Root systems and reflection representations of Coxeter groups

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Preface

The aim of this thesis is to investigate two topics relating to geometric realizations of Coxeter groups. The first is a class of non-orthogonal geometric realizations of Coxeter groups and the second is the study of dominance behaviour in the root systems associated with Coxeter groups extending those examined in [6] and [5].

For arbitrary Coxeter system \((W, R)\), it is well known that \(W\) can be embedded into the orthogonal group of certain bilinear form \(( , )\) on a real vector space \(V\). The root system of \(W\) in \(V\) is a certain \(W\)-stable subset of \(V\) corresponding to the set of reflections in \(W\). If the bilinear form \(( , )\) is non-degenerate then of course \(V\) is \(W\)-isomorphic to its dual \(V^*\), but since the form is not always non-degenerate it is sometimes useful to study both the representation of \(W\) on \(V\) and the contragredient representation on \(V^*\). This motivates the slightly more general approach taken in this thesis, in which we consider a pair of real vector spaces \(V_1\) and \(V_2\) linked by a \(W\)-equivariant bilinear pairing satisfying a few extra conditions (which are guaranteed to hold in the case that \(V_1 = V\) and \(V_2 = V^*\)). We show that in this situation \(W\) embeds (faithfully) in the general linear groups of each of \(V_1\) and \(V_2\), both images being generated by reflections. The classical theory of geometric realizations can be recovered as a special case. We define and study generalized root systems arising in such non-orthogonal geometric realizations of Coxeter systems, and compare them with the root systems arising from the standard geometric realizations. It turns out that it is natural to consider root systems in both \(V_1\) and \(V_2\); these are in \(W\)-equivariant bijective correspondence with each other and with the root system in the classical setting. Familiar properties of simple roots and
positive roots generalize to the non-orthogonal case, although it is no longer necessarily true that the only scalar multiples of a root are itself and its negative.

The investigation of non-orthogonal geometric realizations will occupy Chapter 1 and Chapter 2.

In Chapter 3 we study a partial order in the root systems of Coxeter groups called dominance introduced by B. Brink and R. Howlett. We first examine the dominance of roots in the classical geometric realizations and then generalize these results to the non-orthogonal setting. In [6] and [5] dominance is only defined on the positive roots, but the definition extends naturally to all roots, and it turns out that the geometric characterization of dominance between roots remains unchanged. In [6], it is found that for all finite rank Coxeter groups, the set of positive roots dominating no positive roots other than themselves is finite, and in [5] such sets are explicitly computed. In this thesis we prove that for all natural numbers $n$, the set of roots dominating precisely $n$ positive roots (denoted by $D_n$) is finite for all finite rank Coxeter groups. The set of positive roots is obviously the disjoint union of all these $D_n$'s, and we examine the interaction of this decomposition with the action of $W$. For all infinite Coxeter groups of finite rank and for each $n$, it turns out that $D_n \neq \emptyset$, and we also compute an upper bound of $|D_n|$ (the size of $D_n$). In the classical case we study certain cones related to the Tits cone, and show how they are related to dominance.
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Also, due acknowledgements must be made to my family members. In particular, I wish to thank my mother for her constant love, patience and support. Finally, I would like to dedicate this thesis to my late grandfather Prof Li, Huairen who passed away in 2005, as well as to my
late grandmother Prof Yang, Fengqing and my late father Fu, Zhaolei, both left me in 2008...
Statement of originality

This thesis contains no new material which has been accepted for the award of any other degree or diploma. All work in this thesis, except where duly acknowledged otherwise, is believed to be original.
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Chapter 1

