r_{l-1})$; then $w = \pi(r_1, r_2)u$ with

$$l(w) = l\big(\pi(r_1, r_2)u\big) = l\big(\pi(r_1, r_2)\big) + l(u). \qquad (*)$$

Next, define γ to be $\pi(r_1, r_2)^{-1} \cdot \alpha_0$; since $u^{-1} \cdot \gamma = w^{-1} \cdot \alpha_0 = \alpha_l$ and $\pi(r_1, r_2) \cdot \gamma = \alpha_0$ are both in Π, Lemma (3.28) and $(*)$ force γ to be an elementary root. We show that this cannot be true, and this will be the desired contradiction.

We show first that γ is preceded by $r_1r_2 \cdot \alpha_0$. By definition of π, it is clear that $\pi(r_1, r_2)^{-1}$ equals $w'r_1r_2$ for some $w' \in W_{\{r_1,r_2\}}$ of length $l(\pi(r_1, r_2)^{-1}) - 2$. Now

$$N_+(w', r_1r_2 \cdot \alpha_0) \subseteq r_1r_2 \cdot N_+\big(\pi(r_1, r_2)^{-1}, \alpha_0\big)$$

by (2.17), and since $N\big(\pi(r_1, r_2)^{-1}\big) \subseteq \Phi^+_{\{r_1,r_2\}}$ and $\langle \alpha_0, \alpha_1\rangle$, $\langle \alpha_0, \alpha_2\rangle \leq 0$, it follows that $N_+\big(\pi(r_1, r_2)^{-1}, \alpha_0\big)$ is empty. Therefore $N_+\big(w', r_1r_2 \cdot \alpha_0\big)$ must be empty, and (3.34) yields that $\gamma \succeq r_1r_2 \cdot \alpha_0$. Now

$$r_1r_2 \cdot \alpha_0 = \alpha_0 + \big(4\langle \alpha_0, \alpha_2\rangle\langle \alpha_1, \alpha_2\rangle - 2\langle \alpha_0, \alpha_1\rangle\big)\alpha_1 + \big(-2\langle \alpha_0, \alpha_2\rangle\big)\alpha_2,$$

with $\langle \alpha_0, \alpha_1\rangle$, $\langle \alpha_1, \alpha_2\rangle < 0$ (by construction), and $\langle \alpha_0, \alpha_2\rangle \leq 0$ (as $r_0 \neq r_2$); so the coefficient of α_1 in $r_1r_2 \cdot \alpha_0$ is positive. Lemma (3.35) implies that there exist a, $b \in \mathbb{R}$ with $a > 0$ and $b \geq 0$ such that $\gamma = \alpha_0 + a\alpha_1 + b\alpha_2$. Further,

$$\begin{aligned} \alpha_l = u^{-1} \cdot \gamma &= \big(u^{-1} \cdot \alpha_0\big) + a\big(u^{-1} \cdot \alpha_1\big) + b\big(u^{-1} \cdot \alpha_2\big) \\ &= \big(u^{-1} \cdot \alpha_0\big) + a\big(u^{-1} \cdot \alpha_1\big) + b\alpha_{l-1}, \end{aligned}$$

with $u^{-1} \cdot \alpha_1 \in \Phi^+$ (since $l(r_1u) > l(u)$ by $(*)$). As $a > 0$ and $b \geq 0$, this forces $u^{-1} \cdot \alpha_0$ to be negative; that is, $\alpha_0 \in N(u^{-1})$. Now $N_0(u, \alpha_{l-1})$ is empty by (4.47), and thus (4.46) yields that

$$N(u^{-1}) \cap V_0(\alpha_2) = N_0(u^{-1}, \alpha_2) = N_0(u^{-1}, u \cdot \alpha_{l-1}) = \emptyset.$$

So $\alpha_0 \notin V_0(\alpha_2)$; that is, $\langle \alpha_0, \alpha_2\rangle \neq 0$. Hence r_0 and r_2 are adjoined and $\langle \alpha_0, \alpha_2\rangle < 0$. In particular, the coefficient of α_2 in $r_1r_2 \cdot \alpha_0$ is positive, and so the coefficient of α_2 in γ must also be positive by (3.35); that is, $b > 0$. Now α_0, α_1, $\alpha_2 \in \operatorname{supp}(\gamma)$, and since these form a circuit, (3.39) forces $\gamma \in \Delta$. This contradicts our earlier conclusion that γ is elementary, and thus $l = 2$ and $r_0 = r_2$ after all, as required. $\square$

(4.49) Theorem *The stabilizer of* α_r *in* W *is the semidirect product of* $W_{\Gamma(r)}$ *and* Y_r*. Moreover,* $W_{\Gamma(r)}$ *is a Coxeter group, and* Y_r *is a free group of rank* $e(r) - n(r) + 1$*, where* $e(r)$ *denotes the number of edges and* $n(r)$ *the number of vertices of the connected component of the odd Coxeter graph of* W *containing* r*.*

Chapter 5

An Automatic Structure

The principal result of this chapter is that Coxeter groups with finite distinguished generating sets are automatic. This is proved in [2] under the assumption that the Parallel Wall Theorem is valid. In our proof, the concept of dominance introduced in Chapter 3 replaces the parallel wall property.

For a finite set A, let A^* be the free monoid on A with multiplication $*$. Any subset L of A^* is a *language over the alphabet* A, the elements of A being the *letters*, and the elements of L the *words* of the language. A language is *regular* if and only if there exists a deterministic finite state automaton which accepts the words of the language and rejects words which are not in the language. A *deterministic finite state automaton* is a quintuple $(\mathfrak{S}, A, \mu, \mathfrak{Y}, S_0)$, where $\mathfrak{S}$ is a finite set of *states*, $\mathfrak{Y} \subseteq \mathfrak{S}$ is the set of *accept states*, $S_0 \in \mathfrak{S}$ is the *starting state* and $\mu\colon \mathfrak{S} \times A \to \mathfrak{S}$ is the *transition function*. The automaton reads the letters of a word one at a time, starting from the left and in state S_0, and if it was in state S before reading the letter a, its state after reading a is $\mu(S, a)$. The automaton accepts the word if it is in an accept state after reading the final letter, and rejects it otherwise. We say that $(\mathfrak{S}, A, \mu, \mathfrak{Y}, S_0)$ *recognizes* L.

In order to define automaticity for groups, we shall also have to consider languages over the alphabet $\mathfrak{A} = \big((A \cup \{\$\}) \times (A \cup \{\$\})\big) \setminus \{(\$, \$)\}$, where \$ denotes a symbol which is not in A. For $(a_1, b_1), \ldots, (a_n, b_n)$ in $\mathfrak{A}$ we identify $(a_1, b_1) * \ldots * (a_n, b_n)$ with $(a_1 * \ldots * a_n, b_1 * \ldots * b_n)$. For $(a, b) \in A^* \times A^*$ we define $(a, b)^\$ \in \mathfrak{A}^*$ as follows: if a and b are of the same length, then $(a, b)^\$$ equals (a, b), while if a and b are of unequal length, then as many \$'s are appended to the shorter of a and b as are necessary to make the lengths equal.

A group G is said to be *automatic* if there exists a finite set A of semigroup generators for G, and a language L over A such that the following are satisfied:

(i) the natural homomorphism $\pi\colon L \to G$ is surjective, and

(ii) $L_a = \{(x_1, x_2)^\$ \in \mathfrak{A}^* \mid x_1,\, x_2 \in L \text{ and } \pi(x_1) = \pi(x_2)a\}$ is a regular language over $\mathfrak{A}$ for all $a \in A \cup \{1_G\}$.

We then say that L *yields an automatic structure for* G.

Let W be a Coxeter group with finite distinguished generating set R. We are going to construct a language L over R that yields an automatic structure for W.

If x and y are in R^*, we say that y is a *segment* of x if there exist y', $y'' \in R^*$ such that $x = y' * y * y''$. For $r_1, r_2, \ldots, r_l \in R$ we define the length of $r_1 * r_2 * \cdots * r_l$ to be $\ell(r_1 * r_2 * \cdots * r_l) = l$. Recall that if $w \in W$, then

$$l(w) = \min\bigl\{\, \ell(x) \mid x \in \pi^{-1}(w) \,\bigr\},$$

where $\pi\colon R^* \to W$ denotes the natural homomorphism. An element $x \in R^*$ is called a *reduced word* if $\ell(x) = l\bigl(\pi(x)\bigr)$. Define L' to be the language of all reduced words. Note that if x is in L', then any segment of x must also be in L'.

Now let $\preceq$ be the lexicographical order on R^* for some (arbitrary) ordering of R. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. For $r \in R$ we define R_r to be the set of all simple reflections s with $s \prec r$, and Π_r to be the set of α_s with $s \in R_r$. It is clear that for each $w \in W$ there exists a unique $\nu(w) \in \pi^{-1}(w)$ such that $\nu(w) \in L'$ and $\nu(w) \preceq x$ for all $x \in \pi^{-1}(w) \cap L'$. We define the language L to consist of all these lexicographically minimal reduced words for the various elements of W:

$$L = \bigl\{\, \nu(w) \mid w \in W \,\bigr\} = \bigl\{\, y \in L' \mid y \preceq x \text{ for all } x \in L' \text{ with } \pi(x) = \pi(y) \,\bigr\}.$$

Observe that L coincides with ShortLex as defined in [6]. As above for L', it is clear that if x is in L, each segment of x has to be in L.

(5.50) PROPOSITION *Suppose $w \in W$ and $r \in R$ with $l(wr) = l(w)+1$, and let $r_1, r_2, \ldots, r_l \in R$ with $\nu(w) = r_1 * r_2 * \cdots * r_l$. For $j \in \{1, 2, \ldots, l\}$ define $R_j = R_{r_j}$, and set $R_{l+1} = \{r\}$. Then $\nu(wr) = r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l$,* *where $i \in \{1, 2, \ldots, l+1\}$ is minimal such that there exists an $s \in R_i$ with $(r_l r_{l-1} \cdots r_i) \cdot \alpha_s = \alpha_r$.*

Proof. Since $l(wr) = l(w)+1$ we know that $\nu(w)*r$ is in L', and furthermore that there exist $s_1, s_2, \ldots, s_{l+1} \in R$ with $\nu(wr) = s_1 * s_2 * \cdots * s_{l+1}$. Then

$$\nu(wr) = s_1 * s_2 * \cdots * s_{l+1} \preceq r_1 * \cdots * r_l * r \tag{1}$$

by minimality of $\nu(wr)$ in $L' \cap \pi^{-1}(wr)$. Now $(s_1 s_2 \cdots s_{l+1})r = w$, and by the Exchange Condition there exists an $i \in \{1, \ldots, l+1\}$ such that w equals $s_1 \cdots s_{i-1} s_{i+1} \cdots s_{l+1}$. Thus

$$\nu(w) = r_1 * \cdots * r_l \preceq s_1 * \cdots * s_{i-1} * s_{i+1} * \cdots * s_{l+1}, \tag{2}$$

and it is immediate from (1) and (2) that $r_j = s_j$ for all $j \in \{1, 2, \ldots i-1\}$. We deduce that $s_{i+1} \cdots s_{l+1} = r_i \cdots r_l$, and since both $s_{i+1} * \cdots * s_{l+1}$ and $r_i * \cdots * r_l$ are in L (as these are segments of elements of L), this yields that

$$r_i * \cdots * r_l = s_{i+1} * \cdots * s_{l+1},$$

and thus $r_j = s_{j+1}$ for all $j \in \{i, \ldots, l\}$. Now define $s = s_i$; then $\nu(wr)$ equals $r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l$. If $i = l+1$ it is clear that $s = r$. If $i \leq l$, equation (1) yields that $s \preceq r_i$, and since $s * r_i = s_i * s_{i+1}$ is reduced, it follows that $s \prec r_i$; that is, $s \in R_i$.

Furthermore, $(r_1 r_2 \cdots r_{l-1} r_l)r = wr = r_1 r_2 \cdots r_{i-1} s r_i \cdots r_{l-1} r_l$, and thus

$$(r_i r_{i+1} \cdots r_l) r (r_l \cdots r_{i+1} r_i) = s.$$

Therefore $(r_i r_{i+1} \cdots r_l) \cdot \alpha_r = \pm\alpha_s$, and since α_s and $(r_i r_{i+1} \cdots r_l) \cdot \alpha_r$ are both positive (the latter one because $r_i * r_{i+1} * \cdots * r_l * r$ is reduced), it follows that $(r_i r_{i+1} \cdots r_l) \cdot \alpha_r = \alpha_s$; that is, $(r_l \cdots r_{i+1} r_i) \cdot \alpha_s = \alpha_r$.

Assume for a contradiction that $(r_l r_{l-1} \cdots r_j) \cdot \alpha_t = \alpha_r$ for some $j < i$ and $t \in R_j$. Then $\pi(r_1 * \cdots * r_{j-1} * t * r_j * \cdots * r_l) = wr$, and further

$$r_1 * \cdots * r_{j-1} * t * r_j * \cdots * r_l \prec r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l = \nu(wr),$$

contradicting the minimality of $\nu(wr)$. Hence i is minimal with the above property. ∎

We now describe a finite state automaton $\mathcal{W}$ which recognizes L. The accept states of the automaton $\mathcal{W}$ will be the subsets E of $\mathcal{E}$ such that $E = PLC(E) \cap \mathcal{E}$, where $PLC(E)$ denotes the set of nonnegative linear combinations of roots in E. We denote the set of accept states by $\mathfrak{P}(\mathcal{E})$. There will be one reject state $\mathcal{F}$, and the starting state is the empty set in $\mathfrak{P}(\mathcal{E})$. The transition function $\mu: \mathfrak{S} \times R \to \mathfrak{S}$ is given by

$$\mu(X, r) = \begin{cases} \mathcal{F} & \text{if } X = \mathcal{F}, \\ \mathcal{F} & \text{if } X \in \mathfrak{P}(\mathcal{E}) \text{ and } \alpha_r \in X, \\ PLC\big(r \cdot X \cup r \cdot \Pi_r \cup \{\alpha_r\}\big) \cap \mathcal{E} & \text{if } X \in \mathfrak{P}(\mathcal{E}) \text{ and } \alpha_r \notin X. \end{cases}$$

(5.51) Proposition *The automaton $\mathcal{W}$ recognizes the language L.*

Proof. Let $r_1 * r_2 * \cdots * r_n \in R^*$, and denote the simple root corresponding to r_i by α_i. Set $X_0 = \emptyset$, and for $i \in \{1, \ldots, n\}$ define

$$X_i = PLC\big(r_i \cdot X_{i-1} \cup r_i \cdot \Pi_{r_i} \cup \{\alpha_i\}\big) \cap \mathcal{E}.$$

A straightforward induction yields that $\mathcal{W}$ is either in state X_i or $\mathcal{F}$ after reading $r_1 * r_2 * \cdots * r_i$; moreover, if $\mathcal{W}$ is in $\mathcal{F}$ after reading $r_1 * r_2 * \cdots * r_i$, then there exists an $l \in \{1, \ldots, i\}$ such that $\alpha_l \in X_{l-1}$. We show now that $r_1 * r_2 * \cdots * r_n \in L$ if and only if $\alpha_l \notin X_{l-1}$ for all $l \in \{1, \ldots, n\}$.

Suppose first that $\alpha_l \in X_{l-1}$ for some $l \in \{0, \ldots, n-1\}$. An easy induction yields that X_{l-1} is a subset of

$$PLC\Big(\bigcup_{i=1}^{l-1} (r_{l-1} \cdots r_i) \cdot \Pi_{r_i} \cup \big\{ (r_{l-1} \cdots r_{i+1}) \cdot \alpha_i \mid i \in \{1, \ldots, l-1\} \big\}\Big);$$

thus there exist nonnegative coefficients λ_s^i and μ_i such that

$$\alpha_l = \sum_{i=1}^{l-1} \sum_{\alpha_s \in \Pi_{r_i}} \lambda_s^i (r_{l-1} \cdots r_i) \cdot \alpha_s + \sum_{i=1}^{l-1} \mu_i (r_{l-1} \cdots r_{i+1}) \cdot \alpha_i,$$

and this yields

$$-\alpha_l = r_l \cdot \alpha_l = \sum_{i=1}^{l-1} \sum_{\alpha_s \in \Pi_{r_i}} \lambda_s^i (r_l r_{l-1} \cdots r_i) \cdot \alpha_s + \sum_{i=1}^{l-1} \mu_i (r_l r_{l-1} \cdots r_{i+1}) \cdot \alpha_i.$$

If $(r_l r_{l-1} \cdots r_{i+1}) \cdot \alpha_i$ is negative for some $i \in \{1, \ldots, l-1\}$, it follows by (1.5) that $r_l * r_{l-1} * \cdots * r_{i+1} * r_i$ is not reduced, and thus $r_1 * r_2 * \cdots * r_n$ cannot be in L, as required. Assume next that $r_1 * r_2 * \cdots * r_n$ is reduced. Then $(r_l r_{l-1} \cdots r_{i+1}) \cdot \alpha_i$ is positive for all i, and since λ_s^i, $\mu_i \geq 0$, it follows that $(r_l r_{l-1} \cdots r_i) \cdot \alpha_s$ must be negative for some i and $\alpha_s \in \Pi_{r_i}$. Let j be minimal such that $(r_j r_{j-1} \cdots r_i) \cdot \alpha_s$ is negative; then $(r_{j-1} \cdots r_i) \cdot \alpha_s$ is positive by minimality of j, and thus $(r_{j-1} \cdots r_i) \cdot \alpha_s = \alpha_j$ by (1.4). So

$$(r_{j-1} \cdots r_i) s (r_i \cdots r_{j-1}) = r_j,$$

and it follows that

$$\nu(r_i * r_{i+1} * \cdots * r_j) = r_i \cdots r_{j-1} r_j = s r_i \cdots r_{j-1} = \nu(s * r_i * r_{i+1} * \cdots * r_{j-1}).$$

Since $s \prec r_i$, we find that $r_i * \cdots * r_j$ is not in L; hence $r_1 * \cdots * r_n \notin L$, as required.

Suppose next that $r_1 * r_2 * \cdots * r_n$ is not in L, and let l be minimal in $\{1, \ldots, n\}$ such that $r_1 * r_2 * \cdots * r_l \notin L$. Assume first that $r_1 * r_2 * \cdots * r_l$ is not reduced. Then $l\big(\pi(r_1 * r_2 * \cdots * r_{l-1})\big) = l-1$ by minimality of l, and (1.5) yields that $(r_1 r_2 \cdots r_{l-1}) \cdot \alpha_l$ is negative. Let $i \in \{1, \ldots, l-1\}$ be maximal such that $(r_i r_{i+1} \cdots r_{l-1}) \cdot \alpha_l$ is negative; then $(r_{i+1} \cdots r_{l-1}) \cdot \alpha_l = \alpha_i$ by (1.4) and maximality of i. Since $r_{i+1} * r_{i+2} * \cdots * r_{l-1}$ is reduced (as it is a segment of a reduced word), (3.29) yields that $(r_j r_{j-1} \cdots r_{i+1}) \cdot \alpha_i$ must be in $\mathcal{E}$ for all $j \in \{i+1, \ldots, l\}$. As α_i is in X_i, an easy induction now yields that $(r_j r_{j-1} \cdots r_{i+1}) \cdot \alpha_i$ is in X_{j-1} for all $j \in \{i+1, \ldots, l\}$; in particular, $\alpha_l \in X_{l-1}$, as required.

Assume now that $r_1 * r_2 * \cdots * r_l$ is reduced, but $r_1 * r_2 * \cdots * r_l \notin L$. Then $r_1 * \cdots * r_{l-1} \in L$ by minimality of l, and since $r_1 * r_2 * \cdots * r_{l-1} * r_l$ is reduced but not in L, Proposition (5.50) yields that there exists an $i \in \{1, 2, \ldots, l-1\}$ such that $(r_{l-1} \cdots r_i) \cdot \alpha_s = \alpha_l$ for some $s \in R_{r_i}$. As $r_i * \cdots * r_{l-1}$ is reduced, (3.29) implies that $(r_j r_{j-1} \cdots r_i) \cdot \alpha_s \in \mathcal{E}$ for all $j \in \{i, \ldots, l-1\}$, and since $\alpha_s \in X_i$, it follows by a straightforward induction that $\alpha_l = (r_{l-1} \cdots r_i) \cdot \alpha_s \in X_l$, as required. □

(5.52) Lemma *Suppose $w \in W$ and $r \in R$ with $l(wr) = l(w) - 1$, and let $r_1, r_2, \ldots, r_l \in R$ with $\nu(w) = r_1 * r_2 * \cdots * r_l$. For $i \in \{1, \ldots, l\}$ define $R_i = R_{r_i}$, and set $R_{l+1} = \{r\}$. Then there exists exactly one $i \in \{1, 2, \ldots, l\}$ such that $(r_l \cdots r_{i+1}) \cdot \alpha_{r_i} = \alpha_r$; moreover, $r_i \in R_{i+1}$ and*

$$\nu(wr) = r_1 * \cdots * r_{i-1} * r_{i+1} * \cdots * r_l.$$

Proof. Let $s_1, s_2, \ldots, s_{l-1} \in R$ such that $\nu(wr) = s_1 * s_2 * \cdots * s_{l-1}$. It is clear that $\nu(wr) * r$ is reduced, and (5.50) yields that there exists some $i \in \{1, \ldots, l\}$ such that

$$\nu\big((wr)r\big) = s_1 * \cdots * s_{i-1} * s * s_i * \cdots * s_{l-1},$$

with $(s_{l-1} \cdots s_i) \cdot \alpha_s = \alpha_r$, and $s = r$ if $i = l$, while $s \prec s_i$ if $i \leq l-1$. But $\nu\big((wr)r\big) = \nu(w)$ also equals $r_1 * r_2 * \cdots * r_l$, and hence $s_k = r_k$ for k in $\{1, \ldots, i-1\}$, $r_i = s$ and $r_{j+1} = s_j$ for all $j \in \{i, \ldots, l-1\}$. Thus

$$\nu(wr) = r_1 * \cdots * r_{i-1} * r_{i+1} * \cdots * r_l$$

with $r_i \prec r_{i+1}$ if $i < l$, and $r_i = r$ if $i = l$; that is, $r_i \in R_{i+1}$.

Assume for a contradiction that there exists a $j \in \{1, \ldots, l\} \setminus \{i\}$ with $(r_l \cdots r_{j+1}) \cdot \alpha_{r_j} = \alpha_r$, and suppose without loss of generality that $i < j$. Then $(r_j \cdots r_{i+1}) \cdot \alpha_{r_i} = \alpha_{r_j}$ and thus $r_i r_{i+1} \cdots r_j r_{j+1} = r_{i+1} \cdots r_j$; hence

$$r_1 r_2 \cdots r_l = r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_{j-1} r_{j+1} \cdots r_l,$$

and thus $r_1 r_2 \cdots r_n$ is of length less than l, contradicting our assumption that $r_1 * r_2 * \cdots * r_l$ is reduced. □

Observe that this yields an algorithm which determines $\nu(wr)$ if $\nu(w)$ is given: Suppose that $\nu(w) = r_1 * r_2 * \cdots * r_l$, and set $\beta_l = r_l \cdot \alpha_r$. If $i \geq 2$ and $\beta_i \neq \alpha_{r_{i-1}}$, define $\beta_{i-1} = r_{i-1} \cdot \beta_i$; otherwise

$$\nu(wr) = r_1 * \ldots * r_{i-1} * r_{i+1} \ldots * r_l$$

by the previous lemma, and the algorithm can terminate. If $\beta_i \neq \alpha_{r_{i-1}}$ for all $i \geq 2$, two cases arise. Firstly, if $\beta_i \in \Pi_{r_i}$ for some i, let i be minimal with this property, and let $s \in R_{r_i}$ with $\beta_i = \alpha_s$; then

$$\nu(wr) = r_1 * \ldots * r_{i-1} * s * r_i * \ldots * r_l$$

by (5.50). Secondly, if $\beta_i \notin \Pi_{r_i}$ for all i, then $\nu(wr) = \nu(w) * r$ by (5.50).

Note that if $s \in R$ with $(r_i \cdots r_l) \cdot \alpha_r = \alpha_s$ for some i, then by (3.29), $(r_j \cdots r_l) \cdot \alpha_r$ must be elementary for all $j \in \{i, \ldots, l\}$ since $r_1 * \cdots * r_l$ is reduced. Hence we can stop our search for i according to the above description as soon as β_j is in Δ.

Now let $\mathfrak{R} = \big((R\cup\{\$\})\times(R\cup\{\$\})\big) \setminus \{(\$,\$)\}$, and for each $r \in R\cup\{1\}$ define

$$L_r = \big\{\, (x_1,x_2)^{\$} \in \mathfrak{R}^* \mid x_1, x_2 \in L \text{ and } \pi(x_1)r = \pi(x_2) \,\big\}.$$

Since the words of the language L correspond bijectively to the elements of W, we find that $L_1 = \{\, \big(\nu(w),\nu(w)\big) \mid w \in W \,\}$; therefore a trivial modification of $\mathcal{W}$ will yield a finite state automaton that recognizes L_1. So we only need to show that L_r is regular for $r \in R$. Then

$$\begin{aligned} L_r &= \big\{\big(\nu(w),\nu(wr)\big)^{\$} \mid w \in W\big\} \\ &= \big\{\big(\nu(w)*\$,\nu(wr)\big) \mid w \in W \text{ with } l(wr) = l(w)+1\big\} \\ &\qquad \cup \big\{\big(\nu(w),\nu(wr)*\$\big) \mid w \in W \text{ with } l(wr) = l(w)-1\big\} \\ &= \big\{\big(\nu(wr)*\$,\nu(w)\big) \mid w \in W \text{ with } l(wr) = l(w)-1\big\} \\ &\qquad \cup \big\{\big(\nu(w),\nu(wr)*\$\big) \mid w \in W \text{ with } l(wr) = l(w)-1\big\}. \end{aligned}$$

In particular, $(x_1,x_2) \in L_r$ if and only if $(x_2,x_1) \in L_r$; moreover, (5.52) yields:

(5.53) Corollary *Let $r \in R$ and $(a,b) \in \mathfrak{R}^*$ with $a = s_1 * s_2 * \cdots * s_n$ and $b = t_1 * t_2 * \cdots * t_n$ for some $(s_1,t_1),\ldots,(s_n,t_n) \in \mathfrak{R}$. Furthermore, let l be maximal in $\{1,\ldots,n+1\}$ such that $s_{l-1} = t_{l-1} \in R$. Then $(a,b) \in L_r$ if and only if*

- *(i) $l = n$, $\{s_n,t_n\} = \{r,\$\}$ and $s_1 * \cdots * s_{n-1} * r \in L$, or*
- *(ii) $l < n$, s_l, $t_l \in R$ and $s_l \prec t_l$, $s_i = t_{i-1} \in R$ for all $i \in \{l+1,\ldots,n\}$, $t_n = \$$ and $a \in L$ with $(s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r$, or*
- *(iii) $l < n$, s_l, $t_l \in R$ and $s_l \succ t_l$, $s_{i-1} = r_i \in R$ for all $i \in \{l+1,\ldots,n\}$, $s_n = \$$ and $b \in L$ with $(t_n \cdots t_{l+1}) \cdot \alpha_{t_l} = \alpha_r$.*

We now describe a finite state automaton $\mathcal{W}_r$ which recognizes L_r. The automaton $\mathcal{W}_r$ has one accept state $\mathcal{A}$, and there is one *failure state*, $\mathcal{F}$ (from which there are no transitions to other states). All elements of $\mathfrak{P}(\mathcal{E})$ are states, and the remaining states are the elements of the Cartesian products $\mathfrak{P}(\mathcal{E}) \times \mathcal{E}_r \times R$ and $\mathfrak{P}(\mathcal{E}) \times R \times \mathcal{E}_r$, where $\mathcal{E}_r$ denotes the set of elementary roots that can be written as $w \cdot \alpha_r$ for some $w \in W$. Let $\mathfrak{S}_r$ be the set of all these states, and let $\emptyset \in \mathfrak{P}(\mathcal{E})$ be the starting state. Note that the subset $\mathfrak{P}(\mathcal{E}) \cup \{\mathcal{F}\}$ of $\mathfrak{S}_r$ can be identified with the set of states of $\mathcal{W}$. Next, let μ

be the transition function, described above, for $\mathcal{W}$. The transition function $\mu_r\colon \mathfrak{S}_r \times \mathfrak{R} \to \mathfrak{S}_r$ for the automaton $\mathcal{W}_r$ is defined by the rules listed below. Let $\mathcal{X} \in \mathfrak{S}_r$ and $(s,\, t) \in \mathfrak{R}$, and for brevity let $\mathcal{Y} = \mu_r(\mathcal{X}, (s,t))$.

Case 1: $\mathcal{X} \in \mathfrak{P}(\mathcal{E})$.

(i) If $s = t \in R$, then $\mathcal{Y} = \mu(\mathcal{X}, s)$.

(ii) If either s or t is \$, then $\mathcal{Y} = \begin{cases} \mathcal{A} & \text{if } \{s,t\} = \{r,\$\} \text{ and } \mu(\mathcal{X},r) \neq \mathcal{F}, \\ \mathcal{F} & \{s,t\} \neq \{r,\$\} \text{ or } \mu(\mathcal{X},r) = \mathcal{F}. \end{cases}$

(iii) If $s \prec t \in R$, then $\mathcal{Y} = \begin{cases} \big(\mu(\mathcal{X},s), \alpha_s, t\big) & \text{if } \mu(X,s) \neq \mathcal{F} \text{ and } \alpha_s \in \mathcal{E}_r, \\ \mathcal{F} & \text{if } \mu(X,s) = \mathcal{F} \text{ or } \alpha_s \notin \mathcal{E}_r. \end{cases}$

(iv) If $t \prec s \in R$, then $\mathcal{Y} = \begin{cases} \big(\mu(\mathcal{X},t), s, \alpha_t\big) & \text{if } \mu(X,t) \neq \mathcal{F} \text{ and } \alpha_t \in \mathcal{E}_r, \\ \mathcal{F} & \text{if } \mu(X,t) = \mathcal{F} \text{ or } \alpha_t \notin \mathcal{E}_r. \end{cases}$

Case 2: $\mathcal{X} = (X, \beta, u) \in \mathfrak{P}(\mathcal{E}) \times \mathcal{E}_r \times R$.

Let $Y = \mu(X, s)$ and $\gamma = s \cdot \beta$.

(i) If $Y = \mathcal{F}$, then $\mathcal{Y} = \mathcal{F}$.

(ii) If $s \neq u$, then $\mathcal{Y} = \mathcal{F}$.

(iii) If $s = u$ and $t \in R$ while $Y \neq \mathcal{F}$, then $\mathcal{Y} = \begin{cases} (Y, \gamma, t) & \text{if } \gamma \in \mathcal{E}_r, \\ \mathcal{F} & \text{if } \gamma \notin \mathcal{E}_r. \end{cases}$

(iv) If $(s,t) = (u,\$)$ while $Y \neq \mathcal{F}$, then $\mathcal{Y} = \begin{cases} \mathcal{A} & \text{if } \gamma = \alpha_r, \\ \mathcal{F} & \text{if } \gamma \neq \alpha_r. \end{cases}$

Case 3: $\mathcal{X} = (X, u, \beta) \in \mathfrak{P}(\mathcal{E}) \times R \times \mathcal{E}_r$.

Let $Y = \mu(X, t)$ and $\gamma = t \cdot \beta$.

(i) If $Y = \mathcal{F}$, then $\mathcal{Y} = \mathcal{F}$.

(ii) If $t \neq u$, then $\mathcal{Y} = \mathcal{F}$.

(iii) If $t = u$ and $s \in R$ while $Y \neq \mathcal{F}$, then $\mathcal{Y} = \begin{cases} (Y, s, \gamma) & \text{if } \gamma \in \mathcal{E}_r, \\ \mathcal{F} & \text{if } \gamma \notin \mathcal{E}_r. \end{cases}$

(iv) If $(s,t) = (\$,u)$ while $\mathcal{Y} \neq \mathcal{F}$, then $\mathcal{Y} = \begin{cases} \mathcal{A} & \text{if } \gamma = \alpha_r, \\ \mathcal{F} & \text{if } \gamma \neq \alpha_r. \end{cases}$

Case 4: $\mathcal{X} = \mathcal{A}$ or $\mathcal{F}$.

$\mathcal{Y} = \mathcal{F}$ in all cases.

It can be easily seen that $\mathcal{W}_r$ accepts (x_1, x_2) if and only $\mathcal{W}_r$ accepts (x_2, x_1).

(5.54) Proposition *The automaton $\mathcal{W}_r$ defined above recognizes the language L_r.*

Proof. Let $(s_i, t_i) \in \mathfrak{R}$ and set $a = s_1 * s_2 * \cdots * s_n$ and $b = t_1 * t_2 * \cdots * t_n$. For $i \geq 0$, denote the state of $\mathcal{W}_r$ after reading $(s_1 * \cdots * s_i,\, t_1 * \cdots * t_i)$ by $\mathcal{X}_i$, and let $l \in \{1, \ldots, n+1\}$ be maximal such that $\mathcal{X}_{l-1} = X_{l-1} \in \mathfrak{P}(\mathcal{E})$. It is clear that $s_i = t_i \in R$ for all $i \in \{1, \ldots, l-1\}$.

We show that $\mathcal{X}_n$ equals $\mathcal{A}$ if and only if $(a, b) \in L_r$.

Suppose first that $(a, b) \in L_r$. Then an easy induction shows that $\mathcal{W}$ is in state X_i after reading $s_1 * \cdots * s_i = t_1 * \cdots * t_i$ for $i \in \{0, \ldots, l-1\}$. If $l = n$, then $\{s_n, t_n\} = \{r, \$\}$ by (5.53)(i); moreover, $\mathcal{W}$ is in $\mu(X_{n-1}, r)$ after reading $s_1 * \cdots * s_{n-1} * r$, and since $s_1 * \cdots * s_{n-1} * r$ is in L by (5.53)(i), we deduce that $\mu(X_{n-1}, r) \in \mathfrak{P}(\mathcal{E})$. It follows by rule (ii) of Case 1 that $\mathcal{X}_n = \mathcal{A}$.

Next, suppose that $l < n$. Since (a, b) is in L_r, it follows easily that $s_l, t_l \in R$, and by symmetry of both $\mathcal{W}_r$ and L_r we may assume without loss of generality that $s_l \prec t_l$. Then by (5.53)(ii), $s_i = t_{i+1} \in R$ for all $l \leq i \leq n-1$ and $(s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r$. It follows by (3.29) that $(s_i \cdots s_{l+1}) \cdot \alpha_{s_l}$ is in $\mathcal{E}_r$ for all $i \in \{l, \ldots, n-1\}$, and a straightforward induction yields that

$$\mathcal{X}_i = \big(X_i, (s_i \cdots s_{l+1}) \cdot \alpha_{s_l}, s_{i+1}\big)$$

for $i \in \{l, \ldots, n-1\}$, where X_i denotes the state of $\mathcal{W}$ after reading $s_1 * \cdots * s_i$. In particular,

$$\mathcal{X}_{n-1} = \big(X_{n-1}, (s_{n-1} \cdots s_{l+1}) \cdot \alpha_r, s_n\big).$$

Furthermore, $\mu(X_{n-1}, s_n) \in \mathfrak{P}(\mathcal{E})$ since $s_1 * \cdots * s_n$ is an element of L; as $(s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r$ and $t_n = \$$ by (5.53)(ii), rule (iv) of Case 2 yields that $\mathcal{X}_n = \mathcal{A}$.

It remains to show that $\mathcal{X}_n = \mathcal{A}$ implies $(a, b) \in L_r$. So let $\mathcal{X}_n = \mathcal{A}$; then in particular $\mathcal{X}_i \neq \mathcal{F}$ and $\mathcal{X}_i \neq \mathcal{A}$ for all $i < n$, since there are no transitions from $\mathcal{F}$ or $\mathcal{A}$ into states other than $\mathcal{F}$. Further, $l - 1 < n$ since $\mathcal{A} \notin \mathfrak{P}(\mathcal{E})$.

If $l = n$, then $\mathcal{W}$ is in X_{n-1} after reading $s_1 * s_2 * \cdots * s_{n-1}$, and since $\mathcal{X}_n = \mathcal{A}$, rule (ii) of Case 1 yields $\{s_n, t_n\} = \{r, \$\}$ and $\mu(X_{n-1}, r) \in \mathfrak{P}(\mathcal{E})$. So $\mathcal{W}$ accepts $s_1 * \cdots * s_{n-1} * r$, and it follows by (5.53)(i) that (a, b) is in L_r.

Suppose now that $l < n$. Then $\mathcal{X}_l \neq \mathcal{A}$ and $\mathcal{X}_l \neq \mathcal{F}$, and thus $s_l,\, t_l \in R$ by rule (ii) of Case 1; by symmetry of both $\mathcal{W}_r$ and L_r we may assume without loss of generality that $s_l \prec t_l$. We show now that $r_i = s_{i-1}$ and

$$\mathcal{X}_i = \big(X_i, (s_i \cdots s_{l+1}) \cdot \alpha_{s_l}, t_i\big)$$

for $i \in \{l, \ldots, n-1\}$, where $X_i \in \mathfrak{P}(\mathcal{E})$ denotes the state of $\mathcal{W}$ after reading $s_1 * s_2 * \cdots * s_i$. By rule (iii) of Case 1, $\mathcal{X}_l$ equals $\big(X_l, \alpha_{s_l}, t_l\big)$. Suppose next that $i \in \{l+1, \ldots, n-1\}$, and assume furthermore that

$$\mathcal{X}_{i-1} = \big(X_{i-1}, (s_{i-1} \cdots s_{l+1}) \cdot \alpha_{s_l}, t_{i-1}\big).$$

Since $\mathcal{X}_i \neq \mathcal{F}$ and $\mathcal{X}_i \neq \mathcal{A}$, rule (iii) of Case 2 yields $s_i = t_{i-1}$, $X_i \in \mathfrak{P}(\mathcal{E})$ and further $(s_i \cdots s_{l+1}) \cdot \alpha_{s_l} \in \mathcal{E}_r$; hence $\mathcal{X}_i = \big(X_i, (s_i \cdots s_{l+1}) \cdot \alpha_{s_l}, t_i\big)$, and this finishes the induction. In particular,

$$\mathcal{X}_{n-1} = \big(X_{n-1}, (s_{n-1} \cdots s_{l+1}) \cdot \alpha_{s_l}, t_{n-1}\big),$$

with $X_{n-1} \in \mathfrak{P}(\mathcal{E})$ and $s_i = t_{i-1}$ for all $i \in \{l, \ldots, n-1\}$. Since $\mathcal{X}_n = \mathcal{A}$, rule (iv) of Case 2 implies further that

$$s_n = t_{n-1},\ t_n = \$,\ (s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r \text{ and } \mu(X_{n-1}, s_n) \in \mathfrak{P}(\mathcal{E}).$$

Now $a = s_1 * s_2 * \cdots * s_n \in L$, since $\mu(X_{n-1}, s_n) \in \mathfrak{P}(\mathcal{E})$, while

$$b = s_1 * s_2 * \cdots * s_{l-1} * s_{l+1} * \cdots * s_n * \$$$

with $(s_n \cdots s_{l+1}) \cdot \alpha_{s_l} = \alpha_r$; thus $(a, b) \in L_r$ by (5.53)(ii), as required. □

(5.55) THEOREM *W is automatic, provided that R is finite.*

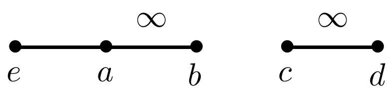
Observe that the automaton $\mathcal{W}_r$ described above is by no means minimal. For example, the state $(X, \beta, u) \in \mathfrak{P}(\mathcal{E}) \times \mathcal{E}_r \times R$ is *inaccessible* if $\beta \notin X$; that is, (X, β, u) cannot be reached from the starting state. Moreover, (X, β, u) is *dead* if $u \cdot \beta \notin \mathcal{E}$; that is, the accept state cannot be reached from (X, β, u). Without changing the language recognized by $\mathcal{W}_r$ we may delete all inaccessible states and amalgamate all dead states with the failure state $\mathcal{F}$, and obtain a *normalized* automaton with fewer states that also recognizes L_r.

A group G is said to be *biautomatic*, if there exists a set A of semigroup generators of G, and a language L over A which yields an automatic structure for G such that, additionally,

(iii) $L^a = \{(x_1, x_2)^\$ \in \mathfrak{A}^* \mid x_1,\, x_2 \in L \text{ and } \pi(x_1) = a\pi(x_2)\}$ is a regular language over $\mathfrak{A}$ for all $a \in A$.

We then say that L *yields a biautomatic structure for* W.

The above constructed language for finitely generated Coxeter groups does not in general yield a biautomatic structure for W. For example, suppose that W has the following Coxeter graph



and let R be ordered alphabetically. Assume for a contradiction that $\mathcal{W}^e$ is a finite state automaton recognizing L^e, and let n be the number of states of $\mathcal{W}^e$. For $i \in \{0, \dots, n\}$ define

$$w(i) = \big((a * b)^i, (c * d)^i\big).$$

Since the number of states of $\mathcal{W}^e$ is less than $n+1$, there exist $i,\, j \in \{0, \dots, n\}$ with $i < j$ such that $\mathcal{W}^e$ is in the same state after reading $w(i)$ as after reading $w(j)$. Now $\mathcal{W}^e$ will certainly accept

$$\begin{aligned}&\big((a * b)^n * (c * d)^n * \$, (c * d)^n * e * (a * b)^n\big)\\ &\quad = \big((a * b)^j, (c * d)^j\big) * \big((a * b)^{n-j} * (c * d)^n * \$, (c * d)^{n-j} * e * (a * b)^n\big),\end{aligned}$$

and thus $\mathcal{W}^e$ is forced to accept

$$\begin{aligned}&\big((a * b)^i, (c * d)^i\big) * \big((a * b)^{n-j} * (c * d)^n * \$, (c * d)^{n-j} * e * (a * b)^n\big)\\ &\quad = \big((a * b)^{n-j+i} * (c * d)^n * \$, (c * d)^{n-j+i} * e * (a * b)^n\big).\end{aligned}$$

But

$$e\pi\big((a * b)^{n-j+i} * (c * d)^n\big) = e(ab)^{n-j+i}(cd)^n$$

and

$$\pi\big((c * d)^{n-j} * e * (a * b)^n\big) = e(ab)^n(cd)^{n-j+i},$$

and these are not equal. Hence $\big((a*b)^{n-j+i}*(c*d)^n, (c*d)^{n-j+i}*e*(a*b)^n\big)$ is not in L^e, and $\mathcal{W}^e$ does not recognize L^e, contradicting our assumption.

Chapter 6

The Set of Elementary Roots

We have seen in Chapter 3 that the set of elementary roots is finite, provided that R is finite. Moreover, the proof of Theorem (3.44) yields that $|\mathcal{E}|$ is bounded by

$$\sum_{d=1}^{c^{|R|}(|R|+1)+1} |R|^d = \frac{|R|}{|R|-1}\big(|R|^{c^{|R|}(|R|+1)+1} - 1\big),$$

where c equals the cardinality of the set

$$\big\{ \cos(n\pi/m_{rs}) \mid r,\, s \in R,\, m_{rs} < \infty \text{ and } n \in \{1, \ldots, m_{rs}-1\} \big\}.$$

This bound, however, is rather large. For example, if

$$W = \langle\, r, s \mid r^2 = s^2 = (rs)^3 = 1 \,\rangle,$$

then $|\mathcal{E}| = 3$, but $|R| = 2$ and $c = 3$, and thus

$$\frac{|R|}{|R|-1}\big(|R|^{c^{|R|}(|R|+1)+1} - 1\big) = 2\big(2^{28} - 1\big).$$

In this chapter we will explicitly determine the set of elementary roots and thus find $|\mathcal{E}|$ precisely.

For $I \subseteq R$, let $\mathcal{E}_I$ denote the set of all elementary roots α with $I(\alpha) = I$; then $\mathcal{E}$ is the disjoint union of all $\mathcal{E}_I$ with $\emptyset \neq I \subseteq R$ finite. Connectedness of the support of a root yields that $\mathcal{E}_I$ is empty if I is not connected; moreover, $\mathcal{E}_I$ is also empty by (3.39), (3.41) if I contains a circuit or an infinite bond.

For $J \subseteq I$, define $\mathcal{E}_I^J$ to be the set of roots α in $\mathcal{E}_I$ such that for $r \in J$ the coefficient of α_r in α equals 1, and for $s \in I \setminus J$ the coefficient of α_s in α is greater than 1; then $\mathcal{E}_I$ is the disjoint union of all $\mathcal{E}_I^J$ with $J \subseteq I$, and thus

$$\mathcal{E} = \bigcup_{I \in \mathcal{I}(R)} \bigcup_{J \subseteq I} \mathcal{E}_I^J = \bigcup_{I \in \mathcal{I}(R)} \bigcup_{J \subset I} \mathcal{E}_I^J \cup \bigcup_{I \in \mathcal{I}(R)} \mathcal{E}_I^I,$$

where $\mathcal{I}(R)$ consists of the finite non-empty connected subsets of R that do not contain any circuits or infinite bonds.

Note that $\sum_{r \in I} \alpha_r$ is an elementary root if $I \in \mathcal{I}(R)$ contains only simple bonds. For if $|I| = 1$, this is trivially true, and we proceed by induction. Suppose that $|I| > 1$. Since I does not contain any circuits, we can choose an $s \in I$ such that s is adjacent to exactly one element of $I \setminus \{s\}$, say t. Then $\sum_{r \in I \setminus \{s\}} \alpha_r$ is an elementary root by induction. Since s is only adjoined to t in $I \setminus \{s\}$, and s and t are adjoined by a simple bond, we find that

$$\Big\langle \sum_{r \in I \setminus \{s\}} \alpha_r,\, \alpha_s \Big\rangle = \langle \alpha_s, \alpha_t \rangle = -\frac{1}{2};$$

hence $\sum_{r \in I} \alpha_r = s \cdot \sum_{r \in I \setminus \{s\}} \alpha_r$ is an elementary root by (3.37).

On the other hand, the next lemma together with (2.26) yield that if r and s are adjoined by a non-simple bond, then no root can have coefficient 1 for both α_r and α_s. So if I does contain non-simple bonds, $\sum_{r \in I} \alpha_r$ cannot be a root, and thus

$$\mathcal{E}_I^I = \begin{cases} \{\sum_{r \in I} \alpha_r\} & \text{if } I \text{ contains only simple bonds,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore

$$|\mathcal{E}| = \sum_{I \in \mathcal{I}(R)} \sum_{J \subset I} |\mathcal{E}_I^J| + n(R),$$

if R is finite, where $n(R)$ denotes the the number of non-empty connected subsets of R that contain only simple bonds and no circuits.

(6.56) Lemma *Let $x_1, x_2, y \in \Pi$ with $x_1 \neq x_2$ such that $\langle x_i, y \rangle$ equals $-\cos(\pi/m_i)$ for $i = 1, 2$. Furthermore, let α and β be positive roots such that β precedes α, and y is not in the support of β. Denote the coefficient of x_i in β by λ_i for $i = 1, 2$. Then the coefficient of y in α equals* 0, *or is greater than or equal to* $2\cos(\pi/m_1)\lambda_1 + 2\cos(\pi/m_2)\lambda_2$.

Proof. Let γ be of maximal depth with $\beta \preceq \gamma \preceq \alpha$ such that $y \notin \operatorname{supp}(\gamma)$. If $\gamma = \alpha$, then $y \notin \operatorname{supp}(\alpha)$, and the assertion is true. So suppose that $\gamma \prec \alpha$, and denote the coefficient of x_i in γ by μ_i; then $\lambda_i \leq \mu_i$. Maximality of γ now yields that $\gamma \prec r_y \cdot \gamma \preceq \alpha$, and the coefficient of y in $r_y \cdot \gamma$ equals $0 - 2\langle \gamma, y\rangle$. This is greater than or equal to

$$-2\big(\langle x_1, y\rangle \mu_1 + \langle x_2, y\rangle \mu_2\big) = 2\cos(\pi/m_1)\mu_1 + 2\cos(\pi/m_2)\mu_2,$$

which in turn is greater than or equal to $2\cos(\pi/m_1)\lambda_1 + 2\cos(\pi/m_2)\lambda_2$. Since $r_y \cdot \gamma \preceq \alpha$, the coefficient of y in α is greater than or equal to the coefficient of y in $r_y \cdot \gamma$ by (3.35), and this finishes the proof. □

By the above we only need to determine $\mathcal{E}_I^J$ for $I \in \mathcal{I}(R)$ and $J \subset I$. We will now further reduce the number of subsets J of I for which we need to calculate $\mathcal{E}_I^J$ (see Theorem (6.5)). Once this is done, we show that we only need to consider $\mathcal{E}_I^J$ for $I \in \mathcal{I}(R)$ containing at most one non-simple bond (see Lemma (6.7)). We then continue by determining $\mathcal{E}_I^J$ in case I contains only simple bonds, and finish this chapter by dealing with the case that I contains exactly one non-simple bond of finite weight.

(6.57) Proposition *Let $I \subseteq R$, $r \in I$ and $K_1, \dots, K_n \subseteq I \setminus \{r\}$ such that $I \setminus \{r\}$ is the disjoint union of $K_1, \dots, K_n$. Suppose further that no element of K_i is adjoined to any element of K_j if $i \neq j$, and set $I_i = K_i \cup \{r\}$ for all $i \in \{1, \dots, n\}$. Then*

$$\phi\colon (\beta_1, \dots, \beta_n) \mapsto \beta_1 + \cdots + \beta_n - (n-1)\alpha_r$$

defines a one-one correspondence between the set of n-tuples in $\Phi_{I_1}^+ \times \cdots \times \Phi_{I_n}^+$ such that the coefficient of α_r in each component equals 1, and the set of roots in Φ_I^+ with coefficient 1 for α_r. Moreover, this map restricts to a one-one correspondence between the set of n-tuples in $\mathcal{E}_{I_1} \times \cdots \times \mathcal{E}_{I_n}$ such that α_r has coefficient 1 in each component, and the set of roots in $\mathcal{E}_I$ with coefficient 1 for α_r.

Before we can show (6.57) we need to prove the next two technical results.

(6.58) Lemma *Let α and β be positive roots with $\alpha \succeq \beta$. Further let $r \in R$, and define I to be the set of simple reflections $s \in R$ which are adjoined to r. Suppose that the coefficient of α_r in α is strictly greater than the coefficient of α_r in β, while for $s \in I$ the coefficients of α_s in α and β coincide. Then $\alpha \succeq r \cdot \beta$.*

Proof. The assertion is trivially true if $r \cdot \beta \preceq \beta$, so suppose that $r \cdot \beta \succ \beta$; that is, $\mathrm{dp}(r \cdot \beta) = \mathrm{dp}(\beta) + 1$. Let γ be of maximal depth with $\alpha \succeq \gamma \succeq \beta$

such that the coefficients of α_r in γ and β coincide, and let $w \in W$ such that $\gamma = w \cdot \beta$ and $l(w) = \mathrm{dp}(\gamma) - \mathrm{dp}(\beta)$. Since the coefficients of α_s in γ and β coincide for all $s \in I \cup \{r\}$, it is clear that $w \in W_{R \setminus (I \cup \{r\})}$. So $rw = wr$ and

$$r \cdot \gamma = rw \cdot \beta = w \cdot (r \cdot \beta).$$

Now $\alpha \succeq r \cdot \gamma \succ \gamma$ by maximality of γ, and thus $\mathrm{dp}(r \cdot \gamma) = \mathrm{dp}(\gamma) + 1$. Hence $l(w) = (r \cdot \gamma) - \mathrm{dp}(r \cdot \beta)$ and $r \cdot \gamma \succeq r \cdot \beta$; transitivity of $\succeq$ yields that α is preceded by $r \cdot \beta$, as required. □

(6.59) LEMMA *Let $\alpha \in \Phi^+$ and $r \in R$ such that the coefficient of α_r in α equals 1. Then $\alpha \succeq \alpha_r$, and thus there exists a $w \in W_{I(\alpha) \setminus \{r\}}$ such that $\alpha = w \cdot \alpha_r$ and $l(w) = \mathrm{dp}(\alpha) - 1$.*

Proof. If α is simple, the assertion is trivially true. Suppose now that α is of depth greater than 1, and assume that all positive roots $\beta \prec \alpha$ are preceded by α_s, whenever $\alpha_s \in \Pi$ has coefficient 1 in β. Let $t \in R$ such that $t \cdot \alpha \prec \alpha$. If $t \neq r$, then $t \cdot \alpha \succeq \alpha_r$ by induction, and thus $\alpha \succ t \cdot \alpha \succeq \alpha_r$, as required. Suppose next that $t = r$. The coefficient of α_r in $r \cdot \alpha$ is less than 1, and thus must be equal to 0 by (2.26). That is, $\alpha_r \notin \mathrm{supp}(r \cdot \alpha)$, and by the connectedness of the support of α there exists an $x \in \mathrm{supp}(r \cdot \alpha)$ such that α_r and x are adjoined by a bond of weight $m \geq 3$. Since $\alpha \succeq r \cdot \alpha$ and the coefficient of α_r in α equals 1, Lemma (6.56) yields that $m = 3$ and that the coefficient of x in $r \cdot \alpha$ equals 1; moreover, (6.56) also yields that α_r cannot be adjacent to any element of $\mathrm{supp}(\alpha) \setminus \{x\}$. Now $r \cdot \alpha \succeq x$ by inductive hypothesis, and since the coefficients of x in α and x coincide, while the coefficient of α_r in α is greater than the coefficient of α_r in x, the previous lemma yields that α is preceded by $r_1 \cdot x_1 = x_1 + \alpha_r$. It is clear that $x_1 + \alpha_r$ is a successor of α_r, hence $\alpha \succeq \alpha_r$ by transitivity of $\succeq$, as required. □

Proof of (6.2). We show first that ϕ is well defined. So let $(\beta_1, \ldots, \beta_n)$ be an n-tuple in $\Phi^+_{I_1} \times \cdots \times \Phi^+_{I_n}$ such that the coefficient of α_r in β_i equals 1 for all i. By the previous lemma there exist $w_i \in W_{K_i}$ with $l(w_i) = \mathrm{dp}(\beta_i) - 1$ such that β_i equals $w_i \cdot \alpha_r$. Observe that the groups W_{K_i} centralize each other by construction. Define $w = w_1 \cdots w_n$ and $\alpha = w \cdot \alpha_r$; then the coefficient of α_r in α equals 1, and a straightforward calculation yields that

$$\alpha = \beta_1 + \cdots + \beta_n - (n-1)\alpha_r.$$

Thus $\phi(\beta_1,\dots,\beta_n)$ is in Φ_I^+, and since the coefficient of α_r in α clearly equals 1, we know that ϕ is well defined.

On the other hand, if $\alpha \in \Phi_I^+$ has coefficient 1 for α_r, Lemma (6.59) yields that there exists a $w \in W_{I\setminus\{r\}}$ with $l(w) = \mathrm{dp}(\alpha) - 1$ such that $\alpha = w \cdot \alpha_r$. Since $W_{I\setminus\{r\}}$ is the direct product of $W_{K_1},\dots,W_{K_n}$, there exist $w_i \in W_{K_i}$ for all $i \in \{1,\dots,n\}$ such that $w = w_1 \cdots w_n$ with length adding. So if we define $\beta_i = w_i \cdot \alpha_r$, then $I(\beta_i) \subseteq I_i$ and α_r has coefficient 1 in β_i. By the above, $\alpha = \beta_1 + \cdots + \beta_n - (n-1)\alpha_r = \phi(\beta_1,\dots,\beta_n)$; whence ϕ is onto. Since $I(\beta_i) \cap I(\beta_j) = \{r\}$ for $i \neq j$, and the coefficient of α_r in all β_i is 1, it is clear that ϕ is one-one, and thus ϕ is a one-one correspondence.

In order to show that ϕ induces a one-one correspondence between the set of n-tuples in $\mathcal{E}_{I_1} \times \cdots \times \mathcal{E}_{I_n}$ such that α_r has coefficient 1 in each component, and the set of roots in $\mathcal{E}_I$ with coefficient 1 for α_r, it suffices to show for $w_1 \in W_{K_1},\dots,w_n \in W_{K_n}$ and $w = w_0 \cdots w_n$ that $w \cdot \alpha_r \in \mathcal{E}$ if and only if $w_i \cdot \alpha_r \in \mathcal{E}$ for all $i \in \{1,\dots,n\}$.

Suppose first that $w_i \cdot \alpha_r \in \Delta$ for some i; by symmetry of the I_i we may assume without loss of generality that $i = 1$. Then $N_+(w_2 \cdots w_n, w_1 \cdot \alpha_r) = \emptyset$; for $N(w_2 \cdots w_n) \subseteq \Phi^+_{I\setminus I_1}$ and $w_1 \cdot \alpha_r \in \Phi^+_{I_1}$, and clearly $\langle \gamma, w_1 \cdot \alpha_r\rangle \leq 0$ for all $\gamma \in \Phi^+_{I\setminus I_1}$. Thus $w \cdot \alpha_r \succeq w_1 \cdot \alpha_r$ by (3.34), and (3.36) implies that $w \cdot \alpha_r \in \Delta$, as required.

For the converse, suppose that $w \cdot \alpha_r$ dominates some $\gamma \in \Phi^+ \setminus \{w \cdot \alpha_r\}$; then $w^{-1} \cdot \gamma \in \Phi^-$ since $\alpha_r \notin \Delta$, and thus $\gamma \in N(w^{-1})$. Now

$$N\big((w_1 \cdots w_n)^{-1}\big) = N(w_1^{-1}) \cup \ldots \cup N(w_n^{-1}),$$

since the W_{K_i} centralize each other, and this union is disjoint; by symmetry we may assume without loss of generality that γ is in $N(w_1^{-1})$ and not in

$$N(w_2^{-1}) \cup \ldots \cup N(w_n^{-1}) = N((w_2 \cdots w_n)^{-1}).$$

So $(w_2 \cdots w_n)^{-1} \cdot \gamma$ is positive, and hence $(w_1 \cdot \alpha_r)\ \mathsf{dom}\ (w_2 \cdots w_n)^{-1} \cdot \gamma$; since $w \cdot \alpha_r \neq \gamma$ it follows that $w_1 \cdot \alpha_r \neq (w_2 \cdots w_n)^{-1} \cdot \gamma$, and thus $w_1 \cdot \alpha_r \in \Delta$, as required. □

Now let $I \in \mathcal{I}(R)$ and $J \subset I$. If $r \in J$, denote the connected components of $I \setminus \{r\}$ by $K_1, \ldots, K_n$. By (6.57) each element of $\mathcal{E}_I^J$ can be written as

$$\alpha_1 + \cdots + \alpha_n - (n-1)\alpha_r,$$

with $\alpha_i \in \mathcal{E}_{\{r\} \cup K_i}$ with coefficient 1 for α_r for all i. Furthermore, if $s \in K_i$, the coefficient of α_s in α_i equals 1 if $s \in J$, and is greater than 1 if $s \notin J$; that is, $\alpha_i \in \mathcal{E}_{\{r\} \cup K_i}^{\{r\} \cup (J \cap K_i)}$. It is clear, that

$$\beta_1 + \cdots + \beta_n - (n-1)\alpha_r$$

is in $\mathcal{E}_I^J$ if $\beta_i \in \mathcal{E}_{\{r\} \cup K_i}^{\{r\} \cup (K_i \cap J)}$ for all i, and an iteration of this procedure yields the following theorem.

(6.60) THEOREM *Let $I \in \mathcal{I}$ and $J \subset I$, and denote the set of connected components of $I \setminus J$ by $\mathcal{K}(I \setminus J)$. Furthermore, for each connected component K of $I \setminus J$ let $X(K, J)$ denote the set of $r \in J$ that are adjoined to some element of K, and define $Y(K, J) = K \cup X(K, J)$. Finally, for $r \in J$ let $n_r(I, J)$ denote the number of $K \in \mathcal{K}(I \setminus J)$ with $r \in X(K, J)$. Then $\mathcal{E}_I^J$ is the set of*

$$\sum_{K \in \mathcal{K}(I \setminus J)} \alpha_K - \sum_{r \in J} (n_r(I, J) - 1)\alpha_r,$$

where $\alpha_K \in \mathcal{E}_{Y(K,J)}^{X(K,J)}$ for all $K \in \mathcal{K}(I \setminus J)$. Hence

$$|\mathcal{E}_I^J| = \prod_{K \in \mathcal{K}(I \setminus J)} |\mathcal{E}_{Y(K,J)}^{X(K,J)}|.$$

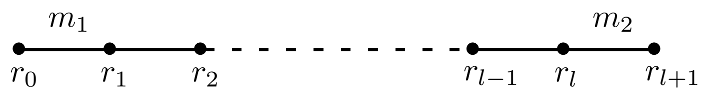
(6.61) *From now on we only need to determine $\mathcal{E}_Y^X$ for $X, Y \subseteq R$ such that*

(i) Y does not contain any circuits or infinite bonds,
(ii) $X \subseteq Y$ and $X \neq Y$,
(iii) $Y \setminus X$ is connected,
(iv) every element of X is adjacent to some element of $Y \setminus X$.

Note that since Y does not contain any circuits, and $Y \setminus X$ is connected, every element of X is adjoined to exactly one element of $Y \setminus X$, and no two elements of X are adjoined.

The following lemma implies that $\mathcal{E}_Y^X$ is empty if Y contains more than one non-simple bond, while X, Y satisfy (6.61); therefore we only need to determine $\mathcal{E}_Y^X$ for X, Y satisfying (6.61) such that Y contains at most one non-simple bond.

(6.62) LEMMA *Let $r_0, \ldots, r_{l+1} \in R$ such that the subgraph of the Coxeter graph corresponding to $\{r_0, \ldots, r_{l+1}\}$ is of the following shape*



with m_1, $m_2 \geq 4$. Denote the simple root corresponding to r_i by x_i, and let α be a root with x_0, $x_{l+1} \in \operatorname{supp}(\alpha)$ such that the coefficients of $x_1, \ldots, x_l$ in α are greater than 1. Then $\alpha \in \Delta$.

Proof. Let $\beta \preceq \alpha$ be a positive root of minimal depth such that x_0 and x_{l+1} are in the support of β, and the coefficients of $x_1, \ldots, x_l$ in β are greater than 1. By (3.36) it suffices to show that β is in Δ, and since this is certainly the case if the support of β contains a circuit (by (3.39)), we can assume without loss of generality that $\operatorname{supp}(\beta)$ does not contain any circuits.

For $i \in \{0, \ldots, l+1\}$, let λ_i denote the coefficient of x_i in β; then $\lambda_1, \ldots, \lambda_l \geq \sqrt{2}$ by (2.26). Next, let $r \in R$ such that $r \cdot \beta \prec \beta$. We show that $\langle r \cdot \beta, \alpha_r \rangle \leq -1$, which then implies $\langle \beta, \alpha_r \rangle \geq 1$; hence $\beta \in \Delta$ by (3.32), since β is clearly of depth greater than $\operatorname{dp}(\alpha_r) = 1$.

Minimality of β forces $r = r_i$ for some $i \in \{0, \ldots, l+1\}$. If $i = 0$ or $l+1$, we may assume without loss of generality that $i = 0$. Then minimality of β yields further that $x_0 \notin \operatorname{supp}(r_0 \cdot \beta)$, and thus

$$\langle r_0 \cdot \beta, x_0 \rangle \leq 0 + \lambda_1 \langle x_1, x_0 \rangle;$$

since $\lambda_1 \geq \sqrt{2}$ and $\langle x_1, x_0 \rangle = \cos(\pi/m_1) \leq -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$, this is less than or equal to -1, as required.

Suppose next that $i \in \{1, \ldots, l\}$. Connectedness of the support of $r_i \cdot \beta$ together with the assumption that $\operatorname{supp}(\beta)$ does not contain any circuits force $x_i \in \operatorname{supp}(r_i \cdot \beta)$, and minimality of β yields that the coefficient of x_i in $r_i \cdot \beta$ is less than or equal to 1; therefore the coefficient of x_i in $r_i \cdot \beta$ equals 1 by

(2.26). Let $K_1, \ldots, K_n$ denote the connected components of $I(\beta) \setminus \{r_i\}$, and for $j \in \{0, \ldots, n\}$ let $\beta_j \in \Phi^+_{K_j \cup \{r_i\}}$ with coefficient 1 for x_i such that

$$r_i \cdot \beta = \beta_1 + \cdots + \beta_n - (n-1)x_i$$

according to (6.57). We may assume without loss of generality that $r_{i-1} \in K_1$.

If $i = 1$, then $\beta_1 \succeq x_1$ by (6.59), and thus $\lambda_0 \geq 2\cos(\pi/m)$ by (6.56); hence

$$\lambda_{i-1}\langle x_{i-1}, x_i\rangle = \lambda_0\langle x_0, x_1\rangle \leq 2\cos(\pi/m_1)(-\cos(\pi/m_1)) \leq -1,$$

as $\cos(\pi/m_1) \geq \frac{1}{\sqrt{2}}$. If $i > 1$, then r_i and r_{i-1} are adjoined by a simple bond, and since $I(\beta)$ does not contain any circuits, r_i is adjoined only to r_{i-1} in $I(\beta_1)$. The coefficient of x_i in $r_i \cdot \beta_0$ equals $-1 + \lambda_{i-1}$, which is greater than 0 since $\lambda_{i-1} > 1$, and thus must be greater than or equal to 1 by (2.26). Therefore $\lambda_{i-1} \geq 2$, and thus again $\lambda_{i-1}\langle x_{i-1}, x_i\rangle \leq -1$. Symmetrical arguments also yield $\lambda_{i+1}\langle x_{i+1}, x_i\rangle \leq -1$ and thus

$$\langle r_i \cdot \beta, x_i\rangle \leq 1 + \lambda_{i-1}\langle x_{i-1}, x_i\rangle + \lambda_{i+1}\langle x_{i+1}, x_i\rangle \leq -1,$$

as required.

□

(6.63) PROPOSITION *Suppose X, Y satisfy (6.61), and assume furthermore that Y contains two or more non-simple bonds. Then $\mathcal{E}_Y^X = \emptyset$.*

§6a Simple bonds only*

For the duration of this section we assume that X, $Y \subseteq R$ satisfy (6.61) and, furthermore, that Y contains only simple bonds.

It is clear that all coefficients of roots in Φ_Y are integers, and thus $\langle \alpha, \beta\rangle$ is an integer multiple of $\frac{1}{2}$ for α, $\beta \in \Phi_Y$. Moreover, $\mathcal{E}_Y^X$ consists of the roots in $\mathcal{E}_Y$ with coefficient 1 for α_r with $r \in X$, and coefficient greater than or equal to 2 for α_s with $s \in Y \setminus X$.

The following is immediate.

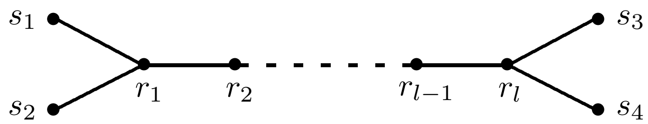
---

* I learned recently that Professor J.-Y. Hée also has a description of the set of elementary roots for the case that the Coxeter graph contains only simple bonds.

(6.64) Lemma *Let $\alpha \in \mathcal{E}$ be of depth greater than 1 with $I(\alpha) \subseteq Y$ and $r \in R$ such that $\alpha \succ r \cdot \alpha$. Then $\langle \alpha, \alpha_r \rangle = \frac{1}{2}$, and the coefficient of α_r in $r \cdot \alpha$ equals the coefficient of α_r in α minus* 1.

The next lemma yields that $\mathcal{E}_Y^X$ is empty if Y contains more than one vertex of valency greater than or equal to 3 (if $l > 1$, namely r_1 and r_l), or one or more vertices of valency greater than or equal to 4 (if $l = 1$, namely r_1).

(6.65) Lemma *Let $r_1, \ldots, r_l$ and s_1, s_2, s_3, s_4 be in Y such that the subgraph of the Coxeter graph corresponding to $\{r_1, \ldots, r_l\} \cup \{s_1, s_2, s_3, s_4\}$ is of the following shape:*



Denote the simple roots corresponding to r_i and s_j by x_i and y_j respectively, and let $\alpha \in \Phi_Y^+$ such that y_1, y_2, y_3, $y_4 \in \operatorname{supp}(\alpha)$, and the coefficients of $x_1, \ldots, x_l$ in α are greater than or equal to 2*. Then $\alpha \in \Delta$.*

Proof. Let β be a positive root of minimal depth preceding α such that y_1, y_2, y_3 and y_4 are in the support of β, and the coefficients of x_i in β are greater than or equal to 2 for all $i \in \{1, \ldots, l\}$. By (3.36) it suffices to show that β is in Δ. Denote the coefficients of x_i and y_j in β by λ_i and μ_j respectively, and let $r \in Y$ such that $r \cdot \beta \prec \beta$. We show that $\langle r \cdot \beta, \alpha_r \rangle \leq -1$, which then implies $\langle \beta, \alpha_r \rangle \geq 1$, and hence $\beta \in \Delta$ by (3.32); (since β is clearly of depth greater than $\operatorname{dp}(\alpha_r) = 1$).

By minimality of β we know that $r = r_i$ or $r = s_j$ for some i, j. If $r = s_j$, we may assume without loss of generality that $r = s_1$; then y_1 cannot be in the support of $s_1 \cdot \beta$ by minimality of β, and since $\lambda_1 \geq 2$, we find that $\langle s_1 \cdot \beta, y_1 \rangle \leq 0 + (-\frac{1}{2})\lambda_1 \leq -1$, as required.

Suppose now that $r = r_i$ for some $i \in \{1, \ldots, l\}$; then the coefficient of x_i in $r_i \cdot \alpha$ has to be less than or equal to 1 by minimality of β. Since $\lambda_{i-1} \geq 2$ if $i \geq 2$, and μ_1, $\mu_2 \geq 1$ if $i = 1$, while $\lambda_{i+1} \geq 2$ if $i \leq l - 1$, and

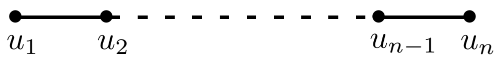
μ_3, $\mu_4 \geq 1$ if $i = l$, this yields that

$$\langle r_i \cdot \beta, x_i\rangle \leq \begin{cases} 1 + (-\frac{1}{2})1 + (-\frac{1}{2})1 + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } i = 1 = l, \\ 1 + (-\frac{1}{2})1 + (-\frac{1}{2})1 + (-\frac{1}{2})2 & \text{if } i = 1 < l, \\ 1 + (-\frac{1}{2})2 + (-\frac{1}{2})2 & \text{if } 1 < i < l, \\ 1 + (-\frac{1}{2})2 + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } 1 < i = l; \end{cases}$$

so $\langle r_i \cdot \beta, x_i\rangle \leq -1$ in all cases, as required. □

An easy calculation yields the following result.

(6.66) Lemma *Suppose that Y equals*

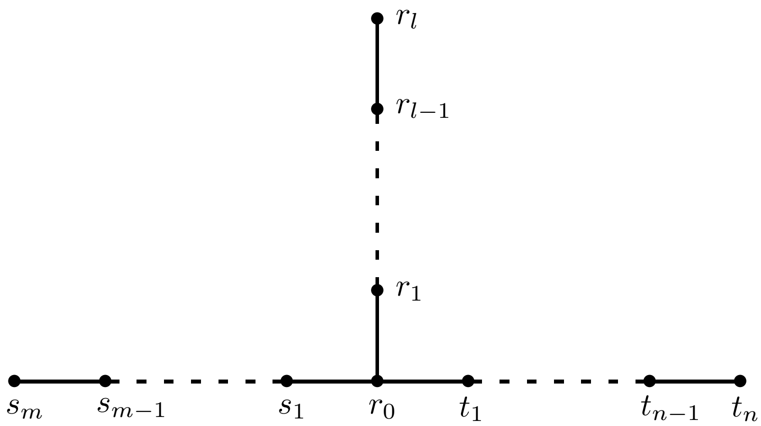


and denote the simple root corresponding to u_i by α_i. Then

$$\Phi^+ = \{\alpha_i + \cdots + \alpha_j \mid 1 \leq i \leq j \leq n\},$$

and thus $\mathcal{E}_Y^X = \emptyset$ if $X \neq Y$.

(6.67) Proposition *Suppose that $Y \subseteq R$ contains only simple bonds and $|Y| > 1$. Further, let $X \subseteq Y$ such that X, Y satisfy (6.61). Then $\mathcal{E}_Y^X$ is empty unless Y equals*



with $l, m, n \geq 1$.

If Y is of the shape described above, and X, Y satisfy (6.61), then X must be contained in $\{r_l, s_m, t_n\}$, since each element of X is adjoined to exactly one element of $Y \setminus X$, and $Y \setminus X$ is connected. It is convenient for us to define s_0 and t_0 to be equal to r_0. In order to avoid double indices, we denote the simple reflections corresponding to r_i, s_j and t_k by x_i, y_j and z_k respectively for $i = 0, \ldots, l$, $j = 0, \ldots, m$ and $k = 0, \ldots, n$.

Define $\rho_{l,m,n}$ to be the root

$$\begin{aligned}(t_{n-1} \cdots t_1)(s_{m-1} \cdots s_1)(r_{l-1} \cdots r_1)r_0(t_n \cdots t_1)(s_m \cdots s_1)(r_l \cdots r_1) \cdot x_0 \\ = x_l + y_m + z_n \\ + 2\big(x_{l-1} + \cdots + x_1 + y_{m-1} + \cdots + y_1 + x_0 + z_1 + \cdots + z_{n-1}\big).\end{aligned}$$

A straightforward calculation yields that

$$(r_i \cdots r_1) \cdot x_0 = x_i + \cdots + x_1 + x_0$$

and $\langle (r_i \cdots r_1) \cdot x_0, x_{i+1} \rangle = -\frac{1}{2}$ for all $i \in \{1, \ldots, l\}$. So by (3.37) we can deduce that $(r_{i+1} \cdots r_1) \cdot x_0$ is elementary if $(r_i \cdots r_1) \cdot x_0$ is elementary. Since $x_0 \in \mathcal{E}$, induction yields that $(r_l \cdots r_1) \cdot x_0$ is elementary. A string of similar arguments yields that $\rho_{l,m,n}$ is elementary, and it follows that $\rho_{l,m,n}$ is an element of $\mathcal{E}_Y^{\{r_l, s_m, t_n\}}$.

The next two assertions will enable us to show that each root in $\mathcal{E}_Y^X$ is preceded by $\rho_{l,m,n}$ if X, Y are of the shape described in (6.67). Note that the next lemma also yields that the depth function coincides with the height function defined by $\sum_{r \in Y} \lambda_r \alpha_r \mapsto \sum_{r \in Y} \lambda_r$ on the set of elementary roots with only simple bonds in their support.

(6.68) LEMMA *Suppose Y contains only simple bonds, and let $\alpha \in \Phi^+$ such that $\alpha = \sum_{r \in Y} \lambda_r \alpha_r$ for some $(\lambda_r)_{r \in Y}$. Then $\mathrm{dp}(\alpha) \leq \sum_{r \in Y} \lambda_r$, with equality if and only if $\alpha \in \mathcal{E}$.*

Proof. If α is of depth 1, then $\alpha = \alpha_r$ for some $r \in Y$, and the assertion is immediate. So suppose now that $\mathrm{dp}(\alpha) > 1$, and assume that for each positive root $\beta \prec \alpha$, the depth of β is less than or equal to the sum of the coefficients of the simple roots in β, with equality if and only if $\beta \in \mathcal{E}$. Further, let $s \in R$ such that $s \cdot \alpha \prec \alpha$.

If α is elementary, (6.64) implies that the coefficient of α_s in $s \cdot \alpha$ equals $\lambda_s - 1$, and since $r \cdot \alpha \in \mathcal{E}$ by (3.36), induction yields that

$$\mathrm{dp}(s \cdot \alpha) = \sum_{r \neq s} \lambda_r + (\lambda_s - 1) = \sum_{r \in Y} \lambda_r - 1;$$

hence $\mathrm{dp}(\alpha) = \mathrm{dp}(s \cdot \alpha) + 1 = \sum_{r \in Y} \lambda_r$.