Non-orthogonal Geometric Realizations of Coxeter Groups

1.1. Paired reflection representations

Let $S$ be an arbitrary set and suppose that each unordered pair \{s, t\} of elements in $S$ is assigned an $m_{st} \in \mathbb{Z} \cup \{\infty\}$, subject to the conditions that $m_{ss} = 1$ for all $s$ in $S$ and $m_{st} \geq 2$ for all distinct $s, t$ in $S$. Suppose that $V_1$ and $V_2$ are vector spaces over the real field $\mathbb{R}$, and suppose that there exists a bilinear map $\langle \ , \ \rangle : V_1 \times V_2 \rightarrow \mathbb{R}$ and sets $\Pi_1 = \{ \alpha_s \mid s \in S \} \subseteq V_1$ and $\Pi_2 = \{ \beta_s \mid s \in S \} \subseteq V_2$ such that the following conditions hold:

1. $\Pi_1$ spans $V_1$ and $\Pi_2$ spans $V_2$;
2. $\langle \alpha_s, \beta_s \rangle = 1$, for all $s \in S$;
3. $\langle \alpha_s, \beta_t \rangle \leq 0$, for all distinct $s, t \in S$;
4. for all $s, t \in S$,
   \[
   \langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle = \begin{cases} 
   \cos^2(\pi/m_{st}) & \text{if } m_{st} \neq \infty, \\
   \gamma^2, \text{ for some } \gamma \geq 1 & \text{if } m_{st} = \infty; 
   \end{cases}
   \]
5. for all $s, t \in S$, $\langle \alpha_s, \beta_t \rangle = 0$ if and only if $\langle \alpha_t, \beta_s \rangle = 0$;
6. $\sum_{s \in S} \lambda_s \alpha_s = 0$ with $\lambda_s \geq 0$ for all $s$ implies $\lambda_s = 0$ for all $s$, and $\sum_{s \in S} \lambda_s \beta_s = 0$ with $\lambda_s \geq 0$ for all $s$ implies $\lambda_s = 0$ for all $s$.

Note that (C4) and (C5) combined imply that $\langle \alpha_s, \beta_t \rangle$ and $\langle \alpha_t, \beta_s \rangle$ are zero if and only if $m_{st} = 2$. We can also express (C6) more compactly as $0 \notin \text{PLC}(\Pi_1)$ and $0 \notin \text{PLC}(\Pi_2)$, where PLC($A$) (the positive linear combinations of $A$) is defined to be

\[
\{ \sum_{a \in A} \lambda_a a \mid \lambda_a \geq 0 \text{ for all } a \in A, \text{ and } \lambda_{a'} > 0 \text{ for some } a' \in A \}.
\]
Definition 1.1.1. In the above situation, if conditions (C1) to (C6) are satisfied then we call $C = (S, V_1, V_2, \Pi_1, \Pi_2, \langle \cdot, \cdot \rangle)$ a Coxeter datum. The $m_{st}$ (for $s, t \in S$) are called the Coxeter parameters of $C$.

Throughout this chapter, $C = (S, V_1, V_2, \Pi_1, \Pi_2, \langle \cdot, \cdot \rangle)$ will be a fixed Coxeter datum with Coxeter parameters $m_{st}$.

Definition 1.1.2. For each $s \in S$ let $\rho_{V_1}(s)$ and $\rho_{V_2}(s)$ be the linear transformations on $V_1$ and $V_2$ defined by

$$\rho_{V_1}(s)(x) = x - 2\langle x, \beta_s \rangle \alpha_s$$

for all $x \in V_1$, and

$$\rho_{V_2}(s)(y) = y - 2\langle \alpha_s, y \rangle \beta_s$$

for all $y \in V_2$. For each $i \in \{1, 2\}$ let $R_i := \{\rho_{V_i}(s) \mid s \in S\}$, and let $W_i$ be the subgroup of $\text{GL}(V_i)$ generated by $R_i$.

Since $\langle \alpha_s, \beta_s \rangle = 1$ for all $s \in S$ (by (C2) above), we find that $\rho_{V_1}(s)(\alpha_s) = -\alpha_s$ and $\rho_{V_2}(s)(\beta_s) = -\beta_s$. It follows that for all $v \in V_1$, $\rho_{V_1}(s)\rho_{V_1}(s)(v) = \rho_{V_1}(s)(v - 2\langle v, \beta_s \rangle \alpha_s) = v - 2\langle v, \beta_s \rangle \alpha_s + 2\langle v, \beta_s \rangle \alpha_s = v$ showing that $\rho_{V_1}(s)$ is an involution. Similarly, $\rho_{V_2}(s)$ is also an involution.