Suppose next that $\alpha \in \Delta$ and $s \cdot \alpha \in \Delta$. Then the coefficient of α_s in $s \cdot \alpha$ is less than or equal to $\lambda_s - 1$; hence

$$\mathrm{dp}(s \cdot \alpha) < \sum_{r \neq s} \lambda_r + (\lambda_s - 1) = \sum_{r \in Y} \lambda_r - 1$$

by induction, and thus $\mathrm{dp}(\alpha) = \mathrm{dp}(s \cdot \alpha) + 1 < \sum_{r \in Y} \lambda_r$.

Finally, suppose that $\alpha \in \Delta$ while $s \cdot \alpha$ is elementary. This is possible only if $\alpha \operatorname{\mathsf{dom}} \alpha_s$, and thus $\langle \alpha, \alpha_s \rangle \geq 1$ by (3.32). The coefficient of α_s in $s \cdot \alpha$ equals $\lambda_s - 2\langle \alpha, \alpha_s \rangle$, and this is less than or equal to $\lambda_s - 2$. Thus

$$\mathrm{dp}(s \cdot \alpha) \leq \sum_{r \neq s} \lambda_r + (\lambda_s - 2) = \sum_{r \in Y} \lambda_r - 2$$

by inductive hypothesis, and therefore

$$\mathrm{dp}(\alpha) = \mathrm{dp}(s \cdot \alpha) + 1 \leq \sum_{r \in Y} \lambda_r - 1 < \sum_{r \in Y} \lambda_r,$$

as required. □

(6.69) PROPOSITION *Suppose Y contains only simple bonds. Let $\beta \in \Phi^+$ and $\alpha \in \mathcal{E}$ with $I(\alpha) \subseteq Y$. Then $\alpha \succeq \beta$ if and only if the coefficients of simple roots in β are less than or equal to the corresponding coefficients in α. (Note that if this is the case, (3.36) yields that $\beta \in \mathcal{E}$.)*

Proof. If $\alpha \succeq \beta$, Lemma (3.35) yields that the coefficients of simple roots in β are less than or equal to the corresponding coefficients in α, and it suffices to show the converse.

Suppose first that $\mathrm{dp}(\beta) = 1$, and let $\gamma \preceq \alpha$ be a positive root of minimal depth such that $\beta \in \mathrm{supp}(\gamma)$. Assume for a contradiction that $\gamma \neq \beta$. Then

$\mathrm{dp}(\gamma) > 1$, since β is in the support of γ, but γ does not equal β. Minimality of γ yields that $r_\beta \cdot \gamma \prec \gamma$ and $\beta \notin \mathrm{supp}(r_\beta \cdot \gamma)$. Since α is elementary, it follows that γ is elementary, and by (6.64), the coefficient of β in γ equals the coefficient of β in $r_\beta \cdot \gamma$ plus 1, that is 1. But then $\gamma \succeq \beta$ by (6.59), and since $\gamma \neq \beta$ we find that $\alpha \succeq \gamma \succ \beta$, contradicting the minimality of γ. So $\gamma = \beta$, as required.

Suppose next that $\mathrm{dp}(\beta) > 1$, and let $(\mu_r)_{r\in Y}$ with $\beta = \sum_{r\in Y} \mu_r \alpha_r$. Furthermore, assume that the assertion is true for all positive roots of depth less than the depth of β. Let α be an elementary root with $(\lambda_r)_{r\in Y}$ such that $\alpha = \sum_{r\in Y} \lambda_r \alpha_r$ and $\lambda_r \geq \mu_r$ for all $r \in Y$. Then $\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta)$ by (6.68).

If $\mathrm{dp}(\alpha) = \mathrm{dp}(\beta)$, Lemma (6.68) together with the hypothesis yield that $\alpha = \beta$. Suppose now that $\mathrm{dp}(\alpha) > \mathrm{dp}(\beta)$ and let $s \in Y$ such that $s \cdot \alpha \prec \alpha$; by (6.64) we know that the coefficient of α_s in $s \cdot \alpha$ equals $\lambda_s - 1$ and that $\langle \alpha, \alpha_s \rangle = \frac{1}{2}$. If $\lambda_s - 1 \geq \mu_s$, induction on $\mathrm{dp}(\alpha) - \mathrm{dp}(\beta)$ yields that $s \cdot \alpha$ is a successor of β, and thus $\alpha \succeq \beta$ by transitivity of $\succeq$. Assume next that $\lambda_s - 1 < \mu_s$; then $\lambda_s = \mu_s$, since $\lambda_s \geq \mu_s$ and λ_s and μ_s are integers. Further, $\lambda_r \geq \mu_r$ and $\langle \alpha_r, \alpha_s \rangle \leq 0$ for $r \neq s$; therefore

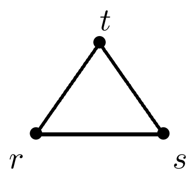
$$\langle \alpha, \alpha_s \rangle = \lambda_s + \sum_{r \neq s} \lambda_r \langle \alpha_r, \alpha_s \rangle \leq \mu_s + \sum_{r \neq s} \mu_r \langle \alpha_r, \alpha_s \rangle = \langle \beta, \alpha_s \rangle. \qquad (*)$$

So $\langle \beta, \alpha_s \rangle > 0$, and thus $s \cdot \beta \prec \beta$. Denote the coefficient of α_s in $s \cdot \beta$ by μ'_s; then μ'_s is less than or equal to $\mu_s - 1 = \lambda_s - 1$, and thus $s \cdot \alpha \succeq s \cdot \beta$ by induction on $\mathrm{dp}(\beta)$.

Assume for a contradiction that $\mu'_s < \mu_s - 1$, and thus $\langle s \cdot \beta, \alpha_s \rangle \leq -1$. Since $s \cdot \alpha$ is an elementary root preceded by $s \cdot \beta$, Lemma (3.38) implies that the coefficients of α_s in $s \cdot \alpha$ and $s \cdot \beta$ coincide; that is, $\mu'_s = \lambda_s - 1$, and thus $\mu'_s = \mu_s - 1$, contradicting our assumption.

So $\mu'_s = \mu_s - 1$ and $\langle \beta, \alpha_s \rangle = \frac{1}{2} = \langle \alpha, \alpha_s \rangle$. Now $(*)$ together with the hypothesis force $\lambda_r = \mu_r$ for all $r \in Y$ such that $\langle \alpha_r, \alpha_s \rangle \neq 0$. So for $r \in R$ adjoined to s, the coefficients of α_r in α and $s \cdot \beta$ coincide, while the coefficient of α_s in α is greater than the coefficient of α_s in $s \cdot \beta$; thus $\alpha \succeq s \cdot (s \cdot \beta) = \beta$ by (6.58), as required. □

Note that (6.69) does not hold in general for arbitrary roots in Φ_Y. For example, if Y equals



then the coefficients in $rs \cdot \alpha_t = 2\alpha_r + \alpha_s + \alpha_t$ are greater than or equal to the corresponding coefficients on α_r, but clearly $\alpha_r \not\preceq rs \cdot \alpha_t$. Note furthermore that (6.69) also does not hold in general for elementary roots for arbitrary Y. For example, suppose that $Y = \{r, s\}$ with $m_{rs} = 4$; then the coefficients in $r \cdot \alpha_s = \sqrt{2}\alpha_r + \alpha_s$ are greater than or equal to the corresponding coefficients in α_r, but clearly $\alpha_r \not\preceq r \cdot \alpha_s$.

If Y is of the shape described in (6.67) and $X \subseteq \{r_l, s_m, t_n\}$, Proposition (6.69) yields that every root in $\mathcal{E}_Y^X$ is a successor of $\rho_{l,m,n}$. It can be easily seen that r_0 is the only element of Y such that $r_0 \cdot \rho_{l,m,n} \succ \rho_{l,m,n}$; that is, $\langle \rho_{l,m,n}, x_0 \rangle < 0$.

If l, m, $n \geq 2$, then

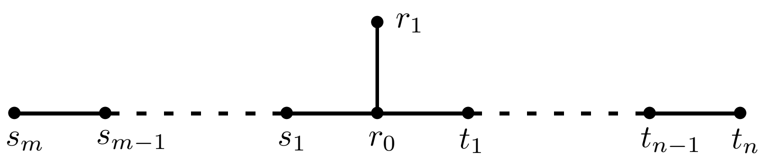
$$\langle r_0 \cdot \rho_{l,m,n}, x_0 \rangle = -\langle \rho_{l,m,n}, x_0 \rangle = -\big(2 + \Big(-\frac{1}{2}\Big)2 + \Big(-\frac{1}{2}\Big)2 + \Big(-\frac{1}{2}\Big)2\big) = 1;$$

thus $r_0 \cdot \rho_{l,m,n} \in \Delta$, and no elementary root can be a successor of $r_0 \cdot \rho_{l,m,n}$.

(6.70) Proposition *Suppose X, Y are of the shape described in (6.67) with $l, m, n \geq 2$. Then*

$$\mathcal{E}_Y^X = \begin{cases} \{\rho_{l,m,n}\} & \text{if } X = \{r_l, s_m, t_n\}, \\ \emptyset & \text{if } X \neq \{r_l, s_m, t_n\}. \end{cases}$$

(6.71) *From now on suppose that Y equals*



with m, $n \geq 1$ and $X \subseteq \{r_1, s_m, t_n\}$.

(6.72) PROPOSITION *Suppose Y is of the shape described in (6.71). Then $\mathcal{E}_Y^{\{r_1,s_m,t_n\}}$ is the set of*

$$\begin{aligned} &x_1 + y_m + 2\big(y_{m-1} + \cdots + y_{j(3)}\big) + 3\big(y_{j(3)-1} + \cdots + y_{j(4)}\big) + \cdots \\ &\quad \cdots + (M-1)\big(y_{j(M-1)-1} + \cdots + y_{j(M)}\big) + M\big(y_{j(M)-1} + \cdots + y_1\big) \\ &\quad + Mx_0 + M\big(z_1 + \cdots + z_{k(M)-1}\big) + \cdots + 2\big(z_{k(3)} + \cdots + z_{n-1}\big) + z_n \end{aligned}$$

with $M \in \{2, \ldots, \min(m,n)+1\}$,

$$m > j(3) > j(4) > \ldots > j(M-1) > j(M) > 0$$

and

$$0 < k(M) < k(M-1) < \ldots < k(4) < k(3) < n.$$

Whence

$$\big|\mathcal{E}_Y^{\{r_1,s_m,t_n\}}\big| = \sum_{M=2}^{\min(m,n)+1} \binom{m-1}{M-2}\binom{n-1}{M-2} = \binom{m+n-1}{m-1}.$$

Before we show (6.72), we prove the next lemma, which yields a more interesting proof for (6.72).

(6.73) LEMMA *Suppose Y is of the shape described in (6.71), and let α be in Φ_Y^+. Denote the coefficients of x_i, y_j, z_k in α by λ_i, μ_j and ν_k respectively, and suppose that $\lambda_0 \geq 1$.*

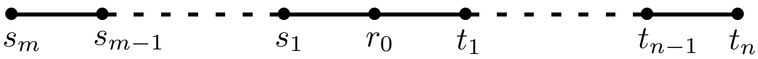
(i) Then $\lambda_1 \leq \lambda_0$, with equality only if $\lambda_0 = \lambda_1 = 1$; furthermore,

$$\mu_m \leq \mu_{m-1} \leq \ldots \leq \mu_1 \leq \lambda_0 \text{ and } \lambda_0 \geq \nu_1 \geq \ldots \geq \nu_{n-1} \geq \nu_n.$$

(ii) There exists a $\beta \preceq \alpha$ such that the coefficient of x_0 in β equals λ_0, while the coefficient of y_1 in β is less than or equal to $\lambda_0 - 1$.

(iii) If $\mu_j \geq \mu_{j+1} + 2$ for some $j \in \{0, \ldots, m-1\}$ (where $\mu_0 = \lambda_0$), there exists a $\beta \preceq \alpha$ such that the coefficient of x_0 in β equals λ_0, while the coefficient of y_1 is less than or equal to $\lambda_0 - 2$.

Proof. The coefficient of x_1 in $r_1 \cdot \alpha$ equals $\lambda_0 - \lambda_1$, and since the coefficient of x_0 in $r_1 \cdot \alpha$ (namely λ_0) is greater than 0, this has to be nonnegative; that is $\lambda_0 \geq \lambda_1$. If $\lambda_0 = \lambda_1$, then $I(r_1 \cdot \alpha)$ is a subset of



and (6.66) yields that $\lambda_0 = 1$. Now let $j \in \{1, \ldots, m\}$. An easy induction yields for $i \in \{j, \ldots, m-1\}$ that y_i has coefficient $\mu_{i+1} + \mu_{j-1} - \mu_j$ in $(s_i \cdots s_j) \cdot \alpha$, and we deduce that the coefficient of y_m in $\beta = (s_l \cdots s_j) \cdot \alpha$ equals $\mu_{j-1} - \mu_j$. Since the coefficient of x_0 in β equals λ_0, and this is positive, β is a positive root, and thus $\mu_{j-1} \geq \mu_j$. Symmetrical arguments yield the inequalities for the ν_k, and this proves (i).

If $\lambda_0 = 1$, then $\alpha \succeq x_0$ by (6.69), and (ii) is certainly true. So suppose that $\lambda_0 \geq 2$, and let $\beta \preceq \alpha$ be of minimal depth such that the coefficient of x_0 in β equals λ_0. Minimality of β implies that $r_0 \cdot \beta \prec \beta$, and the coefficient of x_0 in $r_0 \cdot \beta$ is less than or equal to $\lambda_0 - 1$. Now (i) yields that the coefficient of y_1 in $r_0 \cdot \beta$ is less than or equal to $\lambda_0 - 1$, and so the coefficient of y_1 in β is less than or equal to $\lambda_0 - 1$, as required.

It remains to show (iii). So suppose that $\mu_j \geq \mu_{j+1} + 2$ for some $j \in \{0, \ldots, m-1\}$, and let j be minimal with this property. If $j = 0$ we can choose $\beta = \alpha$, so assume next that $j > 0$, and proceed by induction. By minimality of j we know that $\mu_{j-1} \leq \mu_j + 1$; whence $\mu_{j-1} = \mu_j$ or $\mu_{j-1} = \mu_j + 1$ by (i). If $\mu_{j-1} = \mu_j$, the coefficient of y_j in $s_j \cdot \alpha$ equals μ_{j+1}, and this is less than or equal to $\mu_j - 2 = \mu_{j-1} - 2$; if $\mu_{j-1} = \mu_j + 1$, the coefficient of y_j in $s_j \cdot \alpha$ equals $\mu_{j+1} + 1$, and this is less than or equal to $\mu_j - 2 + 1 = \mu_{j-1} - 2$. So in any case, induction yields that there exists a $\beta \preceq s_j \cdot \alpha \, (\preceq \alpha)$ such that the coefficient of x_0 in β equals λ_0, while the coefficient of y_1 is less than or equal to $\lambda_0 - 2$, and this finishes the proof.

□

Proof of (6.17). Suppose first that α is of the form described above; we show that α is an elementary root, and since the coefficients of α certainly satisfy the required conditions, it will follow that $\alpha \in \mathcal{E}_Y^{\{r_1, s_m, t_n\}}$. If $M = 2$, then $\alpha = \rho_{1,m,n}$ is elementary. Suppose next that $M \geq 3$, and proceed by induction. If $j(M) = k(M) = 1$, then $\langle \alpha, x_0 \rangle = \frac{1}{2}$ and $r_0 \cdot \alpha$ is of the form described above with $M - 1$ in place of M. Thus $r_0 \cdot \alpha \in \mathcal{E}$ by induction on M, and since $\langle r_0 \cdot \alpha, x_0 \rangle = -\frac{1}{2}$, Lemma (3.37) yields that α is elementary.

Suppose next that $j(M) > 1$, and proceed by induction on $j(M)$. Define

$l(i) = j(i)$ for $i \in \{3, \ldots, N-1\}$ and $l(M) = j(M) - 1$; then

$$m > l(3) > l(4) > \ldots > l(M-1) > l(M) > 0.$$

Further, $\langle \alpha, y_{j(M)-1} \rangle = \frac{1}{2}$ and $s_{j(M)-1} \cdot \alpha$ is of the form described above with $l(i)$ in place of $j(i)$ for all i. By induction, $s_{j(M)-1} \cdot \alpha$ is elementary, and as $\langle s_{j(M)-1} \cdot \alpha,\, y_{j(M)-1} \rangle = -\langle \alpha,\, y_{j(M)-1} \rangle = -\frac{1}{2}$, it follows by (3.37) that α is also elementary.

Symmetrical arguments apply if $k(M) > 1$, and it remains to show that we have listed all the elements of $\mathcal{E}_Y^{\{r_1, s_m, t_n\}}$. This could be done by an inductive proof similar to the previous one, but for the sake of variety we choose the following approach.

Let $\mu_{m-1}, \ldots, \mu_1, \lambda_0, \nu_1, \ldots, \nu_{n-1} \geq 2$ such that

$$\alpha = x_1 + y_m + \mu_{m-1} y_{m-1} + \cdots + \mu_1 y_1 + \lambda_0 x_0 + \nu_1 z_1 + \cdots + \nu_{n-1} z_{n-1} + z_n$$

is an elementary root, and assume for a contradiction that there exists a $j \in \{0, \ldots, m-1\}$ with $\mu_j \geq \mu_{j+1} + 2$ (where $\mu_0 = \lambda_0$ and $\mu_m = 1$). Let $\beta \preceq \alpha$ be according to (6.73)(iii) such that the coefficient of x_0 in β equals λ_0, and the coefficient of y_1 in β is less than or equal to $\lambda_0 - 2$. Furthermore, let $\gamma \preceq \beta$ be according to (6.73)(ii) such that the coefficient of x_0 in γ equals λ_0, while the coefficient of z_1 in γ is less than or equal to $\lambda_0 - 1$. Since γ precedes α and β, the coefficient of x_1 in γ is less than or equal to 1, and the coefficient of y_1 in γ is less than or equal to $\lambda_0 - 2$. Hence

$$\langle \gamma, x_0 \rangle \geq \lambda_0 + \Big(-\frac{1}{2}\Big)(\lambda_0 - 2) + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{1}{2}\Big)(\lambda_0 - 1) \geq 1;$$

since the coefficient of x_0 in γ is greater than 1 we find that γ is of depth strictly greater than 1, and thus (3.32) implies that $\gamma \; \mathsf{dom} \; x_0$ and $\gamma \in \Delta$. Now (3.36) forces $\alpha \in \Delta$, contrary to our choice of α. So $\mu_j \leq \mu_{j+1} + 1$ for all $j \in \{0, \ldots, m-1\}$, and thus $\mu_j = \mu_{j+1}$ or $\mu_{j+1} + 1$ by (6.73)(i); symmetrically $\nu_k \in \{\nu_{k+1},\, \nu_{k+1} + 1\}$ for all $k \in \{0, \ldots, n-1\}$, and α is of the desired form.

The binomial identity employed in the latter part of the assertion is known as the Vandermonde-identity. □

(6.74) Proposition *Suppose Y is of the shape described in (6.71). Then $\mathcal{E}_Y^X = \emptyset$ if $X = \{r_1\}$, $\{r_1, s_m\}$, $\{r_1, t_n\}$.*

The next two results are part of the proof of (6.74), but are stated separately since they will be used again later.

(6.75) Lemma *Suppose X and Y are of the shape described in (6.71) and $X \neq \{r_1, s_m, t_n\}$. Furthermore, let α be a minimal element of $\mathcal{E}_Y^X$ with respect to $\preceq$. Then there exists an $r \in \{r_1, s_m, t_n\} \setminus X$ with $r \cdot \alpha \in \mathcal{E}_Y^{X \cup \{r\}}$.*

Proof. Since $X \subseteq \{r_1, s_m, t_n\}$, we know that α is a successor of $\rho_{1,m,n}$, and since $X \neq \{r_1, s_m, t_n\}$ clearly $\alpha \neq \rho_{1,m,n}$. So $\alpha \succ \rho_{1,m,n}$, and there exists an $r \in Y$ with $\alpha \succ r \cdot \alpha \succeq \rho_{1,m,n}$. For $s \in X$, the coefficients of α_s in α and $\rho_{1,m,n}$ coincide, and thus $r \notin X$. Denote the coefficient of α_r in $r \cdot \alpha$ by λ. If $\lambda \geq 2$, then $r \cdot \alpha$ is in $\mathcal{E}_Y^X$, contradicting the minimality of α; therefore $\lambda \leq 1$. The coefficient of α_r in α equals $\lambda + 1$ by (6.64), and this has to be greater than or equal to 2 as $r \in Y \setminus X$; whence $\lambda = 1$ and $r \cdot \alpha \in \mathcal{E}_Y^{X \cup \{r\}}$. Since $r \cdot \alpha$ is a successor of $\rho_{1,m,n}$, the coefficients of $y_{m-1}, \ldots, y_1, x_0, z_1, \ldots, z_{n-1}$ in $r \cdot \alpha$ are greater than or equal to 2, and this forces $r \in \{r_1, s_m, t_n\}$, as required. □

(6.76) Corollary *Suppose X and Y are of the shape described in (6.71) with $X \neq \{r_1, s_m, t_n\}$ such that $\mathcal{E}_Y^{X \cup \{r\}} = \emptyset$ for all $r \in \{r_1, s_m, t_n\} \setminus X$. Then $\mathcal{E}_Y^X = \emptyset$.*

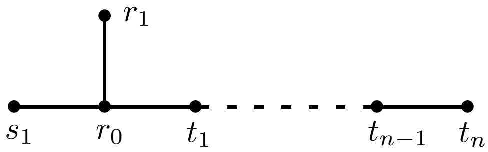
Proof of (6.19). Assume for a contradiction that $\mathcal{E}_Y^{\{r_1, s_m\}} \neq \emptyset$, and let α be an element of minimal depth. As t_n is the only element of $\{r_1, s_m, t_n\} \setminus \{r_1, s_m\}$, the previous lemma yields that $t_n \cdot \alpha \in \mathcal{E}_Y^{\{r_1, s_m, t_n\}}$. By (6.72), the coefficient of z_{n-1} in $t_n \cdot \alpha$ equals 2, while the coefficient of z_n in $t_n \cdot \alpha$ equals 1, and thus

$$\langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -(1 + \Big(-\frac{1}{2}\Big)2) = 0,$$

contradicting $t_n \cdot \alpha \prec \alpha$. So $\mathcal{E}_Y^{\{r_1, s_m\}} = \emptyset$ and symmetrically also $\mathcal{E}_Y^{\{r_1, t_n\}} = \emptyset$. Moreover, $\mathcal{E}_Y^{\{r_1\}} = \emptyset$ by (6.76). □

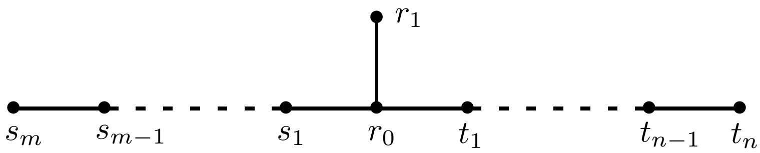
If m also equals 1, symmetrical arguments yield that $\mathcal{E}_Y^{\{s_1,t_n\}}$ and $\mathcal{E}_Y^{\{s_1\}}$ are empty; so $\mathcal{E}_Y^{\{t_n\}}$ is empty by (6.76), and a repeated application of (6.76) yields that $\mathcal{E}_Y^{\emptyset}$ is also empty. (Alternatively, it can be easily verified that $\rho_{1,1,n}$ is the only element of Φ_Y preceded by $\rho_{1,1,n}$.) This yields the next result.

(6.77) Lemma *Suppose Y equals*



with $n \geq 1$ and $X \subseteq \{r_1, s_1, t_n\}$. Then $\mathcal{E}_Y^X = \emptyset$ if $X \neq \{r_1, s_1, t_n\}$ and $\mathcal{E}_Y^{\{r_1,s_1,t_n\}} = \{\rho_{1,1,n}\}$.

(6.78) *From now on suppose that Y equals*



with m, $n \geq 2$ and $X \subseteq \{s_m,\, t_n\}$.

Define $\sigma_{m,n}$ to equal

$$(r_1 r_0) \cdot \rho_{1,m,n} = 2x_1 + y_m + 2(y_{m-1} + \cdots + y_1) + 3x_0 + 2(z_1 + \cdots + z_{n-1}) + z_n.$$

Since $\rho_{1,m,n}$ is elementary and $\langle \rho_{1,m,n},\, x_0 \rangle = \langle r_0 \cdot \rho_{1,m,n}, x_1 \rangle = -\frac{1}{2}$, Lemma (3.37) yields that $\sigma_{m,n}$ is in $\mathcal{E}$, and hence in $\mathcal{E}_Y^{\{s_m,t_n\}}$.

Now let $\alpha \in \mathcal{E}_Y^X$. By (6.73)(i), the coefficient of x_0 in α is strictly greater than the coefficient of x_1 in α, which in turn is greater than or equal to 2. So the coefficient of x_0 in α is greater than or equal to 3, and since for $r \in Y \setminus \{s_m, t_n\}$ the coefficient of α_r in α is greater than or equal to 2, Proposition (6.69) yields that $\alpha \succeq \sigma_{m,n}$.

(6.79) PROPOSITION *Suppose Y is of the shape described in (6.78) with $m, n \geq 3$. Then the elements of $\mathcal{E}_Y^{\{s_m,t_n\}}$ are exactly the following:*

$$(1) \qquad \begin{aligned} &2x_1 + y_m + 2\big(y_{m-1} + \cdots + y_j\big) + 3\big(y_{j-1} + \cdots + y_1\big) + 3x_0 \\ &\quad + 3\big(z_1 + \cdots + z_{k-1}\big) + 2\big(z_k + \cdots + z_{n-1}\big) + z_n, \end{aligned}$$

where $m > j > 0$ and $0 < k < n$,

$$(2) \qquad \begin{aligned} &2x_1 + y_m + 2\big(y_{m-1} + \cdots + y_1\big) + 4x_0 + 4\big(z_1 + \cdots + z_{k(2)-1}\big) \\ &\quad + 3\big(z_{k(2)} + \cdots + z_{k(1)-1}\big) + 2\big(z_{k(1)} + \cdots + z_{n-1}\big) + z_n, \end{aligned}$$

where $0 < k(2) < k(1) < n$, and

$$(3) \qquad \begin{aligned} &2x_1 + y_m + 2\big(y_{m-1} + \cdots + y_{j(1)}\big) + 3\big(y_{j(1)-1} + \cdots + y_{j(2)}\big) \\ &\quad + 4\big(y_{j(2)-1} + \cdots + y_1\big) + 4x_0 + 2\big(z_1 + \cdots + z_{n-1}\big) + z_n, \end{aligned}$$

where $m > j(1) > j(2) > 0$.

Whence $\big|\mathcal{E}_Y^{\{s_m,t_n\}}\big| = (m-1)(n-1) + \binom{m-1}{2} + \binom{n-1}{2} = \binom{m+n-2}{2}$.

Proof. We show first that if α is of type (1), (2) or (3), then $\alpha \in \mathcal{E}$; since the coefficients of α satisfy the required conditions, it follows that $\alpha \in \mathcal{E}_Y^{\{s_m,t_n\}}$.

Suppose first that α is of type (1). If $j = k = 1$, then $\alpha = \sigma_{m,n}$ is certainly an elementary root. So suppose now that $j + k > 2$, and proceed by induction. By symmetry, we may assume without loss of generality that $j > 1$. It follows that $\langle \alpha, y_{j-1} \rangle = \frac{1}{2}$ and $s_{j-1} \cdot \alpha$ is of type (1) with $j - 1$ in place of j; induction yields that $s_{j-1} \cdot \alpha \in \mathcal{E}$ and so α is elementary by (3.37) since $\langle s_{j-1} \cdot \alpha, y_{j-1} \rangle = -\langle \alpha, y_{j-1} \rangle = -\frac{1}{2}$.

Assume next that α is of type (2). If $k(2) = 1$, then $r_0 \cdot \alpha$ is of type (1) (since $k(1) > 1$), and thus an elementary root by the previous paragraph. Now $\langle r_0 \cdot \alpha, x_0 \rangle = -\frac{1}{2}$, so (3.37) yields that α is elementary. Suppose next that $k(2) > 1$, and proceed by induction. Then $\langle \alpha, z_{k(2)-1} \rangle = \frac{1}{2}$ and $t_{k(2)-1} \cdot \alpha$ is of type (2) with $k(2) - 1$ in place of $k(2)$. So $t_{k(2)-1} \cdot \alpha$ is elementary by induction, and (3.37) implies once again that $\alpha \in \mathcal{E}$.

Symmetrical arguments apply if α is of type (3), therefore it remains to show that all the elements of $\mathcal{E}_Y^{\{s_m,t_n\}}$ have been accounted for. Again, this could be done by an inductive proof, but we choose the following one.

Let $\alpha \in \mathcal{E}_Y^{\{s_m,t_n\}}$. Then $\alpha \succeq \sigma_{m,n}$ by the above, whence the coefficient of x_0 in α is greater than or equal to 3. We show first that α is of type (1), (2) or (3) if α has coefficient 3 or 4 for x_0. Suppose now that

$$\alpha = \lambda_1 x_1 + y_m + \mu_{m-1}y_{m-1} + \cdots + \mu_1 y_1 + 3x_0 + \nu_1 z_1 + \cdots + \nu_{n-1}z_{n-1} + z_n.$$

Then $\lambda_1 < 3$ by (6.73)(i), and thus $\lambda_1 = 2$ by hypothesis. If $\mu_{m-1} = 3$, then (6.73)(ii) and (iii) yield that there exists a $\beta \preceq r_1 \cdot \alpha$ with coefficient of x_0 in β equal to 3, and coefficients of x_1, y_1, z_1 less than or equal to 1,1 and 2 respectively; but then

$$\langle \beta, x_0 \rangle \geq 3 + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{1}{2}\Big)2 = 1,$$

forcing $\beta \in \Delta$ and thus $\alpha \in \Delta$, a contradiction. Thus $\mu_{m-1} = 2$, and symmetrically also $\nu_{n-1} = 2$, and by (6.73)(i) it is clear that α is of type (1).

Next suppose that x_0 has coefficient 4 in α, and let β be of maximal depth with $\sigma_{m,n} \preceq \beta \preceq \alpha$ such that the coefficient of x_0 in β is less than 4. Then maximality of β implies that $\beta \prec r_0 \cdot \beta \preceq \alpha$ and that the coefficient of x_0 in $r_0 \cdot \beta$ equals 4. As $\alpha \succeq r_0 \cdot \beta$, it is clear that $r_0 \cdot \beta$ is elementary, and by (6.64) we deduce that $\langle \beta, x_0 \rangle = -\frac{1}{2}$ and that the coefficient of x_0 in β equals 3. As $\alpha \succeq \beta \succeq \sigma_{m,n}$ we can further conclude that β is in $\mathcal{E}_Y^{\{s_m,t_n\}}$. The previous paragraph now yields that β is of type (1). If j, $k \geq 2$, then

$$\langle \beta, x_0 \rangle = 3 + \Big(-\frac{1}{2}\Big)2 + \Big(-\frac{1}{2}\Big)3 + \Big(-\frac{1}{2}\Big)3 \leq -1,$$

contradicting our conclusion that $\langle \beta, x_0 \rangle = -\frac{1}{2}$. So by symmetry we may assume without loss of generality that $k = 1$, and thus $j > 1$ (since $\langle \beta, x_0 \rangle$ equals $-\frac{1}{2}$). Set $j(1) = j$; then

$$\begin{aligned} r_0 \cdot \beta = 2x_1 + y_m + 2\big(y_{m-1} + \cdots + y_{j(1)}\big) + 3\big(y_{j(1)-1} + \cdots + y_1\big) \\ + 4x_0 + 2\big(z_1 + \cdots + z_{n-1}\big) + z_n; \end{aligned}$$

since $\langle r_0 \cdot \beta, z_1 \rangle = -1$, Lemma (3.38) implies that the coefficient of z_1 in α equals 2, and (6.73)(i) forces the coefficient of z_k to be equal to 2 for $k \in \{1, \ldots, n-1\}$. Now let $j(2) < j(1)$ be maximal such that α is preceded by $\gamma = (s_{j(2)-1} \cdots s_1 r_0) \cdot \beta$; then

$$\begin{aligned} \gamma = 2x_1 + y_m + 2\big(y_{m-1} + \cdots + y_{j(1)}\big) + 3\big(y_{j(1)-1} + \cdots + y_{j(2)}\big) \\ + 4\big(y_{j(2)-1} + \cdots + y_1\big) + 4x_0 + 2\big(z_1 + \cdots + z_{n-1}\big) + z_n. \end{aligned}$$

Note that $\gamma \prec r \cdot \gamma$ for $r \in Y$ only if $r = s_{j(1)}$ and $j(1) < n$, or $r = s_{j(2)}$ and $j(2) < j(1) - 1$, or $r = t_1$. But α cannot be preceded by $t_1 \cdot \gamma$ since the coefficient of z_1 in α equals 2, and by maximality of β and $j(2)$, α can also not be preceded by $s_{j(1)} \cdot \gamma$ (if $j(1) < n$) or $s_{j(2)} \cdot \gamma$ (if $j(2) < j(1) - 1$); thus $\alpha = \gamma$, as required.

Now assume for a contradiction that there exists an $\alpha \in \mathcal{E}_Y^{\{s_m, t_n\}}$ such that the coefficient of x_0 in α is greater than or equal to 5. We may assume without loss of generality that α is of minimal depth with this property. Then $\alpha \succ \sigma_{m,n}$, and thus there exists an $r \in Y$ such that $\alpha \succ r \cdot \alpha \succeq \sigma_{m,n}$. Since $r \cdot \alpha$ precedes an elementary root, it must be elementary, and as $r \cdot \alpha$ lies between α and $\sigma_{m,n}$, both of which are in $\mathcal{E}_Y^{\{s_m, t_n\}}$, further $r \cdot \alpha \in \mathcal{E}_Y^{\{s_m, t_n\}}$. Thus $r = r_0$ by minimality of α, and moreover, the coefficient of x_0 in $r \cdot \alpha$ is less than 5. Lemma (6.64) yields that the coefficient of x_0 in $r_0 \cdot \alpha$ equals 4, and thus $r_0 \cdot \alpha$ is of type (2) or (3) by our earlier conclusion. But then

$$\langle r_0 \cdot \alpha, x_0 \rangle \geq 4 + \Big(-\frac{1}{2}\Big)2 + \Big(-\frac{1}{2}\Big)2 + \Big(-\frac{1}{2}\Big)4 = 0,$$

contradicting $r_0 \cdot \alpha \prec \alpha$. Thus there are no roots in $\mathcal{E}_Y^{\{s_m, t_n\}}$ with coefficient for x_0 greater than or equal to 5, and this finishes the proof.

□

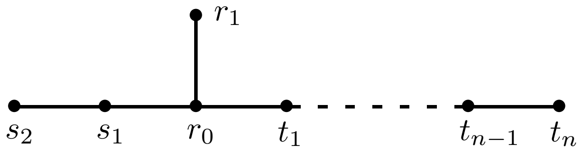
(6.80) Lemma *Suppose Y is of the shape described in (6.78) with $m, n \geq 3$. Then $\mathcal{E}_Y^X = \emptyset$ if X equals $\{s_m\}$, $\{t_n\}$ or $\emptyset$.*

Proof. Assume for a contradiction that $\mathcal{E}_Y^{\{s_m\}} \neq \emptyset$, and let α be an element of minimal depth. Then (6.75) yields that there exists an $r \in \{r_1, t_n\}$ with $r \cdot \alpha \in \mathcal{E}_Y^{\{s_m, r\}}$. Since $\mathcal{E}_Y^{\{s_m, r_1\}}$ is empty by (6.74), we find that $r = t_n$ and $t_n \cdot \alpha \in \mathcal{E}_Y^{\{s_m, t_n\}}$; but (6.79) now forces the coefficient of z_{n-1} in $t_n \cdot \alpha$ to be equal to 2, and thus

$$\langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -(1 + (-\frac{1}{2}) \cdot 2) = 0,$$

contradicting $t_n \cdot \alpha \prec \alpha$. So $\mathcal{E}_Y^{\{s_m\}} = \emptyset$, and symmetrical also $\mathcal{E}_Y^{\{t_n\}} = \emptyset$. Since $\mathcal{E}_Y^{\{r_1\}}$ is also empty by (6.74), Corollary (6.76) implies that $\mathcal{E}_Y^{\emptyset} = \emptyset$. □

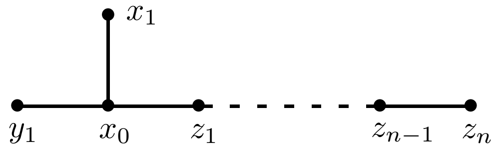
(6.81) *From now on we only need to determine $\mathcal{E}_Y^X$ for Y of the shape*



with $n \geq 2$ and $X \subseteq \{s_2, t_n\}$.

(6.82) Lemma *Suppose Y is of the shape described in (6.81), and let α be a root in Φ_Y. Denote the coefficient of x_0, y_1 and y_2 in α by λ, μ_1 and μ_2 respectively, and suppose that $\lambda \geq 3$. Then $\mu_2 < \mu_1 < \lambda$.*

Proof. Assume for a contradiction that $\mu_1 = \mu_2$ or $\lambda = \mu_1$. Then the support of $s_1 \cdot \alpha$ or $s_2 s_1 \cdot \alpha$ is a subset of



and it can be easily checked using (6.77) that the coefficients in roots with this support are at most 2, contradicting $\lambda \geq 3$. $\square$

(6.83) Proposition *Suppose Y is of the shape described in (6.81). Then the elements of $\mathcal{E}_Y^{\{s_2, t_n\}}$ are:*

$$\begin{aligned}
&\lambda x_1 + y_2 + \mu y_1 + M x_0 \\
&\quad + M\big(z_1 + \cdots + z_{k(M-1)-1}\big) + (M-1)\big(z_{k(M-1)} + \cdots + z_{k(M-2)-1}\big) + \cdots \\
&\qquad \cdots + 3\big(z_{k(3)} + \cdots + z_{k(2)-1}\big) + 2\big(z_{k(2)} + \cdots + z_{n-1}\big) + z_n,
\end{aligned}$$

where

(i) *$M \in \{5, \ldots, n+1\}$ is odd, and $\lambda = \frac{M-1}{2}$, $\mu = \frac{M+1}{2}$, or*

(ii) *$M \in \{3, \ldots, n+1\}$ is odd and $\lambda = \mu = \frac{M+1}{2}$, or*

(iii) *$M \in \{4, \ldots, n+1\}$ is even and $\lambda = \mu = \frac{M}{2}$, or*

(iv) *$M \in \{4, \ldots, n+1\}$ is even and $\lambda = \frac{M}{2}$ and $\mu = \frac{M}{2} + 1$,*

and

$$0 < k(M-1) < k(M-2) < \ldots < k(3) < k(2) < n.$$

Whence

$$\left|\mathcal{E}_Y^{\{s_2,t_n\}}\right| = \binom{n-1}{1} + 2\sum_{M=4}^{n+1}\binom{n-1}{M-2} = 2^n - n - 1.$$

Proof. We show first that α is elementary if α is a vector of the above type, and it follows trivially that $\alpha \in \mathcal{E}_Y^{\{s_2,t_n\}}$. Denote the sum of the coefficients of α by S; then

$$S \geq 2 + 1 + 2 + 3 + 2(n-1) + 1 = 2n + 7.$$

If $S = 2n+7$, then $\alpha = \sigma_{2,n}$ is elementary. Suppose next that $S > 2n+7$, and proceed by induction. If $k(M-1) > 1$, then

$$\langle \alpha, z_{k(M-1)-1}\rangle = M + \Big(-\frac{1}{2}\Big)M + \Big(-\frac{1}{2}\Big)(M-1) = \frac{1}{2},$$

and thus $t_{k(M-1)-1} \cdot \alpha$ is of the form described above with $k(M-1)-1$ in place of $k(M-1)$. By induction this is an elementary root and (3.37) yields that $\alpha \in \mathcal{E}$.

Next assume that $k(M-1) = 1$. Then $M \geq 4$ since $S > 2n+7$, and thus $M - 1 \in \{3, \ldots, n+1\}$. The coefficient of z_1 in α equals $M-1$, and $\lambda + \mu$ equals M or $M+1$.

If $\lambda + \mu$ equals M (that is, in cases (i) and (iii)), we find that

$$\langle \alpha, x_0\rangle = M + \Big(-\frac{1}{2}\Big)M + \Big(-\frac{1}{2}\Big)(M-1) = \frac{1}{2},$$

and the coefficient of x_0 in $r_0 \cdot \alpha$ equals $M-1$. Since $\langle r_0 \cdot \alpha, x_0\rangle = -\frac{1}{2}$, it suffices to show that $r_0 \cdot \alpha$ is of the form described in the assertion; for then $r_0 \cdot \alpha \in \mathcal{E}$ by induction, and (3.37) yields that $\alpha \in \mathcal{E}$. If M is odd, $M-1$ is even; furthermore,

$$\lambda = \frac{M-1}{2} \text{ and } \mu = \frac{M+1}{2} = \frac{M-1}{2} + 1,$$

and these satisfies (iv) for $M-1$ in place of M, as required. Suppose next that M is even, and thus $M-1$ is odd; then

$$\lambda = \mu = \frac{M}{2} = \frac{(M-1)+1}{2},$$

and these satisfies (ii) for $M-1$ in place of M, and $r_0 \cdot \alpha$ is of the required form.

Finally, assume that $\lambda + \mu$ equals $M+1$. If M is even, then $\mu = \frac{M}{2}+1$, $\lambda = \frac{M}{2}$ and $\langle \alpha, y_1 \rangle = \frac{1}{2}$. The coefficient of y_1 in $s_1 \cdot \alpha$ equals $\frac{M}{2}$, while the coefficient of x_1 in $s_1 \cdot \alpha$ equals $\lambda = \frac{M}{2}$; so the coefficients of $s_1 \cdot \alpha$ satisfy (iii). By induction $s_1 \cdot \alpha$ is an elementary root, and (3.37) yields that $\alpha \in \mathcal{E}$. If M is odd, then $\lambda = \mu = \frac{M+1}{2}$, and $\langle \alpha, x_1 \rangle = \frac{1}{2}$. The coefficient of x_1 in $r_1 \cdot \alpha$ equals $\frac{M-1}{2}$, whence the coefficients of $r_1 \cdot \alpha$ satisfy (i). By induction, $r_1 \cdot \alpha \in \mathcal{E}$, and since $\langle r_1 \cdot \alpha, x_1 \rangle = -\frac{1}{2}$, Lemma (3.37) once again implies that α is elementary.

It remains to show that all the elements of $\mathcal{E}_Y^{\{s_2, t_n\}}$ have been enumerated. To do so, we again take the scenic route. So let

$$\alpha = \lambda_1 x_1 + y_2 + \mu_1 y_1 + \lambda_0 x_0 + \nu_1 + \cdots + \nu_{n-1} z_{n-1} + z_n$$

with $\lambda_1, \mu_1, \lambda_0, \nu_1, \ldots, \nu_{n-1} \geq 2$ be an elementary root. Assume for a contradiction that $\nu_k \geq \nu_{k+1} + 2$ for some $k \in \{0, \ldots, n-1\}$ (where $\nu_0 = \lambda_0$ and $\nu_n = 1$). Let $\beta \preceq \alpha$ be according to (6.73)(iii) such that the coefficient of x_0 in β equals λ_0, and the coefficient of z_1 in β is less than or equal to $\lambda_0 - 2$. If the coefficient λ' of x_1 in β is greater than $\frac{\lambda_0}{2}$, then

$$\langle \beta, x_0 \rangle > \frac{\lambda_0}{2} + \Big(-\frac{1}{2}\Big)\lambda_0 = 0$$

and thus $r_0 \cdot \beta \prec \beta$; moreover, the coefficient of x_1 in $r_1 \cdot \beta$ equals $-\lambda' + \lambda_0$, and this is clearly less than $\frac{\lambda_0}{2}$. Thus we may replace β by $r_1 \cdot \beta$, which also precedes α and has coefficient for x_1 less than or equal to $\frac{\lambda_0}{2}$. If the coefficient of y_1 in β is greater than $\frac{\lambda_0}{2} + \frac{1}{2}$, then

$$\langle \beta, y_1 \rangle > \frac{\lambda_0}{2} + \frac{1}{2} + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{1}{2}\Big)\lambda_0 = 0$$

(since the coefficient of y_2 in β is less than or equal to the coefficient of y_2 in α, and thus less than or equal to 1). So $\beta \succ s_1 \cdot \beta$, and we may replace β by $s_1 \cdot \beta$, which also precedes α and has coefficient for y_1 less than or equal to $\frac{\lambda_0}{2} + \frac{1}{2}$. So assume without loss of generality that the sum of the coefficients of x_1 and y_1 in β is less than or equal to $\lambda_0 + \frac{1}{2}$, and thus less than or equal to λ_0 (as the coefficients of x_1 and y_1 in β are integers). Therefore

$$\langle \beta, x_0 \rangle \geq \lambda_0 + \Big(-\frac{1}{2}\Big)(\lambda_0 - 2) + \Big(-\frac{1}{2}\Big)(\lambda_1 + \mu_1) \geq \lambda_0 + \Big(-\frac{1}{2}\Big)(\lambda_0 - 2) + \Big(-\frac{1}{2}\Big)\lambda_0,$$

which equals 1, forcing $\beta \in \Delta$, and contradicting $\alpha \in \mathcal{E}$. So ν_k equals ν_{k+1} or $\nu_{k+1}+1$ for all k by (6.73)(i). In particular, $\lambda_0 \leq n+1$, and we define M to be λ_0. Since $\lambda_1 \geq 2$ further $M \geq 3$ by (6.73)(i).

It remains to show that λ_1 and μ_1 satisfy (i), (ii),(iii) or (iv). First note that since $\langle \alpha, x_1\rangle = \lambda_1 + (-\frac{1}{2})M$ and $\langle \alpha, y_1\rangle = \mu_1 + (-\frac{1}{2})1 + (-\frac{1}{2})M$ have to be less than or equal to $\frac{1}{2}$,

$$\lambda_1 \leq \frac{M+1}{2} \text{ and } \mu_1 \leq \frac{M}{2}+1. \tag{$*$}$$

Next, let $\gamma \preceq \alpha$ be according to (6.73)(ii) such that the coefficient of x_0 in γ equals M, while the coefficient of z_1 in γ is less than or equal to $M-1$, and denote the coefficients of x_1, y_1 in γ by λ' and μ' respectively. By (3.35), $\lambda' \leq \lambda_1$ and $\mu' \leq \mu_1$, and as γ cannot dominate x_0,

$$\frac{1}{2} \geq \langle \gamma, x_0\rangle \geq M + \Big(-\frac{1}{2}\Big)(\lambda'+\mu') + \Big(-\frac{1}{2}\Big)(M-1) \geq \frac{1}{2} + \Big(-\frac{1}{2}\Big)(\lambda_1+\mu_1) + \frac{M}{2};$$

this yields that

$$\lambda_1 + \mu_1 \geq M. \tag{$**$}$$

Suppose first that M is even; then $\lambda_1 \leq \frac{M}{2}$ by $(*)$ since λ_1 is an integer, and thus μ_1 equals $\frac{M}{2}$ or $\frac{M}{2}+1$ by $(*)$ and $(**)$. If $\lambda_1 = \frac{M}{2}$, then λ_1, μ_1 satisfy (iii) or (iv). Assume for a contradiction that $\lambda_1 \leq \frac{M}{2}-1$, and thus $\mu_1 = \frac{M}{2}+1$ and $\lambda_1 = \frac{M}{2}-1$ by the above. So

$$\langle \gamma, x_0\rangle \geq M + \Big(-\frac{1}{2}\Big)(\lambda'+\mu') + \Big(-\frac{1}{2}\Big)(M-1) \geq \frac{1}{2} + \Big(-\frac{1}{2}\Big)(\lambda_1+\mu_1) + \frac{M}{2} = \frac{1}{2},$$

and since $\frac{1}{2} \geq \langle \gamma, x_0\rangle$, we must have equality everywhere; hence λ' must be equal to $\lambda = \frac{M}{2}-1$ and $\mu' = \mu_1 = \frac{M}{2}+1$. Moreover, the coefficient of y_2 in γ is less than or equal to 1. Therefore $\langle \gamma, y_1 + x_0\rangle = \langle \gamma, y_1\rangle + \langle \gamma, x_0\rangle$ is greater than or equal to

$$\begin{aligned}\Big(\frac{M}{2}+1\Big) + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{1}{2}\Big)M \\ + M + \Big(-\frac{1}{2}\Big)\Big(\frac{M}{2}+1\Big) + \Big(-\frac{1}{2}\Big)\Big(\frac{M}{2}-1\Big) + \Big(-\frac{1}{2}\Big)(M-1),\end{aligned}$$

which equals 1, forcing $\gamma \in \Delta$, a contradiction.

Finally assume that M is odd; then $\mu_1 \leq \frac{M+1}{2}$ by $(*)$, and by $(*)$ and $(**)$ we know that λ_1 equals $\frac{M-1}{2}$ or $\frac{M+1}{2}$. Now λ_1, μ_1 satisfy (i) or (ii) if $\mu_1 = \frac{M+1}{2}$, and we assume for a contradiction that $\mu_1 \leq \frac{M-1}{2}$. Then $\lambda_1 = \frac{M+1}{2}$ and $\mu_1 = \frac{M-1}{2}$ by the above; hence

$$\langle \gamma, x_0 \rangle \geq M + \Big(-\frac{1}{2}\Big)(\lambda' + \mu') + \Big(-\frac{1}{2}\Big)(M-1) \geq \frac{1}{2} + \Big(-\frac{1}{2}\Big)(\lambda_1 + \mu_1) + \frac{M}{2} = \frac{1}{2},$$

and since $\frac{1}{2} \geq \langle \gamma, x_0 \rangle$, we must once again have equality everywhere. Thus $\lambda' = \lambda_1 = \frac{M}{2} - 1$ and $\mu' = \mu_1 = \frac{M}{2} + 1$. Now $\langle \gamma, x_1 + x_0 \rangle = \langle \gamma, x_1 \rangle + \langle \gamma, x_0 \rangle$ is greater than or equal to

$$\frac{M+1}{2} + \Big(-\frac{1}{2}\Big)M + M + \Big(-\frac{1}{2}\Big)\frac{M-1}{2} + \Big(-\frac{1}{2}\Big)\frac{M+1}{2} + \Big(-\frac{1}{2}\Big)(M-1),$$

and this equals 1, again forcing $\gamma \in \Delta$, a contradiction. □

(6.84) Lemma *Suppose Y is of the shape described in (6.81). Then $\mathcal{E}_Y^{\{s_2\}}$ is empty.*

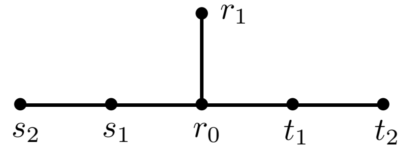
Proof. Assume for a contradiction that $\mathcal{E}_Y^{\{s_2\}}$ is not empty, and let α be an element of minimal depth. By (6.75) there exists an $r \in \{r_1, t_n\}$ such that $r \cdot \alpha \in \mathcal{E}_Y^{\{s_2, r\}}$, and since $\mathcal{E}_Y^{\{r_1, s_2\}}$ is empty by (6.74), we are left with $r = t_n$. The previous proposition now forces the coefficient of z_{n-1} in $t_n \cdot \alpha$ to be equal to 2, and since the coefficient of z_n in $t_n \cdot \alpha$ is 1, this yields

$$\langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -(1 + \Big(-\frac{1}{2}\Big) \cdot 2) = 0,$$

contradicting $t_n \cdot \alpha \prec \alpha$. □

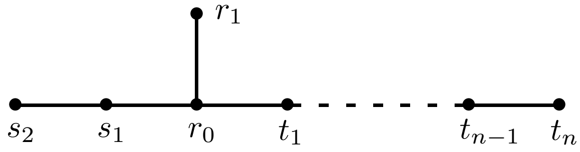
In particular, if $m = n = 2$, symmetrical arguments yield that $\mathcal{E}_Y^{\{t_2\}}$ is also empty; since $\mathcal{E}_Y^{\{r_1\}}$ is empty by (6.74), Corollary (6.76) now implies that $\mathcal{E}_Y^{\emptyset}$ is empty. Alternatively, it can be easily checked that $\sigma_{2,2}$ is the only successor of $\sigma_{2,2}$ if $m = n = 2$, and we get:

(6.85) Lemma *Suppose Y equals*



Then $\mathcal{E}_Y^{\{s_2, t_2\}} = \{\sigma_{2,2}\}$ and $\mathcal{E}_Y^X = \emptyset$ if $X = \{s_2\}$, $\{t_2\}$ or $\emptyset$.

(6.86) *This leaves us to determine $\mathcal{E}_Y^X$ for Y of the shape*



with $n \geq 3$ and $X \subseteq \{t_n\}$.

Define τ_n to be

$$(s_2 s_1 r_0 t_1) \cdot \sigma_{2,n} = 2x_1 + 2y_2 + 3y_1 + 4x_0 + 3z_1 + 2(z_2 + \cdots + z_{n-1}) + z_n.$$

Since $\sigma_{2,n}$ is elementary, and

$$\langle \sigma_{2,n},\, z_1 \rangle = \langle t_1 \cdot \sigma_{2,n},\, x_0 \rangle = \langle (r_0 t_1) \cdot \sigma_{2,n},\, y_1 \rangle = \langle (s_1 r_0 t_1) \cdot \sigma_{2,n},\, y_2 \rangle = -\frac{1}{2},$$

(3.37) yields that τ_n is an elementary root, and it follows easily that τ_n is an element of $\mathcal{E}_Y^{\{t_n\}}$.

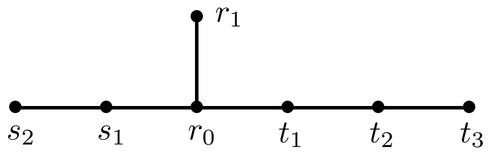
We show now that each root in $\mathcal{E}_Y^X$ with $X \subseteq \{t_n\}$ is a successor of τ_n. So let

$$\alpha = \lambda_1 x_1 + \mu_2 y_2 + \mu_1 y_1 + \lambda_0 x_0 + \nu_1 z_1 + \cdots + \nu_n z_n$$

be an element of $\mathcal{E}_Y^X$. Then $\lambda_1 \geq 2$ by hypothesis, and thus in $\lambda_0 > \lambda_1$ by (6.73)(i); hence in particular, $\lambda_0 \geq 3$, and thus $\mu_2 < \mu_1 < \lambda_0$ by (6.82). Since $\mu_2 \geq 2$ this yields in particular that $\mu_1 \geq 3$ and $\lambda_0 \geq 4$. Clearly $\alpha \succ \sigma_{2,n}$, and an easy calculation yields that t_1 is the only element of Y with $t_1 \cdot \sigma_{2,n} \succ \sigma_{2,n}$; therefore $\alpha \succeq t_1 \cdot \sigma_{2,n}$ and $\nu_1 \geq 3$. Hence $\alpha \succeq \tau_n$ by (6.69), as desired.

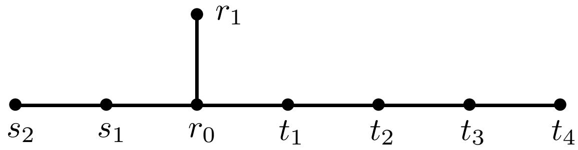
It can be easily verified that the roots listed in the following two lemmas are the only roots preceded by τ_n for $n = 3$, 4, and that these are elementary.

(6.87) Lemma *Suppose Y equals*



Then $\mathcal{E}_Y^{\{t_3\}} = \{\tau_3\}$ and $\mathcal{E}_Y^{\emptyset} = \emptyset$.

(6.88) Lemma *Suppose Y equals*



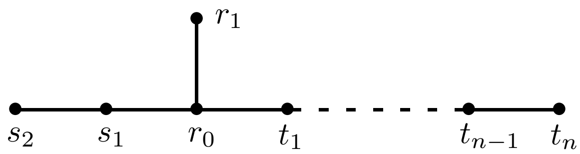
Then $\mathcal{E}_Y^{\{t_4\}}$ consists of the following roots

$$\begin{aligned}
\tau_4 &= 2x_1 + 2y_2 + 3y_1 + 4x_0 + 3z_1 + 2z_2 + 2z_3 + z_4,\\
t_2 \cdot \tau_4 &= 2x_1 + 2y_2 + 3y_1 + 4x_0 + 3z_1 + 3z_2 + 2z_3 + z_4,\\
(t_1t_2) \cdot \tau_4 &= 2x_1 + 2y_2 + 3y_1 + 4x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,\\
(r_0t_1t_2) \cdot \tau_4 &= 2x_1 + 2y_2 + 3y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,\\
(r_1r_0t_1t_2) \cdot \tau_4 &= 3x_1 + 2y_2 + 3y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,\\
(s_1r_0t_1t_2) \cdot \tau_4 &= 2x_1 + 2y_2 + 4y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,\\
(r_1s_1r_0t_1t_2) \cdot \tau_4 &= 3x_1 + 2y_2 + 4y_1 + 5x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,\\
(r_0r_1s_1r_0t_1t_2) \cdot \tau_4 &= 3x_1 + 2y_2 + 4y_1 + 6x_0 + 4z_1 + 3z_2 + 2z_3 + z_4,\\
(t_1r_0r_1s_1r_0t_1t_2) \cdot \tau_4 &= 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 3z_2 + 2z_3 + z_4,\\
(t_2t_1r_0r_1s_1r_0t_1t_2) \cdot \tau_4 &= 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2z_3 + z_4,\\
(t_3t_2t_1r_0r_1s_1r_0t_1t_2) \cdot \tau_4 &= 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 3z_3 + z_4,
\end{aligned}$$

and $\mathcal{E}_Y^{\emptyset}$ has exactly one element, namely

$$(t_4t_3t_2t_1r_0r_1s_1r_0t_1t_2) \cdot \tau_4 = 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 3z_3 + 2z_4.$$

(6.89) *It remains to determine $\mathcal{E}_Y^X$ for Y equal to*



with $n \geq 5$, and $X \subseteq \{t_n\}$.

(6.90) Proposition *Suppose Y is of the shape described in (6.89). Then $\mathcal{E}_Y^{\{t_n\}}$ equals the set of vectors of the following six types:*

$$\begin{aligned}
(1)\qquad & 2x_1 + 2y_2 + 3y_1 + 4x_0 + 4\big(z_1 + \cdots + z_{k(2)-1}\big)\\
& \quad + 3\big(z_{k(2)} + \cdots + z_{k(1)-1}\big) + 2\big(z_{k(1)} + \cdots + z_{n-1}\big) + z_n,
\end{aligned}$$

where $0 < k(2) < k(1) < n$,

$$\begin{aligned}(2)\qquad &\lambda x_1 + 2y_2 + \mu y_1 + 5x_0 \\ &\quad + 5\big(z_1 + \cdots + z_{k(3)-1}\big) + 4\big(z_{k(3)} + \cdots + z_{k(2)-1}\big) \\ &\quad + 3\big(z_{k(2)} + \cdots + z_{k(1)-1}\big) + 2\big(z_{k(1)} + \cdots z_{n-1}\big) + z_n,\end{aligned}$$

where $0 < k(3) < k(2) < k(1) < n$ *and* $\lambda \in \{2,3\}$, $\mu \in \{3,4\}$,

$$\begin{aligned}(3)\qquad &\lambda x_1 + 2y_2 + \mu y_1 + 6x_0 \\ &\quad + 6\big(z_1 + \cdots + z_{k(4)-1}\big) + 5\big(z_{k(4)} + \cdots + z_{k(3)-1}\big) \\ &\quad + 4\big(z_{k(3)} + \cdots + z_{k(2)-1}\big) + 3\big(z_{k(2)} + \cdots + z_{k(1)-1}\big) \\ &\quad + 2\big(z_{k(1)} + \cdots + z_{n-1}\big) + z_n,\end{aligned}$$

where $0 < k(4) < k(3) < k(2) < k(1) < n$ *and* $(\lambda, \mu) = (3,3)$ *or* $(2,4)$,

$$\begin{aligned}(4)\qquad &3x_1 + 2y_2 + 4y_1 + 6x_0 + 4\big(z_1 + \cdots + z_{k(2)-1}\big) \\ &\quad + 3\big(z_{k(2)} + \cdots + z_{k(1)-1}\big) + 2\big(z_{k(1)} + \cdots + z_{n-1}\big) + z_n,\end{aligned}$$

where $1 < k(2) < k(1) < n$,

$$(5)\quad 3x_1+2y_2+4y_1+6x_0+5z_1+3\big(z_2+\cdots+z_{k(1)-1}\big)+2\big(z_{k(1)}+\cdots+z_{n-1}\big)+z_n,$$

where $2 < k(1) < n$ *and*

$$(6)\qquad 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2\big(z_3 + \cdots + z_{n-1}\big) + z_n.$$

Whence $\big|\mathcal{E}_Y^{\{t_n\}}\big| = 2\binom{n+1}{4}$.

Proof. We show first that α is elementary if it is of one of the types (1)-(6); since the coefficients of α certainly satisfy the required conditions, this yields that α is in $\mathcal{E}_Y^{\{t_n\}}$.

Suppose first that α is of type (1); then $k(1)+k(2) \geq 3$. If $k(1) = 2$ and $k(2) = 1$, then $\alpha = \tau_n \in \mathcal{E}$. Suppose next that $k(1) + k(2) > 3$, and proceed by induction. If $k(2) > 1$ we find that $\langle \alpha, y_{k(2)-1} \rangle = \frac{1}{2}$, and that $t_{k(2)-1} \cdot \alpha$ is of type (1) with $k(2) - 1$ in place of $k(2)$. By induction, this is an elementary root, and (3.37) implies that α is elementary. Now suppose that $k(2) = 1$, and thus $k(1) > 2$. Then $\langle \alpha, y_{k(1)-1} \rangle = \frac{1}{2}$ and $t_{k(1)-1} \cdot \alpha$ is of type (1) with

$k(1)-1$ in place of $k(1)$. By induction, this is an elementary root, and since $\langle t_{k(1)-1}\cdot\alpha, z_{(1)-1}\rangle = -\frac{1}{2}$, Lemma (3.37) yields again that $\alpha\in\mathcal{E}$, and this finishes the induction.

Suppose next that α is of type (2), and denote the sum of the coefficients of α by S; then

$$S \geq 2+2+3+5+4+3+2(n-3)+1 = 2n+14.$$

If $S = 2n+14$, then α equals

$$(r_0t_1t_2)\cdot\tau_n = 2x_1+2y_2+3y_1+5x_0+4z_1+3z_2+2(z_3+\cdots+z_{n-1})+z_n,$$

and since τ_n is elementary, it can be easily verified using (3.37) that this is an elementary root. So suppose next that $S > 2n+15$, and proceed by induction. If $k(3)>1$ we find that $\langle\alpha, y_{k(3)-1}\rangle = \frac{1}{2}$, and $t_{k(3)-1}\cdot\alpha$ is of type (2) with $k(3)-1$ in place of $k(3)$. By induction, this is an elementary root, and it follows by (3.37) that α is elementary. If $\lambda = 3$, then $\langle\alpha, x_1\rangle = \frac{1}{2}$ and $r_1\cdot\alpha$ is of type (2) with 2 in place of λ; this is an elementary root by inductive hypothesis, and (3.37) yields that $\alpha\in\mathcal{E}$. If $\mu = 4$, then $\langle\alpha, y_1\rangle = \frac{1}{2}$ and $s_1\cdot\alpha$ is also of type (2); by induction $s_1\cdot\alpha$ is in $\mathcal{E}$, and (3.37) implies again that α is elementary.

Suppose now that $\lambda = 2$, $\mu = 3$ and $k(3) = 1$. Then

$$\langle\alpha, x_0\rangle = 5+\Big(-\frac{1}{2}\Big)3+\Big(-\frac{1}{2}\Big)2+\Big(-\frac{1}{2}\Big)4 = \frac{1}{2},$$

and thus $r_0\cdot\alpha$ is of type (1). So $r_0\cdot\alpha\in\mathcal{E}$ by the above, and since $\langle r_0\cdot\alpha, x_0\rangle$ equals $-\frac{1}{2}$, Lemma (3.37) once again yields that $\alpha\in\mathcal{E}$, and this finishes the induction.

Now let α be of type (3) and denote the sum of the coefficients of α by S; then S is greater than or equal to

$$(\lambda+\mu)+2+6+5+4+3+2(n-4)+1 = 6+2+6+5+4+3+2(n-4)+1 = 2n+19.$$

If $S = 2n+19$, then α is equal to one of the following two roots:

$$\begin{aligned}&(r_0s_1t_1r_0t_2t_1t_2)\cdot\tau_n\\ &\qquad = 2x_1+2y_2+4y_1+6x_0+5z_1+4z_2+3z_3+2(z_4+\cdots+z_{n-1})+z_n,\end{aligned}$$

$$(r_0r_1t_1r_0t_2t_1t_2)\cdot\tau_n$$
$$= 3x_1 + 2y_2 + 3y_1 + 6x_0 + 5z_1 + 4z_2 + 3z_3 + 2(z_4 + \cdots + z_{n-1}) + z_n.$$

As τ_n is elementary, it can be easily verified using (3.37) that these are elementary roots. Suppose next that $S > 2n+19$. If $k(4) > 1$, we find that $\langle\alpha, y_{k(4)-1}\rangle = \frac{1}{2}$ and $t_{k(4)-1}\cdot\alpha$ is of type (3) with $k(4)-1$ in place of $k(4)$. By induction this is an elementary root, and since $\langle t_{k(4)-1}\cdot\alpha, z_{(4)-1}\rangle = -\frac{1}{2}$, this implies that α is elementary. Suppose now that $k(4) = 1$; that is, the coefficient of z_1 in α equals 5. Since the sum of the coefficients of x_1 and y_1 equals 6, we have

$$\langle\alpha, x_0\rangle = 6 + \Big(-\frac{1}{2}\Big)6 + \Big(-\frac{1}{2}\Big)5 = \frac{1}{2},$$

and $r_0\cdot\alpha$ is of type (2). This is an elementary root by the above, and since $\langle r_0\cdot\alpha, x_0\rangle = -\frac{1}{2}$, this implies that α is elementary.

Next, assume that α is of type (4); then $\langle\alpha, x_0\rangle = \frac{1}{2}$, and $r_0\cdot\alpha$ is of type (2) with $k(3) = 1$. By the above, this is an element of $\mathcal{E}$, and since $\langle r_0\cdot\alpha, x_0\rangle = -\frac{1}{2}$ we can deduce that $\alpha\in\mathcal{E}$. Further, if α is of type (5), $\langle\alpha, z_1\rangle = \frac{1}{2}$ and $t_1\cdot\alpha$ is of type (4) with $k(2) = 2$; therefore $t_1\cdot\alpha$ is in $\mathcal{E}$ by the above. As before, this yields that α is an elementary root. Finally, assume that α equals

$$3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2(z_3 + \cdots + z_{n-1}) + z_n.$$

Then $\langle\alpha, z_2\rangle = \frac{1}{2}$ and $t_2\cdot\alpha$ is of type (5) with $k(1) = 3$. By the above, $t_2\cdot\alpha\in\mathcal{E}$, and we can once more conclude that α is an elementary root.

It remains to show that we have listed all the roots in $\mathcal{E}_Y^{\{t_n\}}$. So let $\alpha\in\mathcal{E}_Y^{\{s_m,t_n\}}$. Since $\alpha\succeq\tau_n$ by an earlier remark, we know that the coefficient of x_0 in α is greater than 4. We first show that α is of one of the types (1)-(6) if the coefficient of x_0 in α equals 4, 5 or 6. So let $\lambda_1, \mu_2, \mu_1, \nu_1, \ldots, \nu_{n-1}\geq 2$ such that

$$\alpha = \lambda_1x_1 + \mu_2y_2 + \mu_1y_1 + 4x_0 + \nu_1z_1 + \cdots + \nu_{n-1}z_{n-1} + z_n.$$

Then $2\leq\lambda_1\leq 3$, and since $\langle\alpha, x_1\rangle\leq\frac{1}{2}$, we deduce that $\lambda_1 = 2$. Further, $2\leq\mu_2<\mu_1<4$, and thus $\mu_2 = 2$ and $\mu_1 = 3$.

Assume for a contradiction that $\nu_k\geq\nu_{k+1}+2$ for some $k\in\{0,\ldots,n-1\}$ (where $\nu_0 = 4$ and $\nu_n = 1$), and let $\beta\preceq\alpha$ be according to (6.73)(iii) such that the coefficient of x_0 in β equals 4, and the coefficient of z_1 in β is less than or equal to 2. Denote the coefficients of x_1 and y_1 in β by λ' and μ' respectively. Then $\lambda' \leq \lambda_1$ and $\mu' \leq \mu_1$, since $\beta \preceq \alpha$, and thus $\langle \beta, x_0 \rangle$ is greater than or equal to

$$4 + \Big(-\frac{1}{2}\Big)\mu' + \Big(-\frac{1}{2}\Big)\lambda' + \Big(-\frac{1}{2}\Big)2 \geq 4 + \Big(-\frac{1}{2}\Big)3 + \Big(-\frac{1}{2}\Big)2 + \Big(-\frac{1}{2}\Big)2 = \frac{1}{2};$$

hence $r_0 \cdot \beta \prec \beta$. It is clear that β is an elementary root, and (6.64) yields that $\langle \beta, x_0 \rangle = \frac{1}{2}$ and that the coefficient of x_0 in $r_0 \cdot \beta$ equals 3. But $\langle \beta, x_0 \rangle = \frac{1}{2}$ now forces equality in the above inequality; in particular, $\mu' = 3$, and thus $r_0 \cdot \beta$ has coefficient 3 for x_0 and y_1, contradicting (6.82). So $\nu_k \in \{\nu_{k+1}, \nu_{k+1} + 1\}$ for all $k \in \{0, \ldots, n-1\}$ by (6.73)(i), and it follows that α is of type (1).

Next let $\lambda_1, \mu_2, \mu_1, \nu_1, \ldots, \nu_{n-1} \geq 2$ such that

$$\alpha = \lambda_1 x_1 + \mu_2 y_2 + \mu_1 y_1 + 5x_0 + \nu_1 z_1 + \cdots + \nu_{n-1} z_{n-1} + z_n.$$

Then $2 \leq \lambda_1 \leq 4$, and since $\langle \alpha, x_1 \rangle \leq \frac{1}{2}$ we know that $\lambda_1 \in \{2, 3\}$. Furthermore, $2 \leq \mu_2 < \mu_1 < 5$, and thus $\mu_2 = 2$ and $\mu_1 \in \{3, 4\}$, or $\mu_2 = 3$ and $\mu_1 = 4$. But in the latter case $\langle \alpha, y_2 \rangle = 3 + (-\frac{1}{2})4 = 1$, contradicting $\alpha \in \mathcal{E}$, and thus $\mu_2 = 2$ and $\mu_1 \in \{3, 4\}$.

Assume for a contradiction that $\nu_k \geq \nu_{k+1} + 2$ for some k, and let $\beta \preceq \alpha$ be according to (6.73)(iii) such that the coefficient of x_0 in β equals 5, while the coefficient of z_1 in β is less than or equal to 3. If the coefficient of x_1 in β is 3, α is also preceded by $r_1 \cdot \beta$, and so we may assume without loss of generality that the coefficient of x_1 in β is less than or equal to 2; similarly, if the coefficient of y_1 in β is 4, α is also preceded by $s_1 \cdot \beta$, and so we may further assume without loss of generality that the coefficient of y_1 in β is less than or equal to 3. Thus

$$\langle \beta, x_0 \rangle \geq 5 + \Big(-\frac{1}{2}\Big)3 + \Big(-\frac{1}{2}\Big)2 + \Big(-\frac{1}{2}\Big)3 = 1,$$

forcing $\beta \in \Delta$, and thus $\alpha \in \Delta$, a contradiction. Therefore ν_k equals either ν_{k+1} or $\nu_{k+1} + 1$ for all k by (6.73)(i), and α is of type (2).

Suppose now that $\alpha \in \mathcal{E}_Y^{\{t_n\}}$ has coefficient 6 for x_0, and let β be of maximal depth with $\tau_n \preceq \beta \preceq \alpha$ such that the coefficient of x_0 in β is less than 6. It is clear that β is elementary, and since $\tau_n \preceq \beta \preceq \alpha$, it follows that $\beta \in \mathcal{E}_Y^{\{s_n\}}$. By maximality of β, we deduce that $\beta \prec r_0 \cdot \beta \preceq \alpha$ and that the coefficient of x_0 in $r_0 \cdot \beta$ equals 6; now (6.64) implies that $\langle r_0 \cdot \beta, x_0 \rangle = \frac{1}{2}$ and that the coefficient of x_0 in β equals 5. Since $\beta \in \mathcal{E}_Y^{\{t_n\}}$, the above yields that β is of type (2); that is, β equals

$$\lambda x_1 + 2y_2 + \mu y_1 + 5x_0 + 5\bigl(z_1 + \cdots + z_{k(3)-1}\bigr) + \cdots + 2\bigl(z_{k(1)} + \cdots z_{n-1}\bigr) + z_n,$$

for some $0 < k(3) < k(2) < k(1) < n$ and $\lambda \in \{2,3\}$, $\mu \in \{3,4\}$. As $\langle \beta, x_0 \rangle = -\frac{1}{2}$, we are left with $(\lambda, \mu) \in \{(2,4), (3,3)\}$ and $k(3) > 1$, or $(\lambda, \mu) = (3,4)$ and $k(3) = 1$.

If $(\lambda, \mu) \in \{(2,4), (3,3)\}$ and $k(3) > 1$, then $r_0 \cdot \beta$ equals

$$\begin{aligned} &\lambda x_1 + 2y_2 + \mu y_1 + 6x_0 + 5\bigl(z_1 + \cdots + z_{k(3)-1}\bigr) + 4\bigl(z_{k(3)} + \cdots + z_{k(2)-1}\bigr) \\ &\quad + 3\bigl(z_{k(2)} + \cdots + z_{k(1)-1}\bigr) + 2\bigl(z_{k(1)} + \cdots + z_{n-1}\bigr) + z_n. \end{aligned}$$

Let $k(4) < k(3)$ be the maximal such that α is preceded by $(t_{k(4)-1} \cdots t_1) r_0 \cdot \beta$, and call this root γ. Then

$$\begin{aligned} \gamma = \lambda x_1 &+ 2y_2 + \mu y_1 + 6x_0 \\ &+ 6\bigl(z_1 + \cdots + z_{k(4)-1}\bigr) + 5\bigl(z_{k(4)} + \cdots + z_{k(3)-1}\bigr) \\ &+ 4\bigl(z_{k(3)} + \cdots + z_{k(2)-1}\bigr) + 3\bigl(z_{k(2)} + \cdots + z_{k(1)-1}\bigr) \\ &+ 2\bigl(z_{k(1)} + \cdots + z_{n-1}\bigr) + z_n. \end{aligned}$$

Now assume for a contradiction that α does not equal γ, and let $r \in Y$ such that $\alpha \succeq r \cdot \gamma \succ \gamma$; then $\langle \gamma, \alpha_r \rangle = -\frac{1}{2}$ by (6.64). Since $\lambda \in \{2,3\}$ and $\mu \in \{3,4\}$ clearly $r \neq r_1, s_2, s_1$, and as

$$\langle \gamma, x_0 \rangle \geq 6 + \Bigl(-\frac{1}{2}\Bigr)6 + \Bigl(-\frac{1}{2}\Bigr)(\lambda + \mu) \geq 6 + \Bigl(-\frac{1}{2}\Bigr)6 + \Bigl(-\frac{1}{2}\Bigr)6 \geq 0,$$

furthermore $r \neq r_0$. Now $\langle \gamma, z_{k(4)} \rangle = 0$ if $k(4) = k(3) - 1$ and $r \neq t_{k(4)}$ by maximality of $k(4)$ if $k(4) < k(3) - 1$; hence we are left with $r = t_{k(3)}$ and $k(3) < k(2) - 1$, or $r = t_{k(2)}$ and $k(2) < k(1) - 1$, or $r = t_{k(1)}$ and $k(1) < n$. But then an easy calculation yields that α is preceded by $r \cdot \beta \succ \beta$, contradicting the maximality of β. So $\alpha = \gamma$, and this is of type (3).