The following result follows readily from these definitions:

Proposition 1.1.3. Let $x \in V_1$ and $y \in V_2$. Then for all $s \in S$,

$$\langle \rho_{V_1}(s)(x), \rho_{V_2}(s)(y) \rangle = \langle x, y \rangle.$$

Proof. By Definition 1.1.2 and the bilinearity of $\langle \cdot, \cdot \rangle$, we find that

$$\langle \rho_{V_1}(s)(x), \rho_{V_2}(s)(y) \rangle = \langle x - 2\langle x, \beta_s \rangle \alpha_s, y - 2\langle \alpha_s, y \rangle \beta_s \rangle$$

$$= \langle x, y \rangle - 4\langle x, \beta_s \rangle \langle \alpha_s, y \rangle + 4\langle x, \beta_s \rangle \langle \alpha_s, y \rangle \langle \alpha_s, \beta_s \rangle$$

$$= \langle x, y \rangle. \quad \square$$

The principal result of this section is that $(W_1, R_1)$ and $(W_2, R_2)$ are isomorphic Coxeter systems, in the sense of Definition 3 of Chapitre 4 of [10]. Recall that $(W, R)$ is a Coxeter system if and only if $W$ is a
group and $R$ a generating set for $W$, and in terms of these generators $W$ is defined by a set of relations of the form $(rr')^{m(r,r')} = 1$, where $m(r, r') = m(r', r) > 1$ whenever $r, r'$ are distinct elements of $R$ and $m(r, r')$ is defined, and $m(r, r) = 1$ for all $r \in R$. We shall show that if $R = \{ r_s \mid s \in S \}$ is in bijective correspondence with $S$ and $m(r_s, r_t) = m_{st}$ whenever $m_{st} < \infty$, with $m(r_s, r_t)$ undefined otherwise, then $(W_1, R_1)$ and $(W_2, R_2)$ are both isomorphic to the Coxeter system $(W, R)$. We first show (Proposition 1.1.9 below) that $(W_1, R_1)$ and $(W_2, R_2)$, satisfy the required relations, after which it suffices to prove the following theorem:

Theorem 1.1.4. If $W$ is any group generated by a set $R := \{ r_s \mid s \in S \}$ and satisfying the relations $(r_s r_t)^{m_{st}} = 1$ for all $s, t \in S$ such that $m_{st} < \infty$, and if $f : W \rightarrow W_1$ is a group homomorphism satisfying $f(r_s) = \rho_{\nu_1}(s)$ for all $s \in S$, then $f$ is necessarily injective.

Since $W_1 = \langle R_1 \rangle$, any such homomorphism $f$ must also be surjective. So, choosing $(W, R)$ to be the Coxeter system corresponding to the parameters $m_{st}$, and observing that Proposition 1.1.9 guarantees the existence of a homomorphism $f : W \rightarrow W_1$ satisfying $f(r_s) = \rho_{\nu_1}(s)$ for all $s \in S$, it follows that $f$ is an isomorphism, as required.

Of course analogous statements apply if $W_1$ is replaced by $W_2$; hence the claim that $(W_1, R_1)$ and $(W_2, R_2)$ are isomorphic Coxeter systems will be established once we have proved Theorem 1.1.4 and the analogous result for $W_2$. This will occupy the rest of this section.

Lemma 1.1.5. If $s$ is any element of $S$, then $\alpha_s \notin \text{PLC}(\Pi_1 \setminus \{ \alpha_s \})$ and $\beta_s \notin \text{PLC}(\Pi_2 \setminus \{ \beta_s \})$. Furthermore, if $s, t \in S$ with $s \neq t$, then $\{ \alpha