Suppose next that $(\lambda, \mu) = (3,4)$ and $k(3) = 1$. Then $\gamma = r_0 \cdot \beta$ equals

$$\begin{aligned} &3x_1 + 2y_2 + 4y_1 + 6x_0 + 4\bigl(z_1 + \cdots + z_{k(2)-1}\bigr) \\ &\quad + 3\bigl(z_{k(2)} + \cdots + z_{k(1)-1}\bigr) + 2\bigl(z_{k(1)} + \cdots + z_{n-1}\bigr) + z_n, \end{aligned}$$

and this is of type (4). If α is not equal to γ, let $r \in Y$ such that $\alpha \succeq r \cdot \gamma \succ \gamma$. Then $\langle \gamma, \alpha_r \rangle = -\langle r \cdot \gamma, \alpha_r \rangle = -\frac{1}{2}$ by (6.64), and since

$$\langle \gamma, x_1 \rangle = \langle \gamma, y_2 \rangle = \langle \gamma, y_1 \rangle = \langle \gamma, x_0 \rangle = 0,$$

it follows that $r \in \{t_1, \ldots, t_n\}$, and thus $r \in \{t_1, t_{k(2)}, t_{k(1)}\}$; maximality of β now only leaves us with $r = t_1$. Since $\langle \gamma, z_1 \rangle = -\frac{1}{2}$, we find that $k(2) = 2$; hence $\delta = t_1 \cdot \gamma$ equals

$$3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 3\big(z_2 + \cdots + z_{k(1)-1}\big) + 2\big(z_{k(1)} + \cdots + z_{n-1}\big) + z_n,$$

which is of type (5). If $\alpha \neq \delta$, let $s \in Y$ such that $\alpha \succeq r \cdot \delta \succ \delta$. Then $\langle \delta, \alpha_s \rangle = -\frac{1}{2}$, and since

$$\langle \delta, x_1 \rangle = \langle \delta, y_2 \rangle = \langle \delta, y_1 \rangle = \langle \delta, x_0 \rangle = 0$$

as well as $\langle \delta, y_1 \rangle > 0$, we deduce that $s \in \{t_2, t_{k(1)}\}$; thus $s = t_2$ by maximality of β. Since $\langle \delta, z_2 \rangle = -\frac{1}{2}$, moreover $k(1) = 3$, and

$$t_2 \cdot \delta = 3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2\big(z_3 + \cdots + z_{n-1}\big) + z_n.$$

As $n \geq 5$, it can be easily verified that there are no $t \in Y$ such that $\langle t_2 \cdot \delta, \alpha_t \rangle$ lies in the open interval $(-1, 0)$, and we can deduce from (3.38) that α must be equal to $t_2 \cdot \delta$; that is, α is of type (6).

Now assume for a contradiction that there exists an $\alpha \in \mathcal{E}_Y^{\{t_n\}}$ such that the coefficient of x_0 in α is greater than or equal to 7. We may assume without loss of generality that α is of minimal depth with this property. Then $\alpha \succ \tau_n$, and hence there exists an $r \in Y$ such that $\alpha \succ r \cdot \alpha \succeq \tau_n$. It is clear that $r \cdot \alpha \in \mathcal{E}_Y^{\{t_n\}}$, and minimality of α yields that $r = r_0$ and that the coefficient of x_0 in $r \cdot \alpha$ is less than or equal to 6. By (6.64) the coefficient of x_0 in $r_0 \cdot \alpha$ equals 6, and thus $r \cdot \alpha$ is of type (3), (4), (5) or (6) by the above; but then $\langle r_0 \cdot \alpha, x_0 \rangle \geq 0$, contradicting $\alpha \succ r_0 \cdot \alpha$, and this finishes the proof. □

(6.91) PROPOSITION *Suppose Y is of the shape described in (6.89). Then $\mathcal{E}_Y^{\emptyset} = \emptyset$.*

Proof. Assume for a contradiction that $\mathcal{E}_Y^{\emptyset} \neq \emptyset$, and let α be an element of minimal depth. By (6.75), there exists an $r \in \{r_1, s_2, t_n\}$ such that $r \cdot \alpha$ is in

$\mathcal{E}_Y^{\{r\}}$. Since $\mathcal{E}_Y^{\{r_1\}} = \mathcal{E}_Y^{\{s_2\}} = \emptyset$ by (6.74) and (6.84) respectively, this leaves us with $t_n \cdot \alpha \in \mathcal{E}_Y^{\{t_n\}}$. The elements of $\mathcal{E}_Y^{\{t_n\}}$ are of types (1)-(6) stated in the previous proposition, and so the coefficient of z_{n-1} in $t_n \cdot \alpha$ equals 2, while the coefficient of z_n in $t_n \cdot \alpha$ equals 1; whence

$$\langle \alpha, z_n \rangle = -\langle t_n \cdot \alpha, z_n \rangle = -\left(1 + \left(-\frac{1}{2}\right)2\right) = 0,$$

contradicting $t_n \cdot \alpha \prec \alpha$.

□

§6b One non-simple bond

Henceforth assume that X, $Y \subseteq R$ satisfy (6.61), and that Y contains exactly one non-simple bond of finite weight m. Let r_1, $s_1 \in Y$ be the vertices of the non-simple bond, and denote the simple roots corresponding to r_1, s_1 by x_1 and y_1 respectively. Further, let Y_1 and Y_2 be the connected components of the graph obtained from Y by deleting the non-simple bond, and assume that $r_1 \in Y_1$ and $s_1 \in Y_2$.

We denote $2\cos(\pi/m)$ by c_m; then (2.26) yields that the coefficient of a simple root in any element of Φ_Y^+ equals 0, 1 or c_m, or is greater than or equal to 2.

Each element of $\mathcal{E}_Y$ is preceded by some simple root α_r with $r \in Y$. Suppose $\alpha \in \mathcal{E}_Y$ is preceded by α_r for $r \in Y_1$, and let β be of maximal depth with $\alpha \succeq \beta \succeq \alpha_r$ such that $I(\beta) \subseteq Y_1$. Since $I(\alpha) \not\subseteq Y_1$, maximality of β yields that $\alpha \succeq s_1 \cdot \beta \succ \beta$; then $\langle \beta, y_1 \rangle < 0$ and thus $\langle \beta, y_1 \rangle \in (-1, 0)$ by (3.38). If λ denotes the coefficient of x_1 in β, we find that $\langle \beta, y_1 \rangle = (-\frac{c_m}{2})\lambda$, and hence $0 < \lambda < 2/c_m \leq \sqrt{2}$; thus $\lambda = 1$ by (2.26). If $s \in Y_2 \setminus \{s_1\}$, the coefficient of α_s in $s_1 \cdot \beta$ equals 0, therefore (6.58) yields that

$$\alpha \succeq s_1 \cdot x_1 = x_1 + c_m y_1 \succ x_1.$$

Symmetrical arguments give $\alpha \succeq c_m x_1 + y_1 \succ y_1$ if α is preceded by α_r with $r \in Y_2$.

(6.92) *Observe that $\mathcal{E}_Y^X$ does not depend on R as long as $Y \subseteq R$, hence we may assume without loss of generality that there exists a $t \in R \setminus Y$ such that*

$$m_{tr} = m_{rt} = \begin{cases} 3 & \text{if } r = s_1, \\ 2 & \text{if } r \neq s_1, \end{cases}$$

for $r \in Y$.

(6.93) Proposition *Suppose R satisfies (6.92). Then*

$$\phi\colon \beta \mapsto c_m(\beta - \alpha_t) + x_1$$

defines a one-one correspondence between the set of roots in $\Phi^+_{Y_2 \cup \{t\}}$ with coefficient 1 for α_t, and the set of roots in $\Phi^+_{\{r_1\} \cup Y_2}$ with coefficient 1 for x_1. Moreover, ϕ restricts to a one-one correspondence between the set of roots in $\mathcal{E}_{\{t\} \cup Y_2}$ with coefficient 1 for α_t, and the set of roots in $\mathcal{E}_{\{r_1\} \cup Y_2}$ with coefficient 1 for x_1.

Proof. Observe that for $s \in Y_2$,

$$\begin{aligned} \langle x_1 - c_m\alpha_t, \alpha_s\rangle &= \langle x_1, \alpha_s\rangle - c_m\langle \alpha_t, \alpha_s\rangle \\ &= \begin{cases} 0 - c_m \times 0 & \text{if } s \in Y_2 \setminus \{s_1\}, \\ -\frac{c_m}{2} - c_m(-\frac{1}{2}) & \text{if } s = s_1, \end{cases} \\ &= 0. \end{aligned}$$

Hence for $\beta \in \Phi$ and $\gamma \in \Phi_{Y_2}$,

$$\big\langle x_1 + c_m(\beta - \alpha_t), \gamma\big\rangle = c_m\langle \beta, \gamma\rangle; \tag{$*$}$$

moreover, for $s \in Y_2$,

$$\begin{aligned} s \cdot \big(x_1 + c_m(\beta - \alpha_t)\big) &= x_1 + c_m(\beta - \alpha_t) - 2\big\langle x_1 + c_m(\beta - \alpha_t)\,,\, \alpha_s\big\rangle \\ &= x_1 + c_m(\beta - \alpha_t) - 2c_m\langle \beta, \alpha_s\rangle \\ &= x_1 + c_m(\beta - 2\langle \beta, \alpha_s\rangle - \alpha_t) \\ &= x_1 + c_m(s \cdot \beta - \alpha_t). \end{aligned} \tag{$**$}$$

Now let $\beta \in \Phi^+_{Y_2 \cup \{t\}}$ with coefficient 1 for α_t; then (6.59) implies that there exists a $w \in W_{Y_2}$ with $l(w) = \mathrm{dp}(\beta) - 1$ and $\beta = w \cdot \alpha_t$. The coefficient of x_1 in $w \cdot x_1$ equals 1, and since $x_1 = x_1 + c_m(\alpha_t - \alpha_t)$, a straightforward induction on $l(w)$ using $(**)$ yields that $w \cdot x_1 = x_1 + c_m(w \cdot \alpha_t - \alpha_t)$. Hence ϕ is well defined. By (6.59), every element of $\Phi^+_{\{r_1\} \cup Y_2}$ with coefficient 1 for x_1 can be written as $w \cdot x_1$ for some $w \in W_{Y_2}$, so the above also shows that ϕ is onto. Since ϕ is certainly one-one, ϕ is a one-one correspondence between the given sets, and by the above construction it remains to show for $w \in W_{Y_2}$ that $w \cdot x_1 \in \Delta$ if and only if $w \cdot \alpha_t \in \Delta$.

First, suppose that there exists a $\gamma \in \Phi^+ \setminus \{w \cdot x_1\}$ such that $w \cdot x_1$ dominates γ. Then $w^{-1} \cdot \gamma \in \Phi^-$, since x_1 is not in Δ, and thus γ is an element of $N(w^{-1})$, which is a subset of $\Phi^+_{Y_2}$. Now $\langle w \cdot x_1, \gamma \rangle \geq 1$ by (3.32), and thus by $(*)$

$$\langle w \cdot \alpha_t, \gamma \rangle = \frac{1}{c_m} \langle \alpha, \gamma \rangle \geq \frac{1}{c_m} > \frac{1}{2}.$$

Since $I(w \cdot \alpha_t) \cup I(\gamma)$ contains only simple bonds, $\langle w \cdot \alpha_t, \gamma \rangle$ is an integer multiple of $\frac{1}{2}$, and thus $\langle w \cdot \alpha_t, \gamma \rangle \geq 1$. So $w \cdot \alpha_t \ \mathsf{dom}\ \gamma$ or $\gamma \ \mathsf{dom}\ w \cdot \alpha_t$ by (3.32); but γ cannot dominate $w \cdot \alpha_t$, as $w^{-1} \cdot \gamma$ is negative, while $w^{-1} \cdot (w \cdot \alpha_t) = \alpha_t$ is positive, and thus $w \cdot \alpha_t \in \Delta$.

For the converse, suppose that there exists a $\gamma \in \Phi^+ \setminus \{w \cdot \alpha_t\}$ such that $(w \cdot \alpha_t) \ \mathsf{dom}\ \gamma$. Then $\langle w \cdot x_1, \gamma \rangle = c_m \langle w \cdot \alpha_t, \gamma \rangle \geq c_m \geq 1$ by $(*)$ and (3.32); since $w^{-1} \cdot \gamma$ is negative and $w^{-1} \cdot (w \cdot x_1) = x_1$ is not, we deduce that $w \cdot x_1 \in \Delta$.

□

(6.94) PROPOSITION *Suppose R satisfies (6.92) and $Y_1 = \{r_1\}$. Then*

$$\begin{aligned} \mathcal{E}_Y^{\{r_1\}} &= \big\{ x_1 + c_m(\beta - \alpha_t) \mid \beta \in \mathcal{E}_{Y_2 \cup \{t\}} \text{ has coefficient 1 for } \alpha_t \big\} \\ &= \big\{ x_1 + c_m(\beta - \alpha_t) \mid \beta \in \bigcup_{I \subseteq Y_2} \mathcal{E}^{I \cup \{t\}}_{Y_2 \cup \{t\}} \big\}. \end{aligned}$$

This leaves us to determine $\mathcal{E}_Y^X$ for X with $r_1, s_1 \notin X$; that is, we only need to consider the set of elementary roots with coefficients greater than 1 for x_1 and y_1.

(6.95) Lemma *Suppose $Y = \{r_1, s_1\}$. Then*

$$\mathcal{E}_Y^{\emptyset} = \Big\{ \frac{\sin\big((n+1)\pi/m\big)}{\sin(\pi/m)} x_1 + \frac{\sin\big(n\pi/m\big)}{\sin(\pi/m)} y_1 \mid n \in \{2, \ldots, \frac{m}{2} - 1\} \Big\}$$
$$\cup \Big\{ \frac{\sin\big(n\pi/m\big)}{\sin(\pi/m)} x_1 + \frac{\sin\big((n+1)\pi/m\big)}{\sin(\pi/m)} y_1 \mid n \in \{2, \ldots, \frac{m}{2} - 1\} \Big\}$$

if m is even, and if m is odd,

$$\mathcal{E}_Y^{\emptyset} = \Big\{ \frac{\sin\big((n+1)\pi/m\big)}{\sin(\pi/m)} x_1 + \frac{\sin\big(n\pi/m\big)}{\sin(\pi/m)} y_1 \mid n \in \{2, \ldots, m-3\} \Big\}.$$

(6.96) Proposition *Suppose that $m \geq 6$, $|Y_2| \geq 2$ and $r_1 \notin X$. Then $\mathcal{E}_Y^X$ is empty unless $Y_1 = \{r_1\}$ and $X = \emptyset$. Moreover, if $m \geq 7$ and $Y_1 = \{r_1\}$, then*

$$\begin{aligned} \mathcal{E}_Y^{\emptyset} &= \big\{ (c_m^2 - 1)x_1 + c_m\beta \mid \beta \in \mathcal{E}_{Y_2} \text{ has coefficient 1 for } y_1 \ \big\} \\ &= \big\{ (c_m^2 - 1)x_1 + c_m\beta \mid \beta \in \bigcup_{J \subseteq Y_2 \setminus \{s_1\}} \mathcal{E}_{Y_2}^{J \cup \{s_1\}} \big\}. \end{aligned}$$

Suppose next that $m = 6$ and $Y_1 = \{r_1\}$. We may assume that there exist $t_1, t_2 \in R \setminus Y$ such that

$$m_{rt_1} = m_{rt_2} = \begin{cases} 3 & \text{if } r = s_1, \\ 2 & \text{if } r \neq s_1, \end{cases}$$

for all $r \in Y$. Denote the simple roots corresponding to t_1 and t_2 by z_1 and z_2 respectively. Then

$$\begin{aligned} \mathcal{E}_Y^{\emptyset} &= \big\{ 2x_1 + \sqrt{3}(\beta - z_1 - z_2) \mid \beta \in \mathcal{E}_{\{t_1,t_2\} \cup Y_2} \text{ has coeff. 1 for } z_1 \text{ and } z_2 \big\} \\ &= \big\{ 2x_1 + \sqrt{3}(\beta - z_1 - z_2) \mid \beta \in \bigcup_{J \subseteq Y_2} \mathcal{E}_{\{t_1,t_2\} \cup Y_2}^{\{t_1,t_2\} \cup J} \big\}. \end{aligned}$$

Proof. We show first that if α is an elementary root in Φ_Y preceded by $x_1 + c_m y_1$ with coefficient for x_1 greater than 1, then α is a successor of $(c_m^2 - 1)x_1 + c_m y_1$ and $I(\alpha) \subseteq \{r_1\} \cup Y_2$.

Let β be of maximal depth with $\alpha \succeq \beta \succeq x_1 + c_m y_1$ such that the coefficient of x_1 in β equals 1. Then $\alpha \succeq r_1 \cdot \beta \succ \beta$ by maximality of β; hence $\langle \beta, y_1 \rangle < 0$, and thus $\langle \beta, x_1 \rangle \in (-1, 0)$ by (3.38). Since the coefficient of x_1 in β equals 1, (6.57) together with (6.93) imply that the coefficient of y_1 in β equals kc_m for some $k \in \mathbb{N}$. Now suppose that $r \in Y_1$ is adjoined to r_1, and denote the coefficient of α_r in β by λ; then $\lambda = 0$ or $\lambda \geq 1$ by (2.26). Further

$$\langle \beta, x_1 \rangle \leq 1 + \langle \alpha_r, x_1 \rangle \lambda + \Big(-\frac{c_m}{2}\Big)\mu \leq 1 - \frac{\lambda}{2} - k\frac{c_m^2}{2} \leq 1 - \frac{\lambda}{2} - \frac{3k}{2}$$

as $c_m \geq c_6 = \sqrt{3}$; since $\langle \beta, x_1 \rangle > -1$, we deduce that $k = 1$ and $\lambda = 0$. So the coefficient of y_1 in $r_1 \cdot \beta$ equals c_m, and r_1 is only adjoined to s_1 in $I(\beta)$; therefore $r_1 \cdot \beta \succeq (c_m^2 - 1)x_1 + c_m y_1$ by (6.58), and transitivity of $\succeq$ yields that

$$\alpha \succeq (c_m^2 - 1)x_1 + c_m y_1.$$

Note that connectedness of the support of β implies that $I(\beta) \subseteq \{r_1\} \cup Y_2$, and thus $I(r_1 \cdot \beta) \subseteq \{r_1\} \cup Y_2$. If $r \in Y_1$ is adjoined to r, then

$$\langle r_1 \cdot \beta, \alpha_r \rangle = 0 + \Big(-\frac{1}{2}\Big)(c_m^2 - 1) \leq -1,$$

and since $\alpha \succeq r_1 \cdot \beta$ and $\alpha \in \mathcal{E}$, Lemma (3.38) yields that $\alpha_r \notin \operatorname{supp}(\alpha)$; by the connectedness of the support of α we deduce that $I(\alpha)$ is a subset of $\{r_1\} \cup Y_2$.

Now let $\alpha \in \mathcal{E}_Y^X$. Since $I(\alpha) \not\subseteq Y_1 \cup \{s_1\}$, the above yields that α cannot be preceded by $c_m x_1 + y_1$. So α is preceded by $c_m x_1 + y_1$, and thus by $(c_m^2 - 1)x_1 + c_m y_1$; moreover, $I(\alpha) \subseteq \{r_1\} \cup Y_2$. Hence $\mathcal{E}_Y^X = \emptyset$ unless $Y_1 = \{r_1\}$. By (6.56), the coefficient of α_s in α is greater than or equal to c_m for $s \in Y_2$, and thus $X = \emptyset$. This leaves us to determine $\mathcal{E}^{\emptyset}_{\{r_1\} \cup Y_2}$.

Suppose first that $m \geq 7$ and $Y_1 = \{r_1\}$. Let α be an element of $\mathcal{E}_Y^{\emptyset}$; then $\alpha \succeq (c_m^2 - 1)x_1 + c_m y_1$ by the above. We now show that the coefficient of y_1 in α equals c_m. Let γ be of maximal depth with $\alpha \succeq \gamma \succeq (c_m^2 - 1)x_1 + c_m y_1$ such that γ has coefficient c_m for y_1; then $\alpha = \gamma$ or $\alpha \succeq s_1 \cdot \gamma \succ \gamma$ by maximality of γ.

Assume for a contradiction that $I(\gamma) \subseteq \{r_1, s_1\}$. It follows that γ equals $(c_m^2 - 1)x_1 + c_m y_1$, and since $Y \neq \{r_1, s_1\}$ clearly $\alpha \neq \gamma$; therefore

$$\alpha \succeq s_1 \cdot \gamma = (c_m^2 - 1)x_1 + c_m(c_m^2 - 2)y_1.$$

But since $c_m(c_m^2 - 2) \geq c_7(c_7^2 - 1) \geq 2$, we deduce that $\langle s_1 \cdot \gamma, \alpha_s \rangle \leq -1$ for $s \in Y_2$ adjacent to s_1. So by (3.38), the coefficients of α_s in $s_1 \cdot \gamma$ and α coincide, and by connectedness of the support of α we find that $I(\alpha)$ is contained in $\{r_1, s_1\}$, contradicting $|Y_2| \geq 2$. Thus $I(\gamma) \not\subseteq \{r_1, s_1\}$, and by connectedness of the support of γ there exists an $s \in I(\gamma) \setminus \{r_1, s_1\}$ adjacent to s_1. Denote the coefficient of α_s in γ by μ. Then $\mu \geq c_m$ by (6.56), and thus

$$\langle \gamma, y_1 \rangle \leq c_m + \Big(-\frac{c_m}{2}\Big)(c_m^2 - 1) + \Big(-\frac{1}{2}\Big)c_m = \frac{c_m}{2}(2 - c_m^2) \leq \frac{c_7}{2}(2 - c_7^2) \leq -1.$$

So Lemma (3.38) implies again that the coefficients of y_1 in γ and α coincide; that is, the coefficient of y_1 in α equals c_m.

Note that since the coefficient of y_1 in α equals c_m, and α is a successor of $(c_m^2 - 1)x_1 + c_m y_1$, we can deduce that the coefficient of x_1 in α is $c_m^2 - 1$. Then $r_1 \cdot \alpha$ is an element of $\mathcal{E}_Y^{\{r_1\}}$, and (6.94) yields that $r_1 \cdot \alpha = x_1 + c_m(\beta' - \alpha_t)$ for some $\beta' \in \mathcal{E}_{Y_2 \cup \{t\}}$ with coefficient 1 for α_t (where $t \in R \setminus Y$ according to (6.92)). Since the coefficient of y_1 in $r_1 \cdot \alpha$ equals c_m, we also know that the coefficient of y_1 in β' has to be equal to 1, and as t is only adjoined to s_1 in Y, we deduce that $t \cdot \beta' = \beta' - \alpha_t$. It is clear that $t \cdot \beta'$ is an element of $\mathcal{E}_{Y_2}$ with coefficient 1 for y_1; moreover, $r_1 \cdot \alpha = x_1 + c_m(t \cdot \beta')$, and we conclude that α equals $(c_m^2 - 1)x_1 + c_m(t \cdot \beta')$, as required.

For the converse, let $\beta \in \mathcal{E}_{Y_2}$ with coefficient 1 for y_1. Since t is only adjoined to s_1 (and, moreover, s_1 and t are adjoined by a simple bond), we find that $\langle \beta, \alpha_t \rangle = -\frac{1}{2}$. Now (3.37) implies that $t \cdot \beta = \beta + \alpha_t$ is in $\mathcal{E}_{Y_2 \cup \{t\}}$, and thus

$$x_1 + c_m\beta = x_1 + c_m(t \cdot \beta - \alpha_t)$$

is an elementary root by (6.94). Since $\langle x_1 + c_m\beta, x_1 \rangle = 1 + (-\frac{c_m}{2}) \in (-1, 0)$, (3.37) yields further that

$$r_1 \cdot (x_1 + c_m\beta) = (c_m^2 - 1)x_1 + c_m\beta$$

is also elementary. The coefficients of this root certainly satisfy the required conditions, and thus $(c_m^2 - 1)x_1 + c_m\beta \in \mathcal{E}_Y^{\emptyset}$; this finishes the proof for the case $m = 7$.

Suppose now that $m = 6$; then $c_6 = \sqrt{3}$, and an easy induction yields that for each root in Φ_Y preceded by x_1, the coefficient of α_r in this root is an integer if $r \in Y_1$, and an integer multiple of $\sqrt{3}$ if $r \in Y_2$. Now let $\alpha \in \mathcal{E}_Y^{\emptyset}$; then $\alpha \succeq (c_6^2 - 1)x_1 + c_6 y_1 = 2x_1 + \sqrt{3}y_1$ by the above.

Assume for a contradiction that the coefficient of x_1 in α is greater than 2, and let β be of maximal depth with $\alpha \succeq \beta \succeq 2x_1 + \sqrt{3}y_1$ such that the coefficient of x_1 in β equals 2. Then $\alpha \succeq r_1 \cdot \beta \succ \beta$ by maximality, and (3.38) gives $\langle \beta, x_1 \rangle \in (-1, 0)$. But the coefficient of y_1 in β equals $k\sqrt{3}$ for some $k \in \mathbb{N}$, and r_1 is only adjacent to s_1 in $I(\beta) \setminus \{r_1\}$, and hence

$$\langle \beta, x_1 \rangle = 2 + k\sqrt{3}\big(-\frac{\sqrt{3}}{2}\big) = 2 - \frac{3k}{2} \begin{cases} = \frac{1}{2} & \text{if } k = 1, \\ \leq -1 & \text{if } k \geq 2, \end{cases}$$

contradicting $\langle \beta, x_1 \rangle \in (-1, 0)$. Hence the coefficient of x_1 in α equals 2.

Now observe that if $s \in Y_2$, then

$$\begin{aligned} \langle 2x_1 - \sqrt{3}(z_1 + z_2), \alpha_s \rangle &= 2\langle x_1, \alpha_s \rangle - \sqrt{3}\langle z_1 + z_2, \alpha_s \rangle \\ &= \begin{cases} 2 \times 0 - \sqrt{3} \times 0 & \text{if } s \in Y_2 \setminus \{s_1\}, \\ -\sqrt{3} - \sqrt{3}(-\frac{1}{2} - \frac{1}{2}) & \text{if } s = s_1, \end{cases} \\ &= 0. \end{aligned}$$

Then for $\gamma \in \Phi$ and $\delta \in \Phi_{Y_2}$,

$$\langle 2x_1 + \sqrt{3}\big(\gamma - (z_1 + z_2)\big)\,,\, \delta \rangle = \sqrt{3}\langle \gamma, \delta \rangle; \qquad (*)$$

moreover, for $s \in Y_2$,

$$\begin{aligned} s \cdot \big(2x_1 &+ \sqrt{3}(\gamma - z_1 - z_2)\big) \\ &= 2x_1 + \sqrt{3}(\gamma - z_1 - z_2) - 2\langle 2x_1 + \sqrt{3}(\gamma - z_1 - z_2)\,,\, \alpha_s \rangle \\ &= 2x_1 + \sqrt{3}(\gamma - z_1 - z_2) - 2\sqrt{3}\langle \gamma, \alpha_s \rangle \qquad (**) \\ &= 2x_1 + \sqrt{3}(\gamma - 2\langle \gamma, \alpha_s \rangle - z_1 - z_2) \\ &= 2x_1 + \sqrt{3}(s \cdot \gamma - z_1 - z_2). \end{aligned}$$

Recall now that $\alpha \succeq 2x_1 + \sqrt{3}y_1$, and the coefficient of x_1 in α equals 2 for $\alpha \in \mathcal{E}_Y^{\emptyset}$. Hence there exists a $w \in W_{Y_2}$ such that $\alpha = w \cdot (2x_1 + \sqrt{3}y_1)$. So

$$\alpha = w \cdot \big(2x_1 + \sqrt{3}((z_1 + z_2 + y_1) - z_1 - z_2)\big),$$

and an easy induction on $l(w)$ using $(**)$ yields that α equals

$$2x_1 + \sqrt{3}\big(w \cdot (z_1 + z_2 + y_1) - z_1 - z_2\big) = 2x_1 + \sqrt{3}\big((wt_1t_2) \cdot y_1 - z_1 - z_2\big).$$

Set $\gamma = (wt_1t_2) \cdot y_1$; then clearly $I(\gamma) = \{t_1, t_2\} \cup Y_2$. Assume for a contradiction that γ is not an elementary root, and let $\delta \in \Phi^+ \setminus \{\gamma\}$ be dominated by γ. Then $\langle \gamma, \delta \rangle \geq 1$ by (3.32). Further, $w^{-1} \cdot \delta \in \Phi^-$ since $z_1 + z_2 + y_1$ is elementary, and hence $\delta \in \Phi_{Y_2}$. Now $(*)$ yields that

$$\langle \alpha, \delta \rangle = \sqrt{3}\langle \gamma, \delta \rangle \geq \sqrt{3} \geq 1,$$

and thus $\alpha \;\mathsf{dom}\; \delta$ or $\alpha \;\mathsf{dom}\; \alpha$ by (3.32). Since $w^{-1} \cdot \delta$ is negative, and $w^{-1} \cdot \alpha$ is not, this forces $\alpha \in \Delta$, a contradiction.

Next, let $\beta \in \mathcal{E}_{Y_2 \cup \{t_1, t_2\}}$ with coefficient 1 for z_1 and z_2, and define

$$\alpha = 2x_1 + \sqrt{3}(\beta - z_1 - z_2);$$

we show first that α is in fact a root. Since $I(\beta) = Y_2 \cup \{t_1,\, t_2\}$ contains only simple bonds, Proposition (6.69) yields that β is a successor of $z_1 + z_2 + y_1$; hence $\beta = w \cdot (z_1 + z_2 + y_1)$ with $\mathrm{dp}(\beta) - \mathrm{dp}(z_1 + z_2 + y_1) = l(w)$ for some $w \in W$. The coefficients of z_1 and z_2 in β and $z_1 + z_2 + y_1$ coincide, and since $I(\beta) = Y_2 \cup \{t_1, t_2\}$ we know that $w \in W_{Y_2}$. Now a straightforward induction on $l(w)$ using $(**)$ yields that

$$\begin{aligned}(wr_1s_1) \cdot x_1 &= w \cdot \big(2x_1 - \sqrt{3}(z_1 + z_2) + \sqrt{3}(y_1 + z_1 + z_2)\big)\\ &= 2x_1 + \sqrt{3}\big(w \cdot (z_1 + z_2 + y_1) - z_1 - z_2\big)\\ &= 2x_1 + \sqrt{3}(\beta - z_1 - z_2);\end{aligned}$$

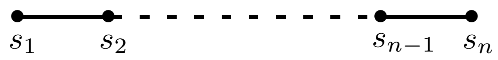
therefore α is in fact a root.

Assume for a contradiction that α dominates some $\delta \in \Phi^+ \setminus \{\alpha\}$. Then $\delta \in N(w^{-1})$, since $(r_1s_1) \cdot x_1 \notin \Delta$, and thus $\delta \in \Phi^+_{Y_2}$. Hence by $(*)$ and (3.32),

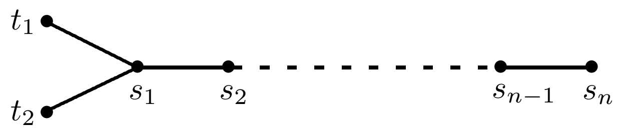
$$\langle \beta, \delta \rangle = \frac{1}{\sqrt{3}}\langle \alpha, \delta \rangle \geq \frac{1}{\sqrt{3}}.$$

Since $I(\beta) \cup I(\delta) = Y_2 \cup \{t_1, t_2\}$ contains only simple bonds, this forces $\langle \beta, \delta \rangle \geq 1$. As $w^{-1} \cdot \delta$ is negative while $w^{-1} \cdot \beta = \gamma$ is positive, δ cannot dominate β, and thus $\beta \in \Delta$, contrary to our choice of β. $\square$

It remains to discuss the cases $m = 4$, 5; before we do so, consider the following consequence of the previous result for $m = 6$: If Y_2 equals



then $\{t_1, t_2\} \cup Y_2$ equals



and if we denote the simple root corresponding to s_j by y_j, we can deduce from (6.77) that the set of roots in $\mathcal{E}_{\{t_1,t_2\}\cup Y_2}$ with coefficient 1 for z_1 and z_2 is

$$\big\{z_1 + z_2 + 2(y_1 + \ldots + y_{j-1}) + y_j + \cdots + y_n \mid j \in \{1, \ldots, n\}\big\};$$

therefore

$$\mathcal{E}_Y^{\emptyset} = \big\{2x_1 + \sqrt{3}\big(2(y_1 + \ldots + y_{j-1}) + y_j + \cdots + y_n\big) \mid j \in \{1, \ldots, n\}\big\},$$

and thus $\big|\mathcal{E}_Y^{\emptyset}\big| = n$.

Next, suppose that Y_2 contains a vertex of valency greater than 2, and let $n \geq 1$ be maximal such that there exist $s_2, \ldots, s_n \in Y_2$ with s_j adjacent only to s_{j-1} and s_{j+1} in Y for $j \in \{1, \ldots, n-1\}$ (where $s_0 = r_1$). Denote the simple root corresponding to s_j by y_j, and let α be an element of $\mathcal{E}_{\{t_1,t_2\}\cup Y_2}$ with coefficient 1 for z_1 and z_2. Since the support of α contains at least two vertices of valency greater than or equal to 3 (if $n > 1$, namely s_1 and s_n) or at least one vertex of valency greater than or equal to 4 (if $n = 1$, namely s_1), (6.65) implies that there exists a $j \in \{1, \ldots, n\}$ such that the coefficient of y_j in α is 1. If we choose j minimal with this property, then (6.57) yields that α equals $\beta + \gamma - y_j$ for some $\beta \in \mathcal{E}_{\{t_1,t_2,s_1,\ldots,s_j\}}$ with coefficient 1 for y_j and coefficient greater than or equal to 2 for $y_1, \ldots, y_{j-1}$, and $\gamma \in \mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j. Since the coefficients of z_1 and z_2 in α equal 1, this yields that $\beta \in \mathcal{E}^{\{t_1,t_2,s_j\}}_{\{t_1,t_2,s_1,\ldots,s_j\}}$; so by (6.77),

$$\beta = z_1 + z_2 + 2(y_1 + \cdots + y_{j-1}) + y_j,$$

and thus every element of $\mathcal{E}_Y^{\emptyset}$ can be written as $2x_1+\sqrt{3}\bigl(2(y_1+\cdots+y_{j-1})+\gamma\bigr)$ for some $\gamma \in \mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j.

Since $z_1 + z_2 + 2(y_1 + \cdots + y_{j-1}) + y_j$ is elementary for $j \in \{1,\ldots,n\}$, it also follows from (6.57) that

$$2x_1 + \sqrt{3}\bigl(2(y_1 + \ldots + y_{j-1}) + \gamma\bigr)$$

is in $\mathcal{E}_Y^{\emptyset}$ for all $\gamma \in \mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j; therefore $\mathcal{E}_Y^{\emptyset}$ is the set

$$\bigcup_{j=1}^{n}\Bigl\{2x_1 + \sqrt{3}\bigl(2(y_1 + \ldots + y_{j-1}) + \gamma\bigr) \mid \gamma \in \bigcup_{J\subseteq Y_2\setminus\{s_1,\ldots,s_j\}} \mathcal{E}^{\{s_j\}\cup J}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}\Bigr\}.$$

Note that by (6.66), $\mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}} = \{y_j + \cdots + y_n\}$ if $|Y_2| = n$; hence the above also applies for the case $|Y_2| = n$.

If $m = 4, 5$ we can use similar arguments to the ones just demonstrated for the case $m = 6$. To do so, we need to develop some more tools. We start with the following variation of (6.57), which is clearly valid for all m.

(6.97) Proposition *Let $r \in Y$ and $J_0,\ldots,J_k \subseteq Y$ such that $Y \setminus \{r\}$ is the disjoint union of $J_0,\ldots,J_k$. Suppose that no element of J_i is adjoined to any element J_j for $i \neq j$, and set $I_j = J_j \cup \{r\}$ for all $j \in \{0,\ldots,k\}$. Assume further that $r_1, s_1 \in I_0$. Then*

$$\phi\colon (\beta_0,\ldots,\beta_k) \mapsto \beta_0 + c_m(\beta_1 + \cdots + \beta_k - k\alpha_r)$$

is a one-one correspondence between the set of $(k+1)$-tuples in $\Phi^+_{I_0} \times\cdots\times \Phi^+_{I_k}$ such that the coefficient of α_r in the first component equals c_m, while for all other components the coefficient of α_r equals 1, and the set of roots in Φ^+_Y with coefficient c_m for α_r. Moreover, ϕ restricts to a one-one correspondence between the set of $(k+1)$-tuples in $\mathcal{E}_{I_0} \times\cdots\times \mathcal{E}_{I_k}$ such that α_r has coefficient c_m in the first component, and 1 in the others, and the set of roots in $\mathcal{E}_Y$ with coefficient c_m for α_r.

Proof. We show first that ϕ is well defined. So let $(\beta_0,\ldots,\beta_k)$ be an element of $\Phi^+_{I_0} \times \cdots \times \Phi^+_{I_k}$ such that the coefficient of α_r in β_0 equals c_m, while the coefficient of α_r in β_j equals 1 for $j \in \{1,\ldots,k\}$. Lemma (6.59) implies that $\beta_j \succeq \alpha_r$ for $j \in \{1, \ldots, k\}$, and hence there exist $w_j \in W_{J_j}$ such that $\beta_j = w_j \cdot \alpha_r$ and $l(w_j) = \mathrm{dp}(\beta_j) - 1$. Define α to be $(w_1 \cdots w_k) \cdot \beta_0$; then the coefficient of α_r in α equals c_m, and it can be easily seen that α equals $\phi(\beta_0, \ldots, \beta_k)$.

As $I(\beta_i) \cap I(\beta_j) = \{r\}$ if $i \neq j$, and the coefficient of α_r in β_i is c_m if $i = 0$, and 1 if $i \in \{1, \ldots, k\}$, it follows that ϕ is one-one, and we show now that ϕ is onto.

Suppose that $\alpha \in \Phi_Y^+$ has coefficient c_m for α_r, and let $\gamma \preceq \alpha$ be of minimal depth such that the coefficient of α_r in α equals c_m. Then $I(\gamma) \subseteq Y$, and as $Y \setminus J_0$ contains only simple bonds, we deduce that $I(\gamma) \cap J_0 \neq \emptyset$. Now $r \cdot \gamma \prec \gamma$ by minimality of γ, and it follows by (2.26) that the coefficient of α_r in $r \cdot \gamma$ equals 0 or 1. In the first case, connectedness of the support of $r \cdot \gamma$ yields that $I(r \cdot \gamma) \subseteq J_0$, and thus $I(\gamma) \subseteq I_0$.

Assume now that the coefficient of α_r in $r \cdot \gamma$ equals 1, and let $K_1, \ldots, K_n$ be the connected components of $I(r \cdot \gamma) \setminus \{r\}$. Assume for a contradiction that $n \geq 2$. By (6.57), there exist roots $\gamma_1, \ldots, \gamma_n$ with $I(\gamma_i) \subseteq K_i$ such that $r \cdot \gamma = \gamma_0 + \cdots + \gamma_n - (n-1)\alpha_r$. For $i \in \{1, \ldots, n\}$, let $t_i \in K_i$ be adjoined to r, and denote the simple root corresponding to t_i by z_i, and the coefficient of z_i in γ_i by ν_i; then $\nu_i \geq 1$ by (2.26), as $t_i \in I(r \cdot \gamma)$. The coefficient of α_r in $r \cdot \gamma$ is

$$1 = c_m - 2\langle \gamma, \alpha_r \rangle = -c_m - 2\langle z_1, \alpha_r \rangle \nu_1 - \cdots - 2\langle z_k, \alpha_r \rangle \nu_k,$$

and we find that $1 \geq -c_m - (\nu_1 + \cdots + \nu_n) = n - c_m$. This forces $n \leq 2$, and thus $n = 2$ by our assumption. Further, $\nu_1 + \nu_2 \leq 1 + c_m$, and by symmetry of K_1 and K_2 we may assume without loss of generality that $\nu_1 \leq \nu_2$. Since ν_i equals 1, or is greater than or equal to c_m by (2.26), we deduce that $\nu_1 = 1$ and $\nu_2 \in \{1, c_m\}$. Since the coefficients of both z_1 and α_r in $r \cdot \gamma$ equal 1, we deduce from (6.56) that $\langle \alpha_r, z_2 \rangle = -\frac{1}{2}$. So the coefficient of α_r in γ equals

$$c_m = 1 - 2(1 + \Big(-\frac{1}{2}\Big)1 + \langle \alpha_r, z_2 \rangle \nu_2) = -2\langle \alpha_r, z_2 \rangle \nu_2.$$

If $\nu_2 = 1$, this forces $\langle \alpha_r, z_2 \rangle = -\frac{c_m}{2}$; but then the coefficients of z_2 and α_r in $r \cdot \gamma$ cannot both be 1 by (6.56) together with (2.26), contrary to our construction. If $\nu_2 = c_m$, the above yields that $\langle \alpha, z_2 \rangle = -\frac{1}{2}$; but then the coefficient of α_r in $r \cdot \gamma_2$ equals $c_m - 1 \in (0, 1)$, and this contradicts (2.26). So $I(r \cdot \gamma)$ has only one connected component, and since $I(r \cdot \gamma) \cap J_0 \neq \emptyset$, we deduce that $I(r \cdot \gamma) \subseteq I_0$, and hence $I(\gamma) \subseteq I_0$.

Since γ has coefficient c_m for α_r, there exists a $w \in W_{Y\setminus\{r\}}$ such that $\alpha = w \cdot \gamma$ and $\mathrm{dp}(\alpha) - \mathrm{dp}(\gamma) = l(w)$. As $W_{Y\setminus\{r\}}$ is the direct product of $W_{J_0}, \ldots, W_{J_k}$, there exist $w_j \in W_{I_j}$ for all $j \in \{0, \ldots, k\}$ such that w equals $w_0 \cdots w_k$ with length adding. Define $\beta = w_0 \cdot \gamma$; this is an element of $\Phi^+_{I_0}$ with coefficient c_m for α_r, and clearly

$$\alpha = (w_1 \cdots w_k) \cdot \beta = \phi(\beta, w_1 \cdot \alpha_r, \ldots, w_k \cdot \alpha_r).$$

The above proves that ϕ is onto, and it remains to show for $\beta \in \Phi^+_{I_0}$ and $w_1 \in W_{I_1}, \ldots, w_k \in W_{I_k}$ that $(w_1 \cdots w_k) \cdot \beta \in \Delta$ if and only if $\beta \in \Delta$ or $w_j \cdot \alpha_r \in \Delta$ for some $j \in \{1, \ldots, k\}$.

Set $w = w_1 \cdots w_k$, and note that for $\delta \in \Phi^+_{J_1}$ clearly $\langle \delta, \alpha_s \rangle = 0$ for all $s \in Y \setminus I_1$; therefore $\langle \beta, \delta \rangle = c_m \langle \alpha_r, \delta \rangle$ and $\langle w_i \cdot \alpha_r, \delta \rangle = \langle \alpha_r, \delta \rangle$ for all i in $\{2, \ldots, k\}$. This implies that

$$\begin{aligned} \langle w \cdot \beta, \delta \rangle &= \langle \beta, \delta \rangle + c_m \langle w_1 \cdot \alpha_r, \delta \rangle + c_m \sum_{i=2}^{k} \langle w_i \cdot \alpha_r, \delta \rangle - k c_m \langle \alpha_r, \delta \rangle \\ &= c_m \langle \alpha_r, \delta \rangle + c_m \langle w_1 \cdot \alpha_r, \delta \rangle + c_m \sum_{i=2}^{k} \langle \alpha_r, \delta \rangle - k c_m \langle \alpha_r, \delta \rangle \\ &= c_m \langle w_1 \cdot \alpha_r, \delta \rangle. \end{aligned} \tag{$*$}$$

Suppose now that $w \cdot \beta \in \Delta$, and let $\delta \in \Phi^+ \setminus \{w \cdot \beta\}$ such that $w \cdot \beta \;\mathsf{dom}\; \delta$. If $w^{-1} \cdot \delta$ is positive, it follows that $\beta \;\mathsf{dom}\; (w^{-1} \cdot \delta)$, and since clearly $\beta \neq w^{-1} \cdot \delta$, we find that $\beta \in \Delta$. Assume next that $\delta \in N(w^{-1})$. An easy calculation yields that

$$N(w^{-1}) = N\big((w_1 \cdots w_k)^{-1}\big) = N(w_1^{-1}) \cup \ldots \cup N(w_k^{-1}),$$

and by symmetry we may assume without loss of generality that $\delta \in N(w_1^{-1})$; then in particular, $I(\delta) \subseteq J_1$. Now (3.32) implies that $\langle w \cdot \beta, \delta \rangle \geq 1$, and thus

$$\langle w_1 \cdot \alpha_r, \delta \rangle \geq \frac{1}{c_m} > \frac{1}{2}$$

by $(*)$; since $I(w_1 \cdot \alpha_r) \cup I(\delta)$ contains only simple bonds, $\langle w_1 \cdot \alpha_r, \delta \rangle$ is an integer multiple of $\frac{1}{2}$, and hence $\langle w_1 \cdot \alpha_r, \delta \rangle \geq 1$. So $w_1 \cdot \alpha_r \;\mathsf{dom}\; \delta$ or

δ dom $w_1 \cdot \alpha_r$ by (3.32). But $w_1^{-1} \cdot \delta$ is negative, and $w_1 \cdot \alpha_r$ is not, and so δ cannot dominate $w_1 \cdot \alpha_r$; this forces $w_1 \cdot \alpha_r \in \Delta$, as required.

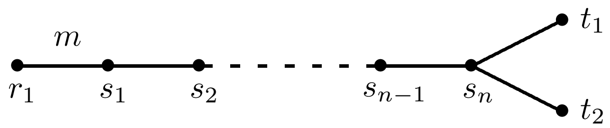
For the converse, suppose first that $\beta \in \Delta$. Since $N(w) \subseteq \Phi^+_{Y \setminus I_0}$, while $\beta \in \Phi^+_{I_0}$, it follows easily that $N_+(w, \beta) = \emptyset$; so $\alpha \succeq \beta$ by (3.34), and thus $\alpha \in \Delta$ by (3.36).

Now assume that $w_1 \cdot \alpha_r \in \Delta$, and let $\delta \in \Phi^+ \setminus \{w_1 \cdot \alpha_r\}$ such that $w_1 \cdot \alpha_r$ dom δ. Then $w_1^{-1} \cdot \delta \in \Phi^-$ since $\alpha_r \notin \Delta$, and thus $\delta \in \Phi^+_{J_1}$. Further, $\langle w_1 \cdot \alpha_r, \delta \rangle \geq 1$ as $w_1 \cdot \alpha_r$ dom δ, and $(*)$ yields that

$$\langle w \cdot \beta, \delta \rangle = c_m \langle w_1 \cdot \alpha_r, \delta \rangle \geq c_m \geq 1.$$

Since $w_1^{-1} \cdot \delta$ is negative and $w^{-1} \cdot (w \cdot \beta)$ is not, (3.32) this yields that $w \cdot \beta \in \Delta$, as required. Symmetrical arguments apply if $w_j \cdot \beta \in \Delta$ for $j \in \{2, \ldots, k\}$, and this finishes the proof. □

(6.98) Lemma *Suppose Y contains the following subgraph*



and denote the simple roots corresponding to s_j, t_k by x, y_j and z_k respectively. Let α be a root in Φ^+_Y with coefficient for x_1 greater than 1, coefficients for $y_1, \ldots, y_n$ greater than or equal to 2 and z_1, $z_2 \in \operatorname{supp}(\alpha)$. Then $\alpha \in \Delta$.

Proof. Let $\beta \preceq \alpha$ be a positive root of minimal depth such that z_1 and z_2 are in the support of β, the coefficient of x_1 in β is greater than 1, and the coefficients of $y_1, \ldots, y_n$ in β are greater than or equal to 2. By (3.36) it suffices to show that β is in Δ. Let $s \in R$ such that $s \cdot \beta \prec \beta$; we show that $\langle s \cdot \beta, \alpha_s \rangle \leq -1$, which then implies $\langle \beta, \alpha_s \rangle \geq 1$, and thus $\beta \in \Delta$ by (3.32); (since β is clearly of depth greater than $\operatorname{dp}(\alpha_s) = 1$).

Denote the coefficients of x_1, y_j, z_k in β by λ, μ_j and ν_k respectively. By minimality of β it follows that s equals r_1, s_j or t_k. If $s = r_1$, minimality of β also implies that the coefficient of x_1 in $r \cdot \beta$ is less than or equal to 1, and thus equals 0 or 1 by (2.26). If the coefficient of x_1 in $r \cdot \beta$ equals 0, then $\langle r_1 \cdot \beta, x_1 \rangle \leq 0 + (-\frac{c_m}{2})\mu_1 \leq -1$ since $\mu_1 \geq 2$, as required. Suppose next that the coefficient of x_1 in $r_1 \cdot \alpha$ equals 1. Then μ_1 is an integer multiple of c_m by (6.57) together with (6.93), and thus $\mu_1 \geq 2c_m$; hence again

$$\langle r_1 \cdot \beta, x_1 \rangle \leq 1 + (-\frac{c_m}{2})\mu_1 \leq 1 - c_m^2 \leq 1 - c_4^2 = 1 - \sqrt{2}^2 = -1.$$

Assume now that $s = s_j$ for some $j \in \{1, \ldots, n\}$, and denote the coefficient of y_j in $t_j \cdot \alpha$ by μ'_j. By minimality of β clearly $\mu'_j < 2$, and by the connectedness of the support of $t_j \cdot \beta$ further $\mu'_j > 0$; thus $\mu'_j \in \{1, c_m\}$ by (2.26). Note that $\mu_{j-1} \geq 2$ if $j > 1$, and $\lambda \geq c_m$ if $j = 1$. Further, $\mu_{j+1} \geq 2$ if $j < n$, and ν_1, $\nu_2 \geq 1$ if $j = n$. So if $\mu'_j = 1$, then

$$\langle s_j \cdot \beta, y_j \rangle \leq \begin{cases} 1 + (-\frac{c_m}{2})c_m + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } 1 = j = n, \\ 1 + (-\frac{c_m}{2})c_m + (-\frac{1}{2})2 & \text{if } 1 = j < n, \\ 1 + (-\frac{1}{2})2 + (-\frac{1}{2})2 & \text{if } 1 < j < n, \\ 1 + (-\frac{1}{2})2 + (-\frac{1}{2})1 + (-\frac{1}{2})1 & \text{if } 1 < j = n, \end{cases}$$

and thus $\langle s_j \cdot \beta, y_j \rangle \leq -1$ in any case, as required. Assume now that $\mu'_j = c_m$. If $j < n$, then μ_{j+1} is an integer multiple of c_m by (6.97) for $r = s_j$, and since $\mu_{j+1} \geq 2$ we know that $\mu_{j+1} \geq 2c_m$; if $j = n$, then ν_1, $\nu_2 \geq c_m$ by (6.97) for $r = s_j$. Therefore

$$\langle s_j \cdot \beta, y_j \rangle \leq \begin{cases} c_m + (-\frac{c_m}{2})c_m + (-\frac{1}{2})c_m + (-\frac{1}{2})c_m & \text{if } 1 = j = n, \\ c_m + (-\frac{c_m}{2})c_m + (-\frac{1}{2})2c_m & \text{if } 1 = j < n, \\ c_m + (-\frac{1}{2})2 + (-\frac{1}{2})2c_m & \text{if } 1 < j < n, \\ c_m + (-\frac{1}{2})2 + (-\frac{1}{2})c_m + (-\frac{1}{2})c_m & \text{if } 1 < j = n, \end{cases}$$

and thus $\langle s_j \cdot \beta, y_j \rangle \leq -1$, as required.

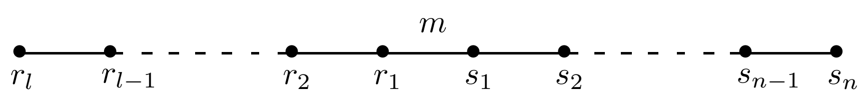
If $s = t_1$, then $z_1 \notin \operatorname{supp}(t_1 \cdot \beta)$ by minimality of β, and

$$\langle t_1 \cdot \beta, z_1 \rangle \leq 0 + (-\frac{1}{2})\mu_n \leq -1$$

since $\mu_n \geq 2$; symmetrical arguments apply if s equals t_2, and this finishes the proof.

□

Now let $l \geq 1$ be maximal such that there exist $r_2, \dots, r_l \in Y$ with r_i adjacent only to r_{i-1} and r_{i+1} in Y for $i \in \{1, \dots, l-1\}$ (where $r_0 = s_1$), and denote $\{r_l, \dots, r_1\}$ by Y_1'. Then either $Y_1 = Y_1'$, or r_l has valency greater than or equal to 3. Similarly, let $n \geq 1$ be maximal such that there exist $s_2, \dots, s_n \in Y_2$ with s_j adjacent only to s_{j-1} and s_{j+1} in Y for $j = 1, \dots, n-1$ (where $s_0 = r_1$), and define $Y_2' = \{s_1, \dots, s_n\}$ and $Y' = Y_1' \cup Y_2'$; then Y' equals



We denote the simple roots corresponding to r_i, s_j by x_i and y_j respectively.

The following result enables us to restrict our main focus to the case $|Y_1| = l$ and $|Y_2| = n$.

(6.99) Lemma *Suppose $r_1 \notin X$ and $|Y_2| > n$ (that is, Y_2 contains a vertex of valency greater than 2). Then $\mathcal{E}_Y^X$ is empty unless $X \subseteq Y_1$. Moreover, if $X \subseteq Y_1$, then $\mathcal{E}_Y^X$ is the set of*

$$\alpha + c_5\beta - c_m y_j$$

with $j \in \{1, \dots, n\}$, $\alpha \in \mathcal{E}^X_{Y_1 \cup \{s_1, \dots, s_{j-1}\}}$ with coefficient c_m for y_j and coefficient greater than or equal to 2 for $y_1, \dots, y_{j-1}$, and $\beta \in \mathcal{E}_{Y_2 \setminus \{s_1, \dots, s_{j-1}\}}$ with coefficient 1 for y_j.

Proof. Let $\gamma \in \mathcal{E}_Y^X$; then (6.98) yields that the coefficients of $y_1, \dots, y_n$ in γ cannot all be greater than or equal to 2, and since X does not contain any of the s_j by (6.61), there must exist a $j \in \{1, \dots, n\}$ such that the coefficient of y_j in γ equals c_m. If we choose j minimal with this property, (6.97) yields that

$$\gamma = \alpha + c_m\beta - c_m y_j$$

for some $\alpha \in \mathcal{E}_{Y_1 \cup \{s_1, \dots, s_{j-1}\}}$ with coefficient c_m for y_j and coefficient greater than or equal to 2 for $y_1, \dots, y_{j-1}$, and $\beta \in \mathcal{E}_{Y_2 \setminus \{s_1, \dots, s_{j-1}\}}$ with coefficient 1 for y_j. Since for $s \in Y_2 \setminus \{s_1, \dots, s_n\}$ the coefficients of α_s in γ are integer multiples of c_m, it follows that $X \subseteq Y_1$. Moreover, it is clear that for $r \in Y_1 \cup \{s_1, \dots, s_{j-1}\}$, the coefficient of α_r in α equals 1 if and only if $r \in X$; that is, $\alpha \in \mathcal{E}^X_{Y_1 \cup \{s_1, \dots, s_{j-1}\}}$.

By (6.97), each root of the form described in the assertion is in $\mathcal{E}_Y^X$ if $X \subseteq Y_1$, and this finishes the proof. □

If $|Y_1| > l$ and $s_1 \notin X$, then symmetrically $\mathcal{E}_Y^X = \emptyset$ unless $X \subseteq Y_2$, and thus $X = \emptyset$; if $|Y_1| = l$ and $X \subseteq Y_1$, then $X \subseteq \{r_l\}$ since each element of X is adjacent to exactly one element of $Y \setminus X$.

(6.100) Proposition *Suppose r_1, $s_1 \notin X$. Then $\mathcal{E}_Y^X$ is empty unless X is empty, or $|Y_1| = l$ and $X = \{r_l\}$, or $|Y_2| = n$ and $\{s_n\}$.*

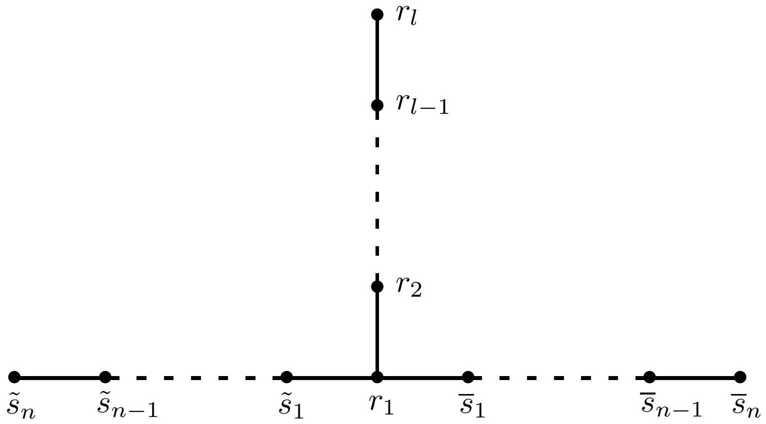
So from now on we only need to determine $\mathcal{E}_Y^{\emptyset}$, $\mathcal{E}_Y^{\{r_l\}}$ (for $|Y_1| = l$), and $\mathcal{E}_Y^{\{s_n\}}$ (for $|Y_2| = n$); by symmetry it suffices to investigate only one of the latter two.

Suppose now that $m = 4$. An easy induction shows for $\alpha \in \Phi_Y^+$ that the coefficient of α_r in α is an integer for all $r \in Y_1$, and an integer multiple of $\sqrt{2}$ for all $r \in Y_2$, or vice versa. This together with (6.99) imply:

(6.101) Proposition *Suppose that $m = 4$. Then $\mathcal{E}_Y^X$ is empty, unless $|Y_1| = l$ and $X \subseteq Y_1$, or $|Y_2| = n$ and $X \subseteq Y_2$.*

If $m = 4$, we assume from now on that $|Y_1| = l$ and $X \subseteq \{r_l\}$. We first determine $\mathcal{E}_Y^X$ for $|Y_2| = n$, and then cope with the case $|Y_2| > n$ using (6.99).

It is clear, that $\mathcal{E}_Y$ is independent of R (as long as $Y \subseteq R$), and so we may assume without loss of generality that R contains $\widetilde{Y}_2 = \{\tilde{s}_1, \ldots, \tilde{s}_n\}$ and $\overline{Y}_2 = \{\overline{s}_1, \ldots, \overline{s}_n\}$ such that $Y_a = Y_1' \cup \widetilde{Y}_2 \cup \overline{Y}_2$ equals



Denote the simple roots corresponding to $\tilde{s}_j$, $\overline{s}_j$ by $\widetilde{y}_j$ and $\overline{y}_j$ respectively, and let $V_{Y'}$ be the subspace of V spanned by $x_l, \ldots, x_1, y_1, \ldots, y_n$, and V_a be the space spanned by $x_l, \ldots, x_1$, $\widetilde{y}_1 + \overline{y}_1, \ldots, \widetilde{y}_n + \overline{y}_n$. Further, define $\phi \colon V_{Y'} \to V_a$ by

$$\phi\Big(\sum_{i=1}^{l} \lambda_i x_i + \sum_{j=1}^{n} \mu_j y_j\Big) = \sum_{i=1}^{l} \lambda_i x_i + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j (\widetilde{y}_j + \overline{y}_j).$$

If $v = \sum_{i=1}^{l} \lambda_i x_i + \sum_{j=1}^{n} \mu_j y_j$ for some λ_i, $\mu_j \in \mathbb{R}$, then

$$\begin{aligned}
\langle \phi(v), x_1 \rangle &= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j \big(\langle \widetilde{y}_j, x_1 \rangle + \langle \overline{y}_j, x_1 \rangle \big) \\
&= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \mu_1 \frac{1}{\sqrt{2}} \big(\langle \widetilde{y}_1, x_1 \rangle \ + \langle \overline{y}_1, x_1 \rangle \big) \\
&= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \mu_1 \Big(-\frac{1}{\sqrt{2}} \Big) \\
&= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \mu_1 \langle y_1, x_1 \rangle \\
&= \sum_{i=1}^{l} \lambda_i \langle x_i, x_1 \rangle + \sum_{j=1}^{n} \mu_j \langle y_j, x_1 \rangle = \langle v, x_1 \rangle,
\end{aligned}$$

while for $i \geq 2$ clearly $\langle \phi(v), x_i \rangle = \langle v, x_i \rangle$. It follows that $\phi(r_i \cdot v) = r_i \cdot \phi(v)$ for all $i \in \{1, \ldots, l\}$. Furthermore,

$$\begin{aligned}
\langle \phi(v), \widetilde{y}_1 \rangle &= \sum_{i=1}^{l} \lambda_i \langle x_i, \widetilde{y}_1 \rangle + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j \big(\langle \widetilde{y}_j, \widetilde{y}_1 \rangle + \langle \overline{y}_j, \widetilde{y}_1 \rangle \big) \\
&= \lambda_1 \langle x_1, \widetilde{y}_1 \rangle + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j \langle \widetilde{y}_j, \widetilde{y}_1 \rangle \\
&= \lambda_1 \Big(-\frac{1}{2} \Big) + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \mu_j \langle y_j, y_1 \rangle \\
&= \frac{1}{\sqrt{2}} \big(\lambda_1 \langle x_1, y_1 \rangle + \sum_{j=1}^{n} \mu_j \langle y_j, y_1 \rangle \big) = \frac{1}{\sqrt{2}} \langle v, y_1 \rangle
\end{aligned}$$

and symmetrically $\langle \phi(v), \overline{y}_1 \rangle = \langle v, y_1 \rangle / \sqrt{2}$; also, for $j \in \{2, \ldots, n\}$ clearly

$$\langle \phi(v), \widetilde{y}_j \rangle = \langle \phi(v), \overline{y}_j \rangle = \frac{1}{\sqrt{2}} \langle v, y_j \rangle.$$

A straightforward calculation now yields that $\phi(s_j \cdot v) = (\tilde{s}_j \overline{s}_j) \cdot \phi(v)$ for $j \in \{1, \ldots, n\}$.

(6.102) Proposition *Let $m = 4$. Then ϕ defines a one-one correspondence between the set of roots in $\mathcal{E}_{Y'} = \mathcal{E}_{\{r_l, \ldots, r_1, s_1, \ldots, s_n\}}$ preceded by x_1, and $\mathcal{E}_{Y_a} \cap V_a$, the set of roots in $\mathcal{E}_{Y_1' \cup \widetilde{Y}_2 \cup \overline{Y}_2}$ with coinciding coefficients for $\widetilde{y}_j$ and $\overline{y}_j$ for all $j \in \{1, \ldots, n\}$.*

Proof. Let $\alpha \in \Phi_{Y'}$ be an elementary root preceded by x_1; then clearly $\phi(\alpha) \in V_a$, and we show now that $\phi(\alpha)$ is an elementary root. If α has depth 1, then $\alpha = x_1$ and $\phi(\alpha) = x_1 \in \mathcal{E}$. Suppose next that α is of depth greater than 1, and assume that $\phi(\beta)$ is an elementary root for all β with $x_1 \preceq \beta \prec \alpha$. Let $r \in Y$ such that $x_1 \preceq r \cdot \alpha \prec \alpha$; then $\phi(r \cdot \alpha)$ is an elementary root by induction. Further, $\langle \alpha, \alpha_r \rangle > 0$, and thus $\langle \alpha, \alpha_r \rangle \in (0, 1)$, since α cannot dominate α_r; that is, $\langle r \cdot \alpha, \alpha_r \rangle \in (-1, 0)$.

If $r = r_i$ for some i, then $\phi(\alpha) = r_i \cdot \phi(r_i \cdot \alpha)$ by the above, and since $\langle \phi(r_i \cdot \alpha), x_i \rangle = \langle r_i \cdot \alpha, x_i \rangle \in (-1, 0)$, it follows by (3.37) that $\phi(\alpha)$ is an elementary root.

Assume next that $r = s_j$ for some j. Then $\phi(\alpha) = (\tilde{s}_j\overline{s}_j) \cdot \phi(s_j \cdot \alpha)$ by the above. Since $\langle \phi(s_j \cdot \alpha), \overline{y}_j \rangle = \langle s_j \cdot \alpha, y_j \rangle / \sqrt{2} \in (-1, 0)$, it follows by (3.37) that $\overline{s}_j \cdot \phi(s_j \cdot \alpha)$ is elementary; furthermore,

$$\langle \overline{s}_j \cdot \phi(s_j \cdot \alpha), \widetilde{y}_j \rangle = \langle \phi(s_j \cdot \alpha), \widetilde{y}_j \rangle = \frac{1}{\sqrt{2}} \langle s_j \cdot \alpha, y_j \rangle \in (-1, 0)$$

and, again by (3.37), we deduce that $(\tilde{s}_j\overline{s}_j) \cdot \phi(s_j \cdot \alpha)$ is elementary; therefore $\phi(\alpha)$ is elementary.

Note that this yields that if $\alpha \in \mathcal{E}_{Y'}$ is preceded by x_1, then $\phi(\alpha)$ is an element of $\mathcal{E}_{Y_a} \cap V_a$.

Now let β be an elementary root in V_a preceded by x_1. If β is of depth 1, then $\beta = x_1 = \phi(x_1)$. Suppose next that $\mathrm{dp}(\beta) > 1$, and assume that every root γ in V_a with $x_1 \preceq \gamma \prec \beta$ equals $\phi(\delta)$ for some elementary root δ in $\Phi_{Y'}$ with $\delta \succeq x_1$. Further, let $t \in Y_a = Y_1' \cup \widetilde{Y}_2 \cup \overline{Y}_2$ with $x_1 \preceq t \cdot \beta \prec \beta$. Since $I(\beta) \subseteq Y_a$ contains only simple bonds, (6.64) implies that $\langle \beta, \alpha_t \rangle = \frac{1}{2}$.

Suppose first that $t = r_i$ for some i. Then $r_i \cdot \beta \in V_a$, and induction yields that $r_i \cdot \beta$ equals $\phi(\alpha)$ for some elementary root $\alpha \in \Phi_{Y'}$ with $\alpha \succeq x_1$. Now $\beta = r_i \cdot \phi(\alpha) = \phi(r_i \cdot \alpha)$; moreover,

$$\langle \alpha, x_i \rangle = \langle \phi(\alpha), x_i \rangle = \langle r_i \cdot \beta, x_i \rangle = -\langle \beta, x_i \rangle = -\frac{1}{2},$$

and thus $r_i \cdot \alpha$ is elementary by (3.37), and $r_i \cdot \alpha \succ \alpha \succ x_1$, as required.

Assume next that $t = \tilde{s}_j$ for some j. By symmetry of β, and since $\tilde{s}_j$ and $\overline{s}_j$ are not adjoined, we know that

$$\langle \overline{s}_j \cdot \beta, \widetilde{y}_j \rangle = \langle \beta, \widetilde{y}_j \rangle = \langle \beta, \overline{y}_j \rangle;$$

therefore $(\tilde{s}_j\overline{s}_j) \cdot \beta \prec \beta$ since $\langle \beta, \widetilde{y}_j \rangle = \frac{1}{2}$. Furthermore, if we denote the coefficient of $\widetilde{y}_j$ in β by μ, the coefficient of $\overline{y}_j$ in $(\tilde{s}_j\overline{s}_j) \cdot \beta$ equals $\mu - 2\langle \beta, \overline{y}_j \rangle$, while the coefficient of $\widetilde{y}_j$ in $(\tilde{s}_j\overline{s}_j) \cdot \beta$ equals

$$\mu - 2\langle \overline{s}_j \cdot \beta, \widetilde{y}_j \rangle = \mu - 2\langle \beta, \overline{y}_j \rangle,$$

and thus $(\tilde{s}_j\overline{s}_j) \cdot \beta \in V_a$. By induction, $(\tilde{s}_j\overline{s}_j) \cdot \beta$ equals $\phi(\alpha)$ for some elementary root $\alpha \in \Phi_{Y'}$ preceded by x_1. So $\beta = \phi(s_j \cdot \alpha)$, and since

$$\langle \alpha, y_j \rangle = \sqrt{2}\langle \phi(\alpha), \widetilde{y}_j \rangle = \sqrt{2}\langle \tilde{s}_j\overline{s}_j \cdot \beta, \widetilde{y}_j \rangle = -\sqrt{2}\langle \overline{s}_j \cdot \beta, \widetilde{y}_j \rangle = -\frac{1}{\sqrt{2}},$$

$s_j \cdot \alpha$ is elementary by (3.37); the above also yields that $s_j \cdot \alpha \succeq \alpha \succeq x_1$, as required. Symmetrical arguments apply if $t \in \overline{Y}_2$.

We can now deduce that ϕ maps the set of roots in $\mathcal{E}_{Y'}$ preceded by x_1 onto the set of roots in $\mathcal{E}_{Y_a} \cap V_a$. For, if β is in $\mathcal{E}_{Y_a} \cap V_a$, then $I(\beta) = Y_a$ and since $I(\beta)$ contains only simple bonds, it follows by (6.69) that β is preceded by x_1. Since ϕ is clearly one-one this finishes the proof.

□

Now let $\alpha \in \mathcal{E}_{Y_a}$ with coefficient 1 for x_i. An easy modification of (6.73)(i) yields that the coefficients of $x_l, \ldots, x_{i+1}$ in α must also be equal to 1, and we deduce that

$$\mathcal{E}_{Y_a} = \bigcup_{i=1}^{l+1} \bigcup_{j=1}^{n+1} \bigcup_{k=1}^{n+1} \mathcal{E}_{Y_a}^{\{r_l,\ldots,r_i\} \cup \{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_k,\ldots,\overline{s}_n\}}.$$

Since $\mathcal{E}_{Y_a}^{\{r_l,\ldots,r_i\} \cup \{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_k,\ldots,\overline{s}_n\}} \cap V_a$ is clearly empty if $j \neq k$, this becomes

$$\mathcal{E}_{Y_a} \cap V_a = \bigcup_{i=1}^{l+1} \bigcup_{j=1}^{n+1} \big(\mathcal{E}_{Y_a}^{\{r_l,\ldots,r_i\} \cup \{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a\big).$$

If $X \subseteq Y_1$, we derive from the definition of ϕ that ϕ induces a one-one correspondence between the set of roots in $\mathcal{E}_Y^X$ preceded by x_1, and

$$\bigcup_{j=1}^{n+1} \big(\mathcal{E}_{Y_a}^{X \cup \{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a\big).$$

(6.103) Proposition *Suppose that $m = 4$, $|Y_1| = l$ and $|Y_2| = n$. If $l = 2$, the elements of $\mathcal{E}_Y^{\{r_2\}}$ are*

$$\begin{aligned} x_2 + Mx_1 + \sqrt{2}\big(M(y_1 + \cdots + y_{j(M-1)-1}) \\ + (M-1)(y_{j(M-1)} + \cdots + y_{j(M-2)-1}) + \cdots \\ \cdots + 2(y_{j(2)} + \cdots + y_{j(1)-1}) + (y_{j(1)} + \cdots + y_n)\big) \end{aligned}$$

with $M \in \{2, \ldots, n+1\}$ and $0 < j(M-1) < j(M-2) < \ldots < j(1) < n+1$. If $l \geq 3$, the elements of $\mathcal{E}_Y^{\{r_l\}}$ are

$$x_l + 2(x_{l-1} + \cdots + x_1) + \sqrt{2}\big(2(y_1 + \cdots + y_{j-1}) + (y_j + \cdots + y_n)\big)$$

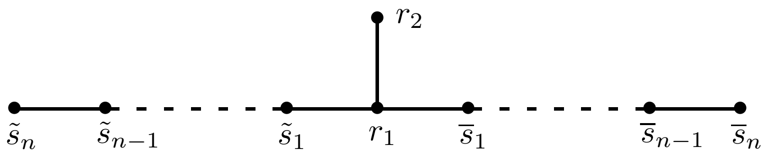
with $j \in \{1, \ldots, n\}$. Thus

$$\left|\mathcal{E}_Y^{\{r_l\}}\right| = \begin{cases} 2^n - 1 & \text{if } l = 2 \text{ and } |Y_2| = n, \\ n & \text{if } l \geq 3 \text{ and } |Y_2| = n. \end{cases}$$

Proof. By (6.59) we know that each root in $\mathcal{E}_Y^{\{r_l\}}$ is preceded by x_l, and it follows (by a remark at the beginning of this section) that each root in $\mathcal{E}_Y^{\{r_l\}}$ is preceded by x_1. Therefore the previous remark yields that

$$\mathcal{E}_Y^{\{r_l\}} = \phi^{-1}\Big(\bigcup_{j=1}^{n+1} \mathcal{E}_{Y_a}^{\{r_l\}\cup\{\tilde{s}_j,\ldots,\tilde{s}_n\}\cup\{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a\Big).$$

If $l = 2$, then $Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2$ equals

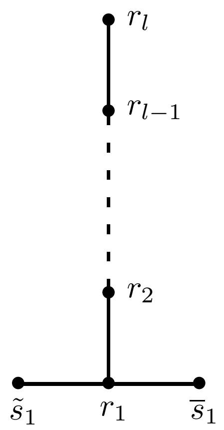


Proposition (6.74) implies that $\mathcal{E}_{Y_a}^{\{r_2\}}$ is empty, and we deduce from (6.72) (together with (6.57)) that the elements of $\mathcal{E}_{Y_a}^{\{r_l\}\cup\{\tilde{s}_j,\ldots,\tilde{s}_n\}\cup\{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a$ for $j \in \{1, \ldots, n\}$ are

$$\begin{aligned} x_2 + &\big(\widetilde{y}_n + \cdots + \widetilde{y}_j\big) + 2\big(y_{j-1} + \cdots + y_{k(3)}\big) + \cdots \\ &\cdots + (M-1)\big(\widetilde{y}_{k(M-1)-1} + \cdots + \widetilde{y}_{k(M)}\big) + M\big(\widetilde{y}_{k(M)-1} + \cdots + \widetilde{y}_1\big) \\ &+ Mx_1 + M\big(\overline{y}_1 + \cdots + \overline{y}_{k(M)-1}\big) + \cdots + \big(\overline{y}_j + \cdots + \overline{y}_m\big), \end{aligned}$$

with $M \in \{2, \ldots, j+1\}$ and $j > k(3) > \ldots > k(M) > 0$. This yields the assertion, if we set $j(1) = j$ and $j(i) = k(i-1)$ for $i \in \{3, \ldots, M\}$.

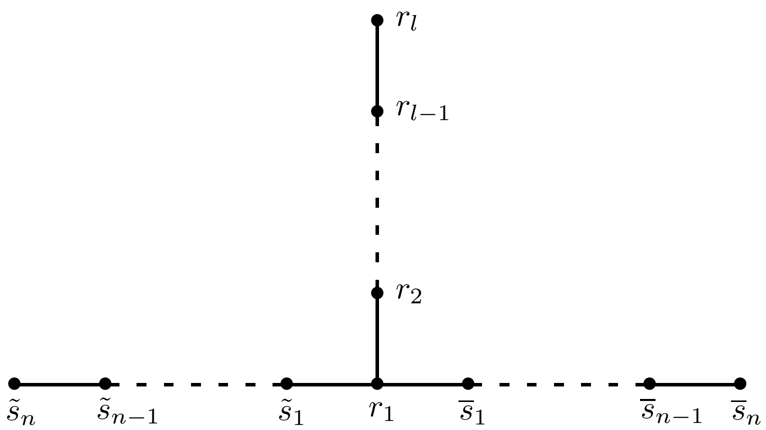
Suppose next that $l \geq 3$. If $n = 1$, then $Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2$ equals



and (6.77) yields that

$$\begin{aligned}\mathcal{E}_Y^{\{r_l\}} &= \big\{\phi^{-1}\big(x_l + 2(x_{l-1} + \cdots + x_1) + \widetilde{y}_1 + \overline{y}_1\big)\big\}\\ &= \big\{x_l + 2(x_{l-1} + \cdots + x_1) + \sqrt{2}y_1\big\},\end{aligned}$$

as required. Suppose now that $n \geq 2$. Then $Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2$ equals



with $l \geq 3$ and $n \geq 2$. Now $\mathcal{E}_{Y'}^{\{r_l\}}$ is empty by (6.70), and by (6.70) (and (6.57)), $\mathcal{E}_{Y_a}^{\{r_l\}\cup\{\tilde{s}_j,\ldots,\tilde{s}_n\}\cup\{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a$ has exactly one element for $j \in$

$\{1,\ldots,n\}$, namely

$$\begin{aligned}x_l+&2(x_{l-1}+\cdots+x_2)+(\widetilde{y}_m+\cdots+\widetilde{y}_j)+2(\widetilde{y}_{j-1}+\cdots+\widetilde{y}_1)\\&+2x_1+2(\overline{y}_1+\cdots+\overline{y}_{j-1})+(\overline{y}_j+\cdots+\overline{y}_m),\end{aligned}$$

and the assertion follws trivially. □

The previous proposition together with (6.99) allow us to conclude the following:

(6.104) PROPOSITION *Suppose that $m=4$, $|Y_1|=l$ and $|Y_2|>n$. If $l=2$, then $\mathcal{E}_Y^{\{r_2\}}$ is the set of*

$$\begin{aligned}x_2+Mx_1+\sqrt{2}\big(&M(y_1+\cdots+y_{j(M-1)-1})\\&+(M-1)(y_{j(M-1)}+\cdots+y_{j(M-2)-1})+\cdots\\&\cdots+3(y_{j(3)}+\cdots+y_{j(2)-1})+2(y_{j(2)}+\cdots+y_{j(1)-1})+\beta\big)\end{aligned}$$

with $M\in\{2,\ldots,n+1\}$,

$$0<j(M-1)<j(M-2)\ldots<j(2)<j(1)<n+1$$

and $\beta\in\mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j(1)-1}\}}$ with coefficient 1 for $y_{j(1)}$.
If $l\geq 3$, then $\mathcal{E}_Y^{\{r_l\}}$ is the set of

$$x_l+2(x_{l-1}+\cdots+x_1)+\sqrt{2}\big(2(y_1+\cdots+y_{j-1})+\beta\big)$$

with $j\in\{1,\ldots,n\}$ and $\beta\in\mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j.
Thus

$$|\mathcal{E}_Y^{\{r_l\}}|=\begin{cases}\sum_{j=1}^n(2^j-1)\sum_{J\subseteq Y_2\setminus\{s_1,\ldots,s_j\}}|\mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}^{\{s_j\}\cup J}| & \text{if } l=2,\\ \sum_{j=1}^n\sum_{J\subseteq Y_2\setminus\{s_1,\ldots,s_j\}}|\mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}^{\{s_j\}\cup J}| & \text{if } l\geq 3.\end{cases}$$

Note that the hypothesis $|Y_2|>n$ is not necessary in (6.104); for if $|Y_2|=n$, then $\mathcal{E}_{Y_2\setminus\{s_1,\ldots,s_{j-1}\}}=\{y_j+\cdots+y_n\}$, and the assertion reduces to (6.103).

(6.105) Proposition *Suppose that $m = 4$, $|Y_1| = l$ and $|Y_2| = n$. Then $\mathcal{E}_Y^{\emptyset}$ is empty unless $l = n = 2$, or $l = 2$ and $n \geq 3$, or $n = 2$ and $l \geq 3$. If $l = 2$ and $n = 2$, then*

$$\mathcal{E}_Y^{\emptyset} = \{2x_2 + 3x_1 + 2\sqrt{2}y_1 + \sqrt{2}y_2,\ \sqrt{2}x_2 + 2\sqrt{2}x_1 + 3y_1 + 2y_2\},$$

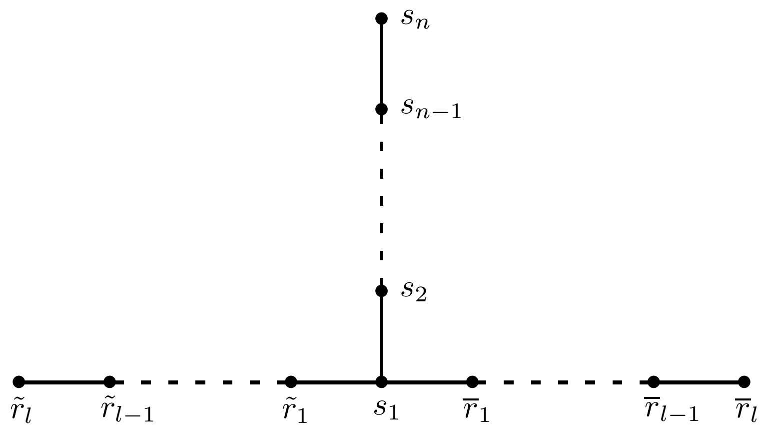
while if $l = 2$ and $n \geq 3$, the elements of $\mathcal{E}_Y^{\emptyset}$ are

$$2x_2 + 3x_1 + \sqrt{2}\big(3(y_1 + \cdots + y_{k-1}) + 2(y_k + \cdots + y_{j-1}) + (y_j + \cdots + y_n)\big)$$

with $1 \leq k < j \leq n$. Hence

$$|\mathcal{E}_Y^{\emptyset}| = \begin{cases} 0 & \text{if } l = 1 \text{ or } n = 1, \\ 2 & \text{if } l = n = 2, \\ \binom{n}{2} & \text{if } l = 2,\ n \geq 3, \\ \binom{l}{2} & \text{if } l \geq 3 \text{ and } n = 2, \\ 0 & \text{if } l,\ n \geq 3. \end{cases}$$

Proof. As $\mathcal{E}_Y$ does not depend on R, we may assume without loss of generality that R contains $\widetilde{Y}_1 = \{\tilde{r}_l, \ldots, \tilde{r}_1\}$ and $\overline{Y}_1 = \{\overline{r}_l, \ldots, \overline{r}_1\}$ such that $\widetilde{Y}_1 \cup \overline{Y}_1 \cup Y_2$ equals



Denote the simple roots corresponding to $\tilde{r}_i$, $\overline{r}_i$ by $\widetilde{x}_i$, $\overline{x}_i$ respectively, and let

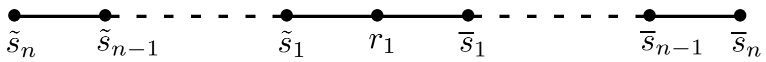
V_b the space spanned by $\widetilde{x}_l + \overline{x}_l$, $\widetilde{x}_1 + \overline{x}_1$, $y_1, \ldots, y_n$. Further, let $\psi\colon V_Y \to V_b$ be defined by

$$\psi\Big(\sum_{i=1}^{l} \lambda_i x_i + \sum_{j=1}^{n} \mu_j y_j\Big) = \frac{1}{\sqrt{2}} \sum_{i=1}^{l} \lambda_i(\widetilde{x}_i + \overline{x}_i) + \sum_{j=1}^{n} \mu_j y_j.$$

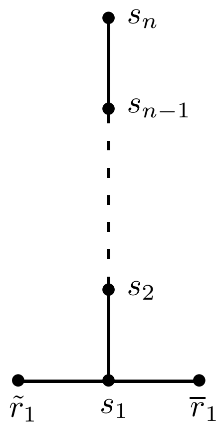
By a remark at the beginning of this section, each root in $\mathcal{E}_Y$ is is preceded by x_1 or y_1 (but certainly not by both), and so the remark following (6.102) yields that $\mathcal{E}_Y^{\{r_l\}}$ is equal to the following disjoint union:

$$\phi^{-1}\Big(\bigcup_{j=1}^{n+1} \mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a\Big) \cup \psi^{-1}\Big(\bigcup_{i=1}^{l+1} \mathcal{E}_{Y_2 \cup \widetilde{Y}_1 \cup \overline{Y}_1}^{\{\tilde{r}_i,\ldots,\tilde{r}_l\} \cup \{\overline{r}_i,\ldots,\overline{r}_l\}} \cap V_b\Big).$$

If $l = 1$, then $Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2$ equals

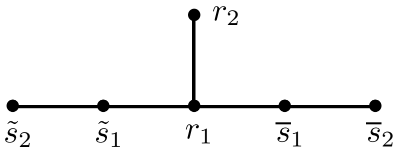


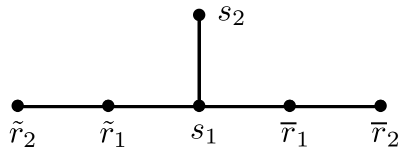
and $\mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_j,\ldots,\overline{s}_n\}}$ is empty for all $j \in \{1, \ldots, n+1\}$ by (6.66); furthermore, $\widetilde{Y}_1 \cup \overline{Y}_1 \cup Y_2$ equals



and $\mathcal{E}_{\widetilde{Y}_1 \cup \overline{Y}_1 \cup Y_2}^{\emptyset}$ and $\mathcal{E}_{\widetilde{Y}_1 \cup \overline{Y}_1 \cup Y_2}^{\{\tilde{r}_1,\bar{r}_1\}}$ are empty by (6.77). Whence $\mathcal{E}_Y^{\emptyset} = \emptyset$ if $l = 1$, and symmetrically $\mathcal{E}_Y^{\emptyset} = \emptyset$ if $n = 1$.

Suppose now that $l,\, n \geq 2$. If both l and n equal 2, $Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2$ and $\widetilde{Y}_1 \cup \overline{Y}_1 \cup Y_2$ equal





respectively, and (6.85) yields that $\mathcal{E}_Y^{\emptyset}$ has two elements; namely

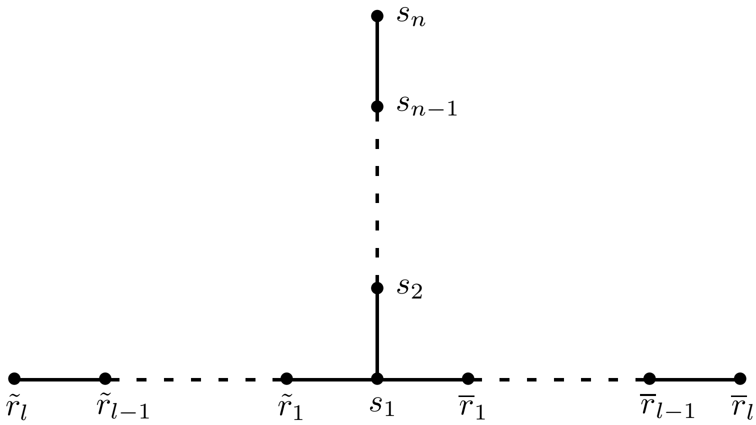
$$\phi^{-1}\big(2x_2 + \widetilde{y}_2 + 2\widetilde{y}_1 + 3x_1 + 2\overline{y}_1 + \overline{y}_2\big) = 2x_2 + 3x_1 + \sqrt{2}(2y_1 + y_2)$$

and

$$\psi^{-1}\big(2y_2 + \widetilde{x}_2 + 2\widetilde{x}_1 + 3y_1 + 2\overline{x}_1 + \overline{x}_2\big) = \sqrt{2}(x_2 + 2x_1) + 3y_1 + 2y_2,$$

as required.

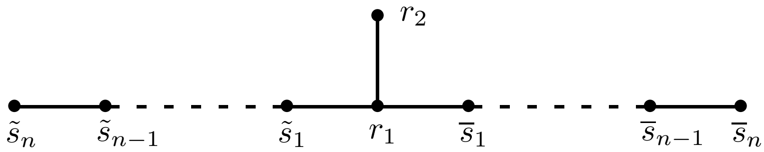
Suppose next that $n \geq 3$. Then $Y_2 \cup \widetilde{Y}_1 \cup \overline{Y}_1$ equals



with $n \geq 3$ and $l \geq 2$, and thus $\mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{r}_l, \ldots, \tilde{r}_i\} \cup \{\overline{r}_l, \ldots, \overline{r}_i\}}$ is empty for all $i \in \{2, \ldots, l+1\}$ by (6.70) (and (6.57)), and furthermore empty by (6.77) for $i = 1$. So

$$\mathcal{E}_Y^{\{r_l\}} = \phi^{-1}\Big(\bigcup_{j=1}^{n+1} \mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_j, \ldots, \tilde{s}_n\} \cup \{\overline{s}_j, \ldots, \overline{s}_n\}} \cap V_a\Big).$$

If l is also greater than or equal to 3, symmetrical arguments yield that $\mathcal{E}_Y^{\emptyset}$ is empty. Assume next that $l = 2$; then $Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2$ equals



with $n \geq 3$. Lemma (6.77) yields that $\mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_1,\ldots,\tilde{s}_n\} \cup \{\overline{s}_1,\ldots,\overline{s}_n\}} = \emptyset$, and (6.80) implies that $\mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\emptyset}$ is also empty. So the above becomes

$$\mathcal{E}_Y^{\{r_l\}} = \phi^{-1}\Big(\bigcup_{j=2}^{n} \mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a\Big).$$

Now $\mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_2,\ldots,\tilde{s}_n\} \cup \{\overline{s}_2,\ldots,\overline{s}_n\}}$ has one element by (6.85), namely

$$2x_1 + \big(\widetilde{y}_n + \cdots + \widetilde{y}_2\big) + 2\widetilde{y}_1 + 3x_0 + +2\overline{y}_1 + \big(\overline{y}_2 + \cdots + \overline{y}_n\big).$$

If $j \in \{3, \ldots, n\}$, (6.79)(1) yields that $\mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a$ is the set of

$$\begin{aligned} 2x_1 + \big(\widetilde{y}_n + \cdots + \widetilde{y}_j\big) + 2\big(\widetilde{y}_{j-1} + \cdots + \widetilde{y}_k\big) + 3\big(\widetilde{y}_{k-1} + \cdots + \widetilde{y}_1\big) + 3x_0 \\ + 3\big(\overline{y}_1 + \cdots + \overline{y}_{k-1}\big) + 2\big(\overline{y}_k + \cdots + \overline{y}_{j-1}\big) + \big(\overline{y}_j + \cdots + \overline{y}_n\big), \end{aligned} \qquad (*)$$

with $k \in \{1, \ldots, j-1\}$. It is clear that $\mathcal{E}_{Y_1 \cup \widetilde{Y}_2 \cup \overline{Y}_2}^{\{\tilde{s}_j,\ldots,\tilde{s}_n\} \cup \{\overline{s}_j,\ldots,\overline{s}_n\}} \cap V_a$ for $j = 2$ is also of the form $(*)$, and one can now easily verify that $\mathcal{E}_Y^{\emptyset}$ is of the required shape.

□

(6.106) PROPOSITION *Suppose that $m = 4$ and $|Y_1| = l$. Then $\mathcal{E}_Y^{\emptyset}$ is empty unless $l = 2$ and $n \geq 2$, or $|Y_2| = 2$ (and thus $|Y_2| = n = 2$) and $l \geq 3$. If $l = 2$, $n \geq 2$ and $|Y_2| \geq 3$, the elements of $\mathcal{E}_Y^{\emptyset}$ are*

$$2x_2 + 3x_1 + \sqrt{2}\big(3(y_1 + \cdots + y_{k-1}) + 2(y_k + \cdots + y_{j-1}) + \beta\big),$$

where $1 \leq k < j \leq n$ *and* $\beta \in \mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ *with coefficient 1 for* y_j.
Hence

$$\left|\mathcal{E}_Y^{\emptyset}\right| = \sum_{j=2}^{n}(j-1) \sum_{J \subseteq Y_2 \setminus \{s_1,\ldots,s_j\}} \left|\mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}^{\{s_j\} \cup J}\right|.$$

Proof. If $|Y_1| = n$,

$$\bigcup_{J \subseteq Y_2 \setminus \{s_1,\ldots,s_j\}} \mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}^{\{s_j\} \cup J} = \{y_j + \cdots + y_n\},$$

and the assertion is proved in (6.105). So assume now that $|Y_2| > n$. By (6.99) we know that every element of $\mathcal{E}_Y^{\emptyset}$ can be written as

$$\alpha + \sqrt{2}\beta - \sqrt{2}y_j$$

with $j \in \{1, \ldots, n\}$, $\alpha \in \mathcal{E}_{Y_1 \cup \{s_1,\ldots,s_j\}}^{\emptyset}$ with coefficient $\sqrt{2}$ for y_j and coefficient greater than or equal to 2 for $y_1, \ldots, y_{j-1}$, and $\beta \in \mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j. But (6.105) yields that $\mathcal{E}_{Y_1 \cup \{s_1,\ldots,s_j\}}^{\emptyset}$ is empty unless $l = 2$ and $j \geq 2$, or $l \geq 3$ and $j = 2$; if $l \geq 3$ and $j = 2$, (6.105) yields further that there are no roots in $\mathcal{E}_{Y_1 \cup \{s_1,\ldots,s_j\}}^{\emptyset}$ with coefficient $\sqrt{2}$ for y_2, and thus $\mathcal{E}_Y^{\emptyset} = \emptyset$ unless $l = 2$ and $j \geq 2$ (and thus $n \geq 2$).

If $l = 2$ and $n \geq 2$, then $\mathcal{E}_{Y_1 \cup \{s_1\}}^{\emptyset} = \emptyset$ by (6.105), and thus $\mathcal{E}_Y^{\emptyset}$ is the set of

$$2x_2 + 3x_1 + \sqrt{2}\big(3(y_1 + \cdots + y_{k-1}) + 2(y_k + \cdots + y_{j-1}) + \beta\big)$$

with $1 \leq k < j \leq n$ and $\beta \in \mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j, and the assertion follows easily. □

This leaves us with $m = 5$. Since

$$\left\{ \frac{\sin(k\pi/n)}{\sin(\pi/n)} \mid n = m_{rs} < \infty \text{ for some } r, s \in Y \text{ and } 1 \leq l \leq \tfrac{k}{2} \right\}$$

equals $\{1, c_5\}$, and $c_5^2 = c_5 + 1$, we know by (2.27) that for $r \in Y$ the coefficient of α_r in any root in $\alpha \in \Phi_Y$ equals $a + bc_5$ for some a, $b \in \mathbb{N}_0$.

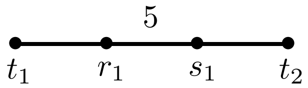
We start off by stating two easy results (the former a modification of (6.73)(i), and the latter a corollary of (3.37)). Trivial though they are, these will make life a lot easier for us.

(6.107) LEMMA *Suppose that α is a root in $\Phi^+_{Y'}$, with $x_1 \in \operatorname{supp}(\alpha)$. Then the coefficient of x_{i+1} in α is less than or equal to the coefficient of x_i in α for all i in $\{1, \ldots, l-1\}$.*

(6.108) COROLLARY *Suppose that $\gamma_1, \ldots, \gamma_k \in \Phi^+$ and $t_1, \ldots, t_{k-1} \in R$ with $\gamma_{i+1} = t_i \cdot \gamma_i$ and $\langle \gamma_i, \alpha_{t_i} \rangle \in (-1, 0)$ for all $i \in \{1, \ldots, k-1\}$. Suppose furthermore that $\gamma_1 \in \mathcal{E}$. Then $\gamma_i \in \mathcal{E}$ for all $i \in \{1, \ldots, k\}$.*

The next lemma together with (6.99) will enable us to restrict our main attention to the case $|Y_1| = 1$ and $|Y_2| = n$.

(6.109) LEMMA *Suppose there exist t_1, $t_2 \in Y$ such that $\{t_1, r_1, s_1, t_2\}$ equals*



Denote the simple roots corresponding to t_1, t_2 by z_1 and z_2 respectively, and let $\alpha \in \Phi_Y$ such that z_1, $z_2 \in \operatorname{supp}(\alpha)$, and the coefficients of x_1 and y_1 in α are greater than or equal to 2. Then $\alpha \in \Delta$.

Proof. Let $\beta \preceq \alpha$ be a positive root of minimal depth such that z_1 and z_2 are in the support of β, and the coefficients of x_1 and y_1 in β are greater than or equal to 2. Further, let $r \in Y$ such that $\beta \succ r \cdot \beta$. It suffices to show that $\langle r \cdot \beta, \alpha_r \rangle \leq -1$; that is, $\langle \beta, \alpha_r \rangle \geq 1$. For then $\beta \in \Delta$ by (3.32), and thus $\alpha \in \Delta$ by (3.36).

Denote the coefficients of z_1, x_1, y_1, z_2 in β by ν_1, λ, μ and ν_2 respectively. By minimality of β it follows that $r \in \{t_1, r_1, s_1, t_2\}$, and by symmetry we may assume without loss of generality that $r \in \{t_1, r_1\}$. Suppose first that $r = t_1$. Then minimality of β also yields that $z_1 \notin \operatorname{supp}(t_1 \cdot \beta)$; that is, the coefficient of z_1 in $t_1 \cdot \beta$ equals 0. Since $\lambda \geq 2$ we deduce that $\langle t_1 \cdot \beta, z_1 \rangle \leq 0 + (-\frac{1}{2})\lambda \leq -1$, as required.

Suppose next that $r = r_1$, and denote the coefficient of x_1 in $r_1 \cdot \beta$ by λ'. Then $\lambda' < 2$ by minimality of β, and $\lambda' > 0$ by connectedness of the support of $r_1 \cdot \beta$; hence λ' equals 1 or c_5. If $\lambda' = 1$, Propositions (6.57) and (6.93) together yield that $\mu = kc_5$ for some $k \in \mathbb{N}_0$, and since $\mu \geq 2$ we know that $k \geq 2$; since $\nu_1 \geq 1$, this implies that

$$\langle r_1 \cdot \beta, x_1 \rangle \leq 1 + \Big(-\frac{1}{2}\Big)\nu_1 + \Big(-\frac{c_5}{2}\Big)\mu \leq 1 - \frac{1}{2} - \frac{kc_5^2}{2} \leq \frac{1}{2} - c_5^2 \leq -1,$$

as required.

Finally, suppose that $\lambda' = c_5$. If $\mu = 2$, let $\gamma \preceq r_1 \cdot \beta$ be of minimal depth such that the coefficients of x_1 and y_1 in γ equal c_5 and 2 respectively; then

$$\langle \gamma, x_1 \rangle \leq c_5 + \Big(-\frac{c_5}{2}\Big)2 = 0 \text{ and } \langle \gamma, y_1 \rangle \leq 2 + \Big(-\frac{c_5}{2}\Big)c_5 = \frac{3}{2} - c_5 \leq 0,$$

and therefore neither $\gamma \succ r_1 \cdot \gamma$ nor $\gamma \succ s_1 \cdot \gamma$, contradicting the minimality of γ. So $\mu > 2$, and since μ equals $ac_5 + b$ for some a, $b \in \mathbb{N}_0$ this forces $\mu \geq c_5 + 1$. Hence

$$\langle r_1 \cdot \beta, x_1 \rangle \leq c_5 + \nu_1\Big(-\frac{1}{2}\Big) + \mu\Big(-\frac{c_5}{2}\Big) \leq c_5 - \frac{1}{2} - \frac{c_5^2}{2} = -1,$$

(since $\nu_1 \geq 1$), as required. □

(6.110) Lemma *Suppose that $m = 5$, $|Y_1| = 1$ and $|Y_2| = n$, and let α be in $\mathcal{E}_Y^{\{s_n\}}$. Then α is preceded by*

$$c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n,$$

and this is an elementary root.

Proof. By (6.59) we know that α is preceded by y_n, and we let β be of maximal depth with $\alpha \succeq \beta \succeq y_n$ such that $I(\beta) \subseteq Y_2$. Maximality of β yields that $\alpha \succeq r_1 \cdot \beta \succ \beta$, and thus $\langle \beta, x_1 \rangle \in (-1, 0)$ by (3.38); if we denote the coefficient of y_1 in β by μ, we find that $\langle \beta, x_1 \rangle = -\frac{c_5}{2}\mu$, and (2.26) yields that $\mu = 1$. As $s_1, s_n \in I(\beta)$, we deduce that $I(\beta) = \{s_1, \ldots, s_n\}$. Now β is elementary, and $I(\beta)$ contains only simple bonds, therefore (6.69) implies that $r_1 \cdot \beta \succ \beta \succeq y_1 + \cdots + y_n$; furthermore, the coefficients of y_1 in $r_1 \cdot \beta$ and $y_1 + \cdots + y_n$ coincide, and thus

$$r_1 \cdot \beta \succeq r_1 \cdot (y_1 + \cdots + y_n) = c_5 x_1 + y_1 + \cdots + y_n$$

by (6.58). It follows by transitivity of $\succeq$ that α is preceded by

$$\gamma_1 = c_5 x_1 + y_1 + \cdots + y_n;$$

since $y_1+\cdots+y_n$ is clearly elementary, and $\langle y_1+\cdots+y_n, x_1\rangle = -\frac{c_5}{2} \in (-1,0)$, (3.37) yields that γ_1 is elementary. For $j \in \{1,\ldots,n-1\}$, let γ_{j+1} be equal to $s_j \cdot \gamma_j$. Then a straightforward calculation yields that

$$\gamma_j = c_5 x_1 + (c_5+1)(y_1+\cdots+y_{j-1}) + y_j + \cdots + y_n$$

and $\langle \gamma_j, y_j\rangle = -\frac{1}{2} \in (-1,0)$; since γ_1 is elementary, (6.108) implies that γ_j is elementary for all $j \in \{1,\ldots,n\}$. In particular $\gamma_n \in \mathcal{E}$, and it remains to show that α is preceded by γ_n.

By the above $\alpha \succeq \gamma_1$, and the proof is finished if $n=1$; so suppose that $n \geq 2$. The coefficient of y_1 in γ_1 equals 1, while the coefficient of y_1 in α is greater than 1, and we let δ be a root with $\gamma_1 \preceq \delta \preceq \alpha$ such that the coefficient of y_1 in δ equals 1, and $\delta \prec s_1 \cdot \delta \preceq \alpha$. As α is elementary, clearly $s_1 \cdot \delta \in \mathcal{E}$, and thus $\langle \delta, y_1\rangle > -1$ by (3.38). The coefficient of y_2 in δ equals 1 by (6.107), and we denote the coefficient of y_1 in δ by λ. Then

$$\langle \delta, y_1\rangle \leq 1 + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{c_5}{2}\Big)\lambda,$$

and thus $\lambda < \frac{3}{c_5} \leq 2$; since $\lambda \geq c_5$ as $\delta \succeq \gamma_1$, this yields $\lambda = c_5$. Hence the coefficients of x_1 in $s_1 \cdot \delta$ and γ coincide; as $|Y_2| = n$, and the coefficients of y_2 in $s_1 \cdot \delta$ and γ both equal 1, (6.58) yields that $s_1 \cdot \delta$ is a successor of $s_1 \cdot \gamma_1$, which equals γ_2.

If $n=2$, this finishes the proof, so suppose $n > 2$, and assume that $\alpha \succeq \gamma_j$ for some $j \in \{2,\ldots,n-1\}$. The coefficient of y_j in γ_j equals 1, while the coefficient of y_j in α is greater than 1, and (again) we let δ be a root with $\gamma_j \preceq \delta \preceq \alpha$ such that the coefficient of y_j in δ equals 1, and $\delta \prec r_j \cdot \delta \preceq \alpha$. As α is elementary, clearly $s_j \cdot \delta \in \mathcal{E}$, and thus $\langle \delta, y_j\rangle > -1$ by (3.38). The coefficient of y_{j+1} in δ equals 1 by (6.107), and we denote the coefficient of y_{j-1} in δ again by λ; then

$$\langle \delta, y_j\rangle \leq 1 + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{1}{2}\Big)\lambda = \frac{1}{2}(1-\lambda),$$

and we deduce that $\lambda < 3$. But $\lambda \geq c_5 + 1$ as $\delta \succeq \gamma_j$, and this only leaves us with $\lambda = c_5 + 1$ (since λ equals $ac_5 + b$ for some a, $b \in \mathbb{N}_0$). Hence the coefficients of y_{j-1} in $s_j \cdot \delta$ and γ coincide again. As $|Y_2| = n$, and the coefficients of y_{j+1} in $s_j \cdot \delta$ and γ both equal 1, (6.58) yields that $s_j \cdot \delta$ is a successor of $s_j \cdot \gamma_j$, which equals γ_{j+1}. So by induction $\alpha \succeq \gamma_n$, as required. □

(6.111) PROPOSITION *Suppose $m=5$, $|Y_1|=1$ and $|Y_2|=n$.*

(i) If $n=2$, then

$$\mathcal{E}_Y^{\{s_2\}} = \{c_5x_1+(c_5+1)y_1+y_2, (c_5+1)x_1+(c_5+1)y_1+y_2\}.$$

(ii) If $n=3$, then

$$\begin{aligned}\mathcal{E}_Y^{\{s_3\}} = \{&c_5x_1+(c_5+1)y_1+(c_5+1)y_2+y_3,\\ &(c_5+1)x_1+(c_5+1)y_1+(c_5+1)y_2+y_3,\\ &(c_5+1)x_1+(2c_5+1)y_1+(c_5+1)y_2+y_3,\\ &(2c_5+1)x_1+(2c_5+1)y_1+(c_5+1)y_2+y_3,\\ &(2c_5+1)x_1+(2c_5+2)y_1+(c_5+1)y_2+y_3,\\ &(2c_5+1)x_1+(2c_5+2)y_1+(c_5+2)y_2+y_3\}.\end{aligned}$$

(iii) If $n \geq 4$, the elements of $\mathcal{E}_Y^{\{s_n\}}$ are exactly the following:

$$\alpha^{(1)} = c_5x_1+(c_5+1)(y_1+\cdots+y_{n-1})+y_n,$$

$$\alpha_j^{(2)} = (c_5+1)x_1+(2c_5+1)(y_1+\cdots+y_{j-1})+(c_5+1)(y_j+\cdots+y_{n-1})+y_n,$$

with $j \in \{1,\ldots,n-1\}$,

$$\alpha_j^{(3)} = (2c_5+1)x_1+(2c_5+1)(y_1+\cdots+y_{j-1})+(c_5+1)(y_j+\cdots+y_{n-1})+y_n,$$

with $j \in \{2,\ldots,n-1\}$, and

$$\alpha^{(4)} = (2c_5+1)x_1+(2c_5+2)y_1+(c_5+1)(y_2+\cdots+y_{n-1})+y_n.$$

Hence

$$|\mathcal{E}_Y^{\{r_l\}}| = \begin{cases} 2 & \text{if } |Y_1|=1 \text{ and } n=2,\\ 6 & \text{if } |Y_1|=1 \text{ and } n=3,\\ 2n-1 & \text{if } |Y_1|=1 \text{ and } n\geq 4.\end{cases}$$

Proof. If $n=2,3$ it can be easily verified that we have in fact enumerated all roots preceded by $c_5x_1+(c_5+1)(y_1+\cdots+y_{n-1})+y_n$ with coefficient 1 for y_n, and that these are elementary. It remains to show (iii).

By (6.110) we know that $\alpha^{(1)}$ is elementary. Since $\langle\alpha^{(1)}, x_1\rangle = -\frac{1}{2}$, it follows by (3.37) that $r_1 \cdot \alpha^{(1)}$ is also elementary; that is, $\alpha_1^{(2)} \in \mathcal{E}$. Now $\alpha_{j+1}^{(2)} = s_j \cdot \alpha_j^{(2)}$ for $j \in \{1, \ldots, n-1\}$ and further $\langle\alpha_j^{(2)}, y_j\rangle = -\frac{c_5}{2} \in (-1, 0)$; hence $\alpha_j^{(2)} \in \mathcal{E}$ for all $j \in \{1, \ldots, n\}$ by (6.108). For $j \in \{2, \ldots, n\}$ clearly $\alpha_j^{(3)} = r_1 \cdot \alpha_j^{(2)}$ and $\langle\alpha_j^{(2)}, x_1\rangle = -\frac{c_5}{2}$, thus it follows by (3.37) that $\alpha_j^{(3)}$ is elementary for all $j \in \{2, \ldots, n\}$. Finally, $\langle\alpha_2^{(3)}, y_1\rangle = -\frac{1}{2}$ and therefore $\alpha^{(4)} = s_1 \cdot \alpha_2^{(3)} \in \mathcal{E}$ by (3.37). So the above listed vectors are in $\mathcal{E}$, and thus certainly in $\mathcal{E}_Y^{\{s_n\}}$.

We prove now that all elements of $\mathcal{E}_Y^{\{s_n\}}$ have been accounted for. So let $\beta \in \mathcal{E}_Y^{\{s_n\}}$, and assume for a contradiction that β is not equal to any of the roots listed above. By (6.110) we know that β is preceded by $\alpha^{(1)}$. As $\beta \neq \alpha^{(1)}$, and the coefficients of y_n in $\alpha^{(1)}$ and β coincide, there exists an $r \in \{r_1, s_1, \ldots, s_{n-1}\}$ with $\beta \succeq r \cdot \alpha^{(1)}$; that is, $\langle\alpha^{(1)}, \alpha_r\rangle < 0$. This forces $r = r_1$, and thus $\beta \succeq r_1 \cdot \alpha^{(1)} = \alpha_1^{(2)}$. Now let $j \in \{1, \ldots, n-1\}$ be maximal such that $\beta \succeq \alpha_j^{(2)}$; that is, $\beta \succ \alpha_j^{(2)}$ by assumption. Then there exists an $s \in \{r_1, s_1, \ldots, s_{n-1}\}$ with $\beta \succeq s \cdot \alpha_j^{(2)}$, and maximality of j forces $s = r_1$ and $j \geq 2$. Whence $\beta \succeq r_1 \cdot \alpha_j^{(2)} = \alpha_j^{(3)}$, and hence $\beta \succ \alpha_j^{(3)}$ by assumption. Now let $t \in \{r_1, s_1, \ldots, s_{n-1}\}$ with $\beta \succeq t \cdot \alpha_j^{(3)}$. Then $\langle\alpha_j^{(3)}, \alpha_t\rangle < 0$, and thus $\langle\alpha_j^{(3)}, \alpha_t\rangle \in (-1, 0)$, as β is elementary. We find that $t = s_1$ and $j = 2$; so β is preceded by $s_1 \cdot \alpha_2^{(3)} = \alpha^{(4)}$. Since $\langle\alpha^{(4)}, x_1\rangle$, $\langle\alpha^{(4)}, y_2\rangle \leq -1$ and $\langle\alpha^{(4)}, \alpha_r\rangle \geq 0$ for $r \in Y \setminus \{r_1, s_1\}$, we deduce from (3.38) that $\beta = \alpha^{(4)}$, contrary to our assumption, and this finishes the proof. $\square$

(6.112) PROPOSITION *Suppose $m = 5$, $|Y_2| = n \geq 2$ and $|Y_1| > 1$. Then $\mathcal{E}_Y^{\{s_n\}}$ equals*

$$\begin{aligned}&\bigl\{c_5\alpha + (c_5+1)(y_1 + \cdots + y_{n-1}) + y_n \mid \alpha \in \mathcal{E}_{Y_1} \text{ with coefficient 1 for } x_1\bigr\}\\ &\quad= \bigl\{c_5\alpha + (c_5+1)(y_1 + \cdots + y_{n-1}) + y_n \mid \alpha \in \bigcup_{J \subseteq Y_1 \setminus \{r_1\}} \mathcal{E}_{Y_1}^{J \cup \{r_1\}}\bigr\}.\end{aligned}$$

Hence

$$\bigl|\mathcal{E}_Y^{\{s_n\}}\bigr| = \sum_{J \subseteq Y_1 \setminus \{r_1\}} \bigl|\mathcal{E}_{Y_1}^{J \cup \{r_1\}}\bigr|.$$

In particular, if $|Y_1| = l$,

$$\mathcal{E}_Y^{\{s_n\}} = \{c_5(x_l + \cdots + x_1) + (c_5+1)(y_1 + \cdots + y_{n-1}) + y_n\}.$$

Proof. Since $c_5 x_1 + (c_5+1)(y_1 + \cdots + y_{n-1}) + y_n$ is elementary by (6.110), Proposition (6.97) yields that

$$c_5\alpha + (c_5+1)(y_1 + \cdots + y_{n-1}) + y_n$$

is an elementary root for $\alpha \in \mathcal{E}_{Y_1}$ with coefficient 1 for x_1, and thus certainly in $\mathcal{E}_Y^{\{s_n\}}$. It remains to show that all elements of $\mathcal{E}_Y^{\{s_n\}}$ are of this form.

Let $\gamma \in \mathcal{E}_Y^{\{s_n\}}$. Since the coefficient of y_n in γ equals 1, we deduce from (6.97) that the coefficients of $y_1, \ldots, y_{n-1}$ in γ cannot be equal to c_5, and thus must be greater than or equal to 2. In particular, the coefficient of y_1 in γ is greater than or equal to 2; since $n \geq 2$ and $|Y_1| \geq 2$, Lemma (6.109) now forces the coefficient of x_1 in γ to be less than 2, and thus equal to c_5. So by (6.97),

$$\gamma = c_5\alpha + \beta - c_5 x_1$$

for some $\alpha \in \mathcal{E}_{Y_1}$ with coefficient 1 for x_1, and $\beta \in \mathcal{E}_{\{r_1\}\cup Y_2}$ with coefficient c_5 for x_1. It is clear that β must be an element of $\mathcal{E}_{\{r_1\}\cup Y_2}^{\{s_n\}}$, and since the coefficient of x_1 in β equals c_5, we deduce from (6.111) that

$$\beta = c_5 x_1 + (c_5+1)(y_1 + \cdots + y_{n-1}) + y_n;$$

whence γ is of the desired shape.

□

This leaves us with $X = \emptyset$.

(6.113) Lemma *Suppose that $m = 5$, $|Y_1| = l$ and $|Y_2| = n$, and let $\alpha \in \mathcal{E}_Y^{\emptyset}$. Then*

$$\alpha \succeq c_5 x_1 + c_5(y_1 + \cdots + y_n)$$

or

$$\alpha = (2c_5+1)x_1 + (2c_5+2)y_1 + (c_5+2)y_2 + (c_5+1)y_3.$$

Proof. If $n = 1$, the assertion is certainly true. Now suppose that $n \geq 2$ and assume that the assertion is true for $n-1$. Let β be of minimal depth with

$\beta \preceq \alpha$ such that $\beta \in \mathcal{E}_Y^{\emptyset}$, and denote the coefficients of x_1, y_j in β by λ and μ_j respectively. Let $r \in Y$ such that $r \cdot \beta \prec \beta$; then $\langle r \cdot \beta, \alpha_r \rangle \in (-1, 0)$ by (3.38). Moreover, $r \cdot \beta$ is elementary, and since $r \cdot \beta \notin \mathcal{E}_Y^{\emptyset}$ by minimality of β, the coefficient of α_r in $r \cdot \beta$ must be less than or equal to 1, and thus equal to 0 or 1 by (2.26).

Suppose first that $r = r_1$. If the coefficient of x_1 in $r_1 \cdot \beta$ equals 0, we find that $\langle r_1 \cdot \beta, x_1 \rangle = -\frac{c_5}{2}\mu_1 \leq -1$; for $\mu_1 > 1$, and thus $\mu_1 \geq \sqrt{2}$ by (2.26). But this contradicts our conclusion that $\langle r \cdot \beta, \alpha_r \rangle \in (-1, 0)$, and so the coefficient of x_1 in $r_1 \cdot \beta$ equals 1. By (6.93) (together with (6.57)), μ_1 equals kc_5 for some $k \in \mathbb{N}_0$, and

$$\langle r_1 \cdot \beta, x_1 \rangle = 1 + \Big(-\frac{c_5}{2}\Big)k = 1 - \Big(c_5 + \frac{1}{2}\Big)k$$

forces $k = 1$ and $\mu_1 = c_5$. So by (6.97), $r_1 \cdot \beta$ equals $x_1 + c_5\gamma$ for some $\gamma \in \mathcal{E}_{Y_2}$ with coefficient 1 for y_1. Since $I(\gamma)$ contains only simple bonds, (6.69) yields that $\gamma \succeq y_1 + \cdots + y_n$, and by definition of $\succeq$ there exists a $w \in W_{Y_2}$ with $\gamma = w \cdot (y_1 + \cdots + y_n)$ and $N(w) = N_-(w, y_1 + \cdots + y_n)$. Since the coefficients of y_1 in γ and $y_1 + \cdots + y_n$ coincide, we conclude that $w \in W_{Y_2 \setminus \{s_1\}}$, and thus $N(w) \subseteq \Phi^+_{Y_2 \setminus \{s_1\}}$. So if $\delta \in N(w)$, then

$$\big\langle c_5x_1 + c_5(y_1 + \cdots + y_n)\,,\, \delta\big\rangle = c_5\big\langle y_1 + \cdots + y_n\,,\, \delta\big\rangle < 0,$$

as $\delta \in N_-(w, y_1 + \cdots + y_n)$; thus $N_-\big(w, c_5x_1 + c_5(y_1 + \cdots + y_n)\big) = N(w)$. Now

$$\begin{aligned}\beta &= c_5x_1 + c_5\gamma = c_5x_1 + c_5\big(w \cdot (y_1 + \cdots + y_n)\big)\\ &= w \cdot \big(c_5x_1 + c_5(y_1 + \cdots + y_n)\big)\end{aligned}$$

and by definition of $\succeq$ this is preceded by $c_5x_1 + c_5(y_1 + \cdots + y_n)$, as required.

Suppose next that $r = s_j$ for some $j \in \{1, \ldots, n\}$. Since the coefficient of y_j in $s_j \cdot \beta$ equals 0 or 1, while the coefficient of α_s in $s_j \cdot \beta$ is greater than 1 for $s \in Y \setminus \{s_j\}$, Lemma (6.107) yields that $j = n$. Suppose first that the coefficient of y_n in $s_n \cdot \beta$ equals 1, and thus $s_n \cdot \beta \in \mathcal{E}_Y^{\{s_n\}}$; then

$$s_n \cdot \beta \succeq c_5x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n$$

by (6.111). If $\mu_{n-1} = c_5 + 1$, (6.58) implies that

$$\begin{aligned}\beta &\succeq s_n \cdot \big(c_5x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + y_n\big)\\ &= c_5x_1 + (c_5 + 1)(y_1 + \cdots + y_{n-1}) + c_5y_n\\ &= (s_{n-1} \cdots s_1) \cdot \big(c_5x_1 + c_5(y_1 + \cdots + y_n)\big),\end{aligned}$$

and it can be easily verified that this is preceded by $c_5x_1 + c_5(y_1 + \cdots + y_n)$. Assume now that $\mu_{n-1} \neq c_5 + 1$; then (6.111) gives that $n = 3$ and

$$s_3 \cdot \beta = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + y_3.$$

If $\alpha = \beta = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + (c_5 + 1)y_3$ there is nothing left to show, so suppose that $\alpha \neq \beta$. Since $\langle \beta, x_1 \rangle = 0$ and $\langle \beta, y_1 \rangle$ as well as $\langle \beta, y_3 \rangle$ are positive, we deduce that α must be a successor of $s_2 \cdot \beta$. Now

$$\begin{aligned} s_2 \cdot \beta &= (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (2c_5 + 1)y_2 + (c_5 + 1)y_3 \\ &= (s_1r_1s_2s_1r_1s_1s_2s_3) \cdot \big(c_5x_1 + c_5(y_1 + y_2 + y_3)\big), \end{aligned}$$

and it can be easily verified that this is preceded by $c_5(x_3 + x_2 + x_1) + c_5y_1$.

Finally, suppose that the coefficient of y_n in $s_n \cdot \beta$ equals 0. Then $\mu_n \geq c_5$ as $\beta \in \mathcal{E}^{\emptyset}_Y$; furthermore $0 = \mu_n - 2\langle \beta, y_n \rangle$, and as β cannot dominate y_n, this yields that $1 > \langle \beta, y_n \rangle \geq \frac{\mu_n}{2}$, and thus $\mu_n = c_5$ by (2.26). Since $s_n \cdot \beta$ is certainly an element of $\mathcal{E}^{\emptyset}_{Y \setminus \{s_n\}}$, induction yields that either

$$s_n \cdot \alpha \succeq c_5x_1 + c_5(y_1 + \cdots + y_{n-1}),$$

or $n = 4$ and $s_4 \cdot \beta = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 2)y_2 + (c_5 + 1)y_3$. The coefficient of y_{n-1} in $s_n \cdot \beta$ equals c_5, and so the latter is impossible, while the former case together with (6.58) yield that

$$\beta \succeq s_n \cdot \big(c_5x_1 + c_5(y_1 + \cdots + y_{n-1})\big) = c_5x_1 + c_5(y_1 + \cdots + y_n),$$

and this finishes the proof. □

Observe that (6.108) implies that $c_5x_1 + c_5(y_1 + \cdots + y_n)$ is an elementary root. For $\delta_1 = c_5x_1 + c_1y_1$ is clearly elementary, and if we define $\delta_j = s_j \cdot \delta_{j-1}$ for $j \in \{2, \ldots, n\}$, then an easy calculation yields that

$$\delta_j = c_5x_1 + c_5(y_1 + \cdots + y_j)$$

and $\langle \delta_{j-1}, y_j \rangle = -\frac{c_5}{2} \in (-1, 0)$, and thus $\delta_n = c_5x_1 + c_5(y_1 + \cdots + y_n) \in \mathcal{E}$ by (6.108).

Next let $\beta^{(1)}_1 = \delta_n$, and for $j \in \{1, \ldots, n-1\}$ define $\beta^{(1)}_{j+1} = s_j \cdot \beta^{(1)}_j$. Then a straightforward calculation yields that

$$\beta^{(1)}_j = c_5x_1 + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n),$$

and $\langle \beta^{(1)}_j, y_j \rangle = -\frac{1}{2} \in (-1, 0)$; whence (6.108) implies that $\beta^{(1)}_j \in \mathcal{E}$ for all $j \in \{1, \ldots, n\}$.

Note also that $\beta^{(1)}_1, \ldots, \beta^{(1)}_n$ are the only elements of $\mathcal{E}^{\emptyset}_{\{r_1\} \cup \{s_1, \ldots, s_n\}}$ with coefficient c_5 for x_1, and this enables us to prove the next result.

(6.114) PROPOSITION *Suppose that $m = 5$ and $|Y_1|$, $|Y_2| \geq 2$. Then $\mathcal{E}_Y^{\emptyset}$ is the set of*

$$c_5\alpha + (c_5+1)\big(x_{i-1} + \cdots + x_1\big) + c_5\beta$$

with $i \in \{1,\ldots,l\}$, $\alpha \in \mathcal{E}_{Y_1 \setminus \{r_1,\ldots,r_{i-1}\}}$ with coefficient 1 for x_i, and $\beta \in \mathcal{E}_{Y_2}$ with coefficient 1 for y_1, and

$$c_5\alpha + (c_5+1)(y_1 + \cdots + y_{j-1}) + c_5\beta$$

with $j \in \{1,\ldots,n\}$, $\alpha \in \mathcal{E}_{Y_1}$ with coefficient 1 for x_1 and $\beta \in \mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j. Hence

$$\begin{aligned}
\big|\mathcal{E}_Y^{\emptyset}\big| = &\Big(\sum_{J \subseteq Y_1 \setminus \{r_1\}} \big|\mathcal{E}_{Y_1\}}^{J\cup\{r_1\}}\big|\Big) \times \Big(\sum_{K \subseteq Y_2 \setminus \{s_1\}} \big|\mathcal{E}_{Y_2}^{\{s_1\}\cup K}\big|\Big) \\
&+ \Big(\sum_{i=2}^{l} \sum_{J \subseteq Y_1 \setminus \{r_1,\ldots,r_i\}} \big|\mathcal{E}_{Y_1 \setminus \{r_{i-1},\ldots,r_1\}}^{J\cup\{r_i\}}\big|\Big) \times \Big(\sum_{K \subseteq Y_2 \setminus \{s_1\}} \big|\mathcal{E}_{Y_2}^{\{s_1\}\cup K}\big|\Big) \\
&+ \Big(\sum_{J \subseteq Y_1 \setminus \{r_1\}} \big|\mathcal{E}_{Y_1\}}^{J\cup\{r_1\}}\big|\Big) \times \Big(\sum_{j=2}^{n} \sum_{K \subseteq Y_2 \setminus \{s_1,\ldots,s_j\}} \big|\mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}^{\{s_j\}\cup K}\big|\Big).
\end{aligned}$$

Proof. First, let $\alpha \in \mathcal{E}_{Y_1}$ with coefficient 1 for x_1, $j \in \{1,\ldots,n\}$ and β in $\mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j. Since

$$c_5x_1 + (c_5+1)(y_1 + \cdots + y_{j-1}) + c_5y_j$$

is in $\mathcal{E}_{\{r_1,s_1,\ldots,s_j\}}$ by the remark preceding this proposition, (6.97) yields that

$$c_5\alpha + (c_5+1)(y_1 + \cdots + y_{j-1}) + c_5y_j$$

is an element of $\mathcal{E}_{Y_1 \cup \{s_1,\ldots,s_j\}}$, and a repeated application of (6.97) implies that

$$c_5\alpha + (c_5+1)(y_1 + \cdots + y_{j-1}) + c_5\beta$$

is an element of $\mathcal{E}_Y$, and hence clearly in $\mathcal{E}_Y^{\emptyset}$. Symmetrical arguments apply for $i \in \{1,\ldots,l\}$, $\alpha \in \mathcal{E}_{Y_1 \setminus \{r_1,\ldots,r_{i-1}\}}$ with coefficient 1 for x_i, and $\beta \in \mathcal{E}_{Y_2}$ with coefficient 1 for y_1; hence it remains to show that all elements of $\mathcal{E}_Y^{\emptyset}$ can be obtained in this way.

Let $\gamma \in \mathcal{E}_Y^{\emptyset}$. Since $|Y_1|, |Y_2| \geq 2$, Lemma (6.109) gives that the coefficient of x_1 or y_1 in γ equals c_5, and by symmetry we may assume without loss of generality that the coefficient of x_1 in γ equals c_5. Then

$$\gamma = c_5\alpha + \beta - c_5 x_1$$

for some $\alpha \in \mathcal{E}_{Y_1}$ with coefficient 1 for x_1, and $\beta \in \mathcal{E}_{\{r_1\}\cup Y_2}$ with coefficient c_5 for x_1.

If $|Y_2| = n$, the remark preceding ths proposition yields that β equals $\beta_j^{(1)}$ for some $j \in \{1, \ldots, n\}$, and thus

$$\gamma = c_5\alpha + (c_5+1)(y_1 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n);$$

as $y_j + \cdots + y_n$ is certainly an element of $\mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j, it follows that γ is of the required form.

Suppose next that $|Y_1| > n$. Then (6.99) yields that

$$\beta = \beta_1 + x_5\beta_2 - c_5 y_j$$

for some $\beta_1 \in \mathcal{E}_{\{r_1,s_1,\ldots,s_j\}}$ with coefficient c_5 for y_j, and coefficient greater than or equal to 2 for $y_1, \ldots, y_{j-1}$, and $\beta_2 \in \mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ with coefficient 1 for y_j. Since the coefficient of x_1 in β_1 must equal c_5, and the coefficients of $y_1, \ldots, y_{j-1}$ in β have to be greater than or equal to 2, we find that

$$\beta_1 = c_5 x_1 + (c_5+1)(y_1 + \cdots + y_{j-1}) + c_5 y_j.$$

So

$$\gamma = c_5\alpha + (c_5+1)(y_1 + \cdots + y_{j-1}) + c_5\beta_2,$$

as required. $\square$

(6.115) COROLLARY *Suppose that $m = 5$, $|Y_1| = l$ and $|Y_2| = n$. Then $\mathcal{E}_Y^{\emptyset}$ is the set of*

$$c_5\big(x_l + \cdots + x_i\big) + (c_5+1)\big(x_{i-1} + \cdots + x_1\big) + c_5\big(y_1 + \cdots + y_n\big)$$

with $i \in \{1, \ldots, l\}$ and

$$c_5\big(x_l + \cdots + x_1\big) + (c_5+1)\big(y_1 + \cdots + y_{j-1}\big) + c_5\big(y_j + \cdots + y_n\big)$$

with $j \in \{1, \ldots, n\}$. *Hence* $|\mathcal{E}_Y^{\emptyset}| = l + n - 1$.

(6.116) Corollary *Suppose* $m = 5$, $|Y_1| = l$ *and* $|Y_2| > n$. *Then* $\mathcal{E}_Y^{\emptyset}$ *is the set of*

$$c_5\big(x_l + \cdots + x_i\big) + (c_5+1)\big(x_{i-1} + \cdots + x_1\big) + c_5\beta$$

with $i \in \{1, \ldots, l\}$ *and* $\beta \in \mathcal{E}_{Y_2}$ *with coefficient 1 for* y_1, *and*

$$c_5\big(x_l + \cdots + x_1\big) + (c_5+1)\big(y_1 + \cdots + y_{j-1}\big) + c_5\beta$$

with $j \in \{2, \ldots, n\}$ *and* $\beta \in \mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}$ *with coefficient 1 for* y_j. *Hence*

$$|\mathcal{E}_Y^{\emptyset}| = l \times \sum_{K \subseteq Y_2 \setminus \{s_1\}} \big|\mathcal{E}_{Y_2}^{\{s_1\} \cup K}\big| + \sum_{j=2}^{n} \sum_{K \subseteq Y_2 \setminus \{s_1, \ldots, s_j\}} \big|\mathcal{E}_{Y_2 \setminus \{s_1, \ldots, s_{j-1}\}}^{\{s_j\} \cup K}\big|.$$

Assume from now on that $|Y_1| = 1$. It is a tedious but finite task to verify that the next lemma lists all the roots in Φ_Y preceded by

$$c_5x_1 + c_5(y_1 + \cdots + y_n),$$

or equal to $(2c_5+1)x_1 + (2c_5+2)y_1 + (c_5+2)y_2 + (c_5+1)y_3$ for $|Y_2| = n \leq 3$, and that these are elementary roots.

(6.117) Lemma *Suppose that* $m = 5$, $|Y_1| = 1$ *and* $|Y_2| = n$.

(i) If $n = 1$, *then* $\mathcal{E}_Y^{\emptyset} = \{c_5x_1 + c_5y_1\}$.

(ii) If $|Y_2| = n = 2$, *then*

$$\begin{aligned}\mathcal{E}_Y^{\emptyset} = &\big\{c_5x_1 + c_5(y_1 + y_2), c_5x_1 + (c_5+1)y_1 + c_5y_2,\\ &(c_5+1)x_1 + (c_5+1)y_1 + c_5y_2,\ (c_5+1)x_1 + 2c_5y_1 + c_5y_2\big\}.\end{aligned}$$

(iii) If $|Y_2| = n = 3$,

$$(2c_5+1)x_1 + (2c_5+2)y_1 + (c_5+2)y_2 + (c_5+1)y_3$$

is an element of $\mathcal{E}_Y^{\emptyset}$, *and we denote this by means of the following diagram:*

$$\underset{2c+1}{\bullet}\overset{5}{\text{------}}\underset{2c+2}{\bullet}\text{------}\underset{c+2}{\bullet}\text{------}\underset{c+1}{\bullet}$$

The remaining elements of $\mathcal{E}_Y^{\emptyset}$ are represented by the following diagrams:

$$\underset{c}{\bullet}\overset{5}{\text{------}}\underset{c}{\bullet}\text{------}\underset{c}{\bullet}\text{------}\underset{c}{\bullet}$$

$$\underset{c}{\bullet}\overset{5}{\text{------}}\underset{c+1}{\bullet}\text{------}\underset{c}{\bullet}\text{------}\underset{c}{\bullet}$$

$$\underset{c+1}{\bullet}\overset{5}{\text{------}}\underset{c+1}{\bullet}\text{------}\underset{c}{\bullet}\text{------}\underset{c}{\bullet} \qquad \underset{c}{\bullet}\overset{5}{\text{------}}\underset{c+1}{\bullet}\text{------}\underset{c+1}{\bullet}\text{------}\underset{c}{\bullet}$$

$$\underset{c+1}{\bullet}\overset{5}{\text{------}}\underset{2c}{\bullet}\text{------}\underset{c}{\bullet}\text{------}\underset{c}{\bullet} \qquad \underset{c+1}{\bullet}\overset{5}{\text{------}}\underset{c+1}{\bullet}\text{------}\underset{c+1}{\bullet}\text{------}\underset{c}{\bullet}$$

$$\underset{c+1}{\bullet}\overset{5}{\text{------}}\underset{2c}{\bullet}\text{------}\underset{2c}{\bullet}\text{------}\underset{c}{\bullet} \qquad \underset{c+1}{\bullet}\overset{5}{\text{------}}\underset{2c+1}{\bullet}\text{------}\underset{c+1}{\bullet}\text{------}\underset{c}{\bullet}$$

$$\underset{c+1}{\bullet}\overset{5}{\text{------}}\underset{2c+1}{\bullet}\text{------}\underset{2c}{\bullet}\text{------}\underset{c}{\bullet} \qquad \underset{2c+1}{\bullet}\overset{5}{\text{------}}\underset{2c+1}{\bullet}\text{------}\underset{c+1}{\bullet}\text{------}\underset{c}{\bullet}$$

$$\underset{2c+1}{\bullet}\overset{5}{\text{------}}\underset{2c+1}{\bullet}\text{------}\underset{2c}{\bullet}\text{------}\underset{c}{\bullet} \qquad \underset{2c+1}{\bullet}\overset{5}{\text{------}}\underset{2c+2}{\bullet}\text{------}\underset{c+1}{\bullet}\text{------}\underset{c}{\bullet}$$

$$\underset{2c+1}{\bullet}\overset{5}{\text{------}}\underset{3c+1}{\bullet}\text{------}\underset{2c}{\bullet}\text{------}\underset{c}{\bullet} \qquad \underset{2c+1}{\bullet}\overset{5}{\text{------}}\underset{2c+2}{\bullet}\text{------}\underset{2c+1}{\bullet}\text{------}\underset{c}{\bullet}$$

$$2c+2 \overset{5}{\text{———}} 3c+1 \text{———} 2c \text{———} c$$

$$2c+1 \overset{5}{\text{———}} 2c+2 \text{———} 2c+1 \text{———} c+1$$

$$2c+1 \overset{5}{\text{———}} 3c+1 \text{———} 2c+1 \text{———} c$$

$$2c+2 \overset{5}{\text{———}} 3c+1 \text{———} 2c+1 \text{———} c$$

$$2c+1 \overset{5}{\text{———}} 3c+1 \text{———} 2c+1 \text{———} c+1$$

$$2c+2 \overset{5}{\text{———}} 3c+2 \text{———} 2c+1 \text{———} c$$

$$2c+2 \overset{5}{\text{———}} 3c+1 \text{———} 2c+1 \text{———} c+1$$

$$3c+1 \overset{5}{\text{———}} 3c+2 \text{———} 2c+1 \text{———} c$$

$$2c+2 \overset{5}{\text{———}} 3c+2 \text{———} 2c+1 \text{———} c+1$$

$$3c+1 \overset{5}{\text{———}} 3c+2 \text{———} 2c+1 \text{———} c+1$$

$$2c+2 \overset{5}{\text{———}} 3c+2 \text{———} 2c+2 \text{———} c+1$$

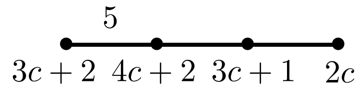
$$3c+1 \overset{5}{\text{———}} 3c+2 \text{———} 2c+2 \text{———} c+1$$

$$3c+1 \overset{5}{\text{———}} 3c+3 \text{———} 2c+2 \text{———} c+1$$

$$3c+2 \overset{5}{\text{———}} 3c+3 \text{———} 2c+2 \text{———} c+1$$

$$3c+2 \overset{5}{\text{———}} 4c+2 \text{———} 2c+2 \text{———} c+1$$

$$3c+2 \overset{5}{\text{———}} 4c+2 \text{———} 3c+1 \text{———} c+1$$



Hence

$$|\mathcal{E}_Y^{\emptyset}| = \begin{cases} 1 & \textit{if } n = 1, \\ 4 & \textit{if } n = 2, \\ 32 & \textit{if } n = 3. \end{cases}$$

(6.118) Proposition *Suppose $m = 5$, $|Y_1| = 1$ and $|Y_2| = n \geq 4$. Then $\mathcal{E}_Y^{\emptyset}$ consists of the following roots*

$$\beta_j^{(1)} = c_5 x_1 + (c_5 + 1)\big(y_1 + \cdots + y_{j-1}\big) + c_5\big(y_j + \cdots + y_n\big),$$

with $1 \leq j \leq n$,

$$\begin{aligned} \beta_{j,k}^{(2)} = (c_5 + 1)x_1 &+ (2c_5 + 1)\big(y_1 + \cdots + y_{k-1}\big) \\ &+ (c_5 + 1)\big(y_k + \cdots + y_{j-1}\big) + c_5\big(y_j + \cdots + y_n\big), \end{aligned}$$

with $1 \leq k < j \leq n$,

$$\begin{aligned} \beta_{j,k}^{(3)} = (2c_5 + 1)x_1 &+ (2c_5 + 1)\big(y_1 + \cdots + y_{k-1}\big) \\ &+ (c_5 + 1)\big(y_k + \cdots + y_{j-1}\big) + c_5\big(y_j + \cdots + y_n\big), \end{aligned}$$

with $2 \leq k < j \leq n$,

$$\beta_j^{(4)} = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 1)\big(y_2 + \cdots + y_{j-1}\big) + c_5\big(y_j + \cdots + y_n\big),$$

with $3 \leq j \leq n$,

$$\begin{aligned} \beta_{j,k}^{(5)} = (c_5 + 1)x_1 &+ (2c_5 + 1)\big(y_1 + \cdots + y_{k-1}\big) \\ &+ 2c_5\big(y_k + \cdots + y_{j-1}\big) + c_5\big(y_j + \cdots + y_n\big), \end{aligned}$$

with $1 \leq k < j \leq n$,

$$\begin{aligned} \beta_{j,k}^{(6)} = (2c_5 + 1)x_1 &+ (2c_5 + 1)\big(y_1 + \cdots + y_{k-1}\big) \\ &+ 2c_5\big(y_k + \cdots + y_{j-1}\big) + c_5\big(y_j + \cdots + y_n\big), \end{aligned}$$

with $2 \leq k < j \leq n$,

$$\beta_j^{(7)} = (2c_5+1)x_1 + (3c_5+1)y_1 + 2c_5\big(y_2+\cdots+y_{j-1}\big) + c_5\big(y_j+\cdots+y_n\big),$$

with $3 \leq j \leq n$,

$$\beta_j^{(8)} = (2c_5+2)x_1 + (3c_5+1)y_1 + 2c_5\big(y_2+\cdots+y_{j-1}\big) + c_5\big(y_j+\cdots+y_n\big),$$

for $3 \leq j \leq n$, *and*

$$\beta_i^{(9)} = \alpha_i + c_5\big(y_4+\cdots+y_n\big),$$

where $i \in \{1,\ldots,5\}$ *and*

$$\alpha_1 = (2c_5+1)x_1 + (2c_5+2)y_1 + (2c_5+1)y_2 + c_5y_3,$$

$$\alpha_2 = (2c_5+1)x_1 + (3c_5+1)y_1 + (2c_5+1)y_2 + c_5y_3,$$

$$\alpha_3 = (2c_5+2)x_1 + (3c_5+1)y_1 + (2c_5+1)y_2 + c_5y_3,$$

$$\alpha_4 = (2c_5+1)x_1 + (3c_5+2)y_1 + (2c_5+1)y_2 + c_5y_3,$$

$$\alpha_5 = (3c_5+1)x_1 + (3c_5+2)y_1 + (2c_5+1)y_2 + c_5y_3.$$

Hence $|\mathcal{E}_Y^{\emptyset}| = 2n^2+1$.

Proof. We show first that the above listed vectors are elementary roots, and it follows trivially that they are in $\mathcal{E}_Y^{\emptyset}$. We saw before that $\beta_j^{(1)}$ is elementary for $j \in \{1,\ldots,n\}$. Since $\beta_{j,1}^{(2)} = r_1 \cdot \beta_j^{(1)}$ and $\langle \beta_j^{(1)}, x_1 \rangle = -\frac{1}{2} \in (-1,0)$ for $j \in \{2,\ldots,n\}$, it follows by (3.37) that $\beta_{j,1}^{(2)}$ is elementary. Furthermore, $\beta_{j,k+1}^{(2)} = s_k \cdot \beta_{j,k}^{(2)}$ and $\langle \beta_{j,k}^{(2)}, y_k \rangle = -\frac{c_5}{2}$ for $k \in \{1,\ldots,j-2\}$, and thus (6.108) implies that $\beta_{j,k}^{(2)}$ is elementary for $k \in \{1,\ldots,j-1\}$, as required.

If $j \in \{3,\ldots,n\}$ and $k \in \{2,\ldots,j-1\}$, then $\beta_{j,k}^{(3)} = r_1 \cdot \beta_{j,k}^{(2)}$ and $\langle \beta_{j,k}^{(2)}, x_1 \rangle = -\frac{c_5}{2}$, and it follows by (3.37) that $\beta_{j,k}^{(3)} \in \mathcal{E}$. Next, $\beta_j^{(4)} = s_1 \cdot \beta_{2,j}^{(3)}$ and $\langle \beta_{2,j}^{(3)}, y_1 \rangle = -\frac{1}{2}$ for $j \in \{3,\ldots,n\}$; therefore $\beta_j^{(4)}$ is elementary for all $j \in \{3,\ldots,n\}$ by (3.37).

Clearly $\beta^{(5)}_{2,1} = s_1 \cdot \beta^{(2)}_{2,1}$ and $\langle \beta^{(2)}_{2,1}, y_1 \rangle = \frac{1-c_5}{2}$, and (3.37) yields that $\beta^{(5)}_{2,1}$ is elementary. For $j \in \{2, \dots, n-1\}$ we find that $\beta^{(5)}_{j+1,1} = s_j \cdot \beta^{(5)}_{j,1}$ and $\langle \beta^{(5)}_{j,1}, y_j \rangle = \frac{1-c_5}{2}$; hence $\beta^{(5)}_{j,1}$ is elementary for all $j \in \{2, \dots, n\}$ by (6.108). Further, $\beta^{(5)}_{j,k+1} = s_k \cdot \beta^{(2)}_{j,k}$ and $\langle \beta^{(2)}_{j,k}, y_k \rangle = -\frac{1}{2}$ for $k \in \{1, \dots, j-2\}$, therefore $\beta^{(5)}_{j,k}$ is elementary for all $k \in \{1, \dots, j-1\}$ by (6.108).

If $j \in \{3, \dots, n\}$ and $k \in \{2, \dots, j-1\}$, then $\beta^{(6)}_{j,k} = r_1 \cdot \beta^{(5)}_{j,k}$ and $\langle \beta^{(5)}_{j,k}, x_1 \rangle = -\frac{c_5}{2}$, and it follows by (3.37) that $\beta^{(6)}_{j,k}$ is elementary.

Next, $\beta^{(7)}_{j} = s_1 \cdot \beta^{(6)}_{j,2}$ and $\langle \beta^{(6)}_{j,2}, y_1 \rangle = -\frac{c_5}{2}$ for $j \in \{3, \dots, n\}$, and thus $\beta^{(7)}_{j} \in \mathcal{E}$ by (3.37). For $j \in \{3, \dots, n\}$ also $\beta^{(8)}_{j} = r_1 \cdot \beta^{(7)}_{j}$, and $\langle \beta^{(7)}_{j}, y_1 \rangle = -\frac{1}{2}$, and hence $\beta^{(8)}_{j}$ is clearly elementary by (3.37).

Finally, since $y_3 + \cdots + y_n$ is in $\mathcal{E}_{\{s_3, \dots, s_n\}}$ with coefficient 1 for y_3, and $\alpha_i \in \mathcal{E}_{\{r_1, s_1, s_2, s_3\}}$ by (6.117)(iii) with coefficient c_5 for y_3, Proposition (6.97) yields furthermore that $\beta^{(9)}_{i}$ is elementary for $i \in \{1, \dots, 5\}$.

It remains to show that all elements of $\mathcal{E}^{\emptyset}_{Y}$ have been listed. Suppose $\alpha \in \mathcal{E}^{\emptyset}_{Y}$; then $\alpha \succeq \beta^{(1)}_{1}$ by (6.113) as $n \geq 4$. If $\alpha = \beta^{(1)}_{1}$, there is nothing left to show. So assume next that $\alpha \succ \beta^{(1)}_{1}$, and proceed by induction. Let $r \in Y$ with $\alpha \succ r \cdot \alpha \succeq \beta^{(1)}_{1}$; then $\langle r \cdot \alpha, \alpha_r \rangle \in (-1, 0)$, as α is elementary. Furthermore, $r \cdot \alpha$ is elementary, and since $r \cdot \alpha \succeq \beta^{(1)}_{1}$ clearly $r \cdot \alpha \in \mathcal{E}^{\emptyset}_{Y}$. By induction this gives rise to the following cases:

Case 1: $r \cdot \alpha = c_5 x_1 + (c_5 + 1)(y_1 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$ for some $j \in \{1, \dots, n\}$; then

(i) $r = r_1$ and $j \geq 2$, and thus $\alpha = \beta^{(2)}_{j,1}$, or

(ii) $r = s_j$ and $j \leq n-1$, hence $\alpha = \beta^{(1)}_{j+1}$.

Case 2:
$$r \cdot \alpha = (c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + (c_5 + 1)(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$$
for some $1 \leq k < j \leq n$; then

(i) $r = r_1$ and $k \geq 2$, and hence $\alpha = \beta^{(3)}_{j,k}$, or

(ii) $r = s_1$ with $j = 2$ and $k = 1$, and $\alpha = \beta^{(5)}_{2,1}$, or

(iii) $r = s_k$ with $k \leq j-2$, and $\alpha = \beta^{(2)}_{j,k+1}$, or

(iv) $r = s_j$ with $j < n$, and thus $\alpha = \beta^{(2)}_{j+1,k}$.

Case 3: $r \cdot \alpha = (2c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + (c_5 + 1)(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$ for some $2 \leq k < j \leq n$; then

(i) $r = s_1$ and $k = 2$, and thus $\alpha = \beta^{(4)}_j$, or

(ii) $r = s_k$ and $k \leq j - 2$, and $\alpha = \beta^{(3)}_{j,k+1}$, or

(iii) $r = s_j$ and $j < n$, and thus $\alpha = \beta^{(3)}_{j+1,k}$.

Case 4: $r \cdot \alpha = (2c_5 + 1)x_1 + (2c_5 + 2)y_1 + (c_5 + 1)(y_2 + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$, for some $j \in \{3, \ldots, n\}$; then $r = s_2$ with $j = 3$, and $\alpha = \beta^{(9)}_1$.

Case 5: $r \cdot \alpha = (c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + 2c_5(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$ for some $1 \leq k < j \leq n$; then

(i) $r = r_1$ and $k \geq 2$, and hence $\alpha = \beta^{(6)}_{j,k}$, or

(ii) $r = s_k$ with $k \leq j - 2$, and $\alpha = \beta^{(5)}_{j,k+1}$, or

(iii) $r = s_j$ and $j < n$, and thus $\alpha = \beta^{(5)}_{j+1,k}$.

Case 6: $r \cdot \alpha = (2c_5 + 1)x_1 + (2c_5 + 1)(y_1 + \cdots + y_{k-1}) + 2c_5(y_k + \cdots + y_{j-1}) + c_5(y_j + \cdots + y_n)$ for some $1 \leq k < j \leq n$,; then

(i) $r = s_1$ and $k = 1$, and $\alpha = \beta^{(7)}_j$, or

(ii) $r = s_k$ with $k \leq j - 2$, and $\alpha = \beta^{(6)}_{j,k+1}$, or

(iii) $r = s_j$ and $j < n$, and thus $\alpha = \beta^{(6)}_{j+1,k}$.

Case 7: $r \cdot \alpha = (2c_5+1)x_1+(3c_5+1)y_1+2c_5(y_2+\cdots+y_{j-1})+c_5(y_j+\cdots+y_n)$, for some $j \in \{3, \ldots, n\}$; then

(i) $r = r_1$ and $\alpha = \beta^{(8)}_j$, or

(ii) $r = s_j$ and $j < n$, and hence $\alpha = \beta^{(7)}_{j+1}$.

Case 8: $r\cdot\alpha = (2c_5+2)x_1+(3c_5+1)y_1+2c_5(y_2+\cdots+y_{j-1})+c_5(y_j+\cdots+y_n)$ for some $j\in\{3,\ldots,n\}$; then $r=s_2$ and $j=3$, and $\alpha=\beta_3^{(9)}$.

Case 9: $r\cdot\alpha=\alpha_i+c_5(y_4+\cdots+y_n)$ for some $i\in\{1,\ldots,5\}$; then

(i) $i=1$, $r=s_1$ and $\alpha=\beta_2^{(9)}$, or

(ii) $i=2$, $r=r_1$ and $\alpha=\beta_3^{(9)}$, or

(iii) $i=3$, $r=s_1$ and $\alpha=\beta_4^{(9)}$, or

(iv) $i=4$, $r=r_1$ and $\alpha=\beta_5^{(9)}$,

and this completes the proof. □

Lemma (6.99) now yields the following.

(6.119) PROPOSITION *Suppose $m=5$, $|Y_1|=1$ and $|Y_2|>n$.*

(i) If $n=1$, the elements of $\mathcal{E}_Y^{\emptyset}$ are

$$c_5x_1+c_5\alpha$$

with $\alpha\in\mathcal{E}_{Y_2}$ with coefficient 1 for y_1.

(ii) If $n=2$, the elements of $\mathcal{E}_Y^{\emptyset}$ are the roots in (i) plus, additionally,

$$\beta-c_5y_2+c_5\alpha,$$

with $\alpha\in\mathcal{E}_{Y_2\setminus\{s_1\}}$ with coefficient 1 for y_2, and β equal to one of the 3 roots in (6.117)(ii) with coefficient c_5 for y_2 and coefficient greater than or equal to 2 for y_1.

(iii) If $n=3$, the elements of $\mathcal{E}_Y^{\emptyset}$ are the roots named in (ii), plus, additionally, the roots

$$\beta+c_5\alpha-c_5y_3$$

with $\alpha\in\mathcal{E}_{Y_2\setminus\{s_1,s_2\}}$ with coefficient 1 for y_3, and β equal to one of the 15 roots in (6.117)(iii) with coefficient c_5 for y_3 and coefficient greater than or equal to 2 for y_1 and y_2.

(iv) If $n\geq 4$, the elements of $\mathcal{E}_Y^{\emptyset}$ are the roots named in (iii), plus, additionally,

$$\beta-c_5(y_j+\cdots+y_n)+c_5\alpha,$$

with $j \in \{4, \ldots, n\}$, $\alpha \in \mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}$ *with coefficient 1 for* y_j, *and* β *equal to* $\beta_j^{(1)}$, $\beta_j^{(4)}$, $\beta_j^{(7)}$, $\beta_j^{(8)}$, *or* $\beta_{k,j}^{(2)}$, $\beta_{k,j}^{(5)}$ *with* $k \in \{1, \ldots, j-1\}$, *or* $\beta_{k,j}^{(3)}$, $\beta_{k,j}^{(6)}$ *with* $k \in \{2, \ldots, j-1\}$ *as defined in (6.118).*

Hence

$$|\mathcal{E}_Y^{\emptyset}| = \sum_{j=1}^{n} M(j) \sum_{J \subseteq Y_2 \setminus \{s_1,\ldots,s_j\}} \left|\mathcal{E}_{Y_2 \setminus \{s_1,\ldots,s_{j-1}\}}^{\{s_j\} \cup J}\right|,$$

with

$$M(j) = \begin{cases} 1 & \textit{if } j = 1, \\ 3 & \textit{if } j = 2, \\ 15 & \textit{if } j = 3, \\ 4j - 2 & \textit{if } j \geq 4. \end{cases}$$

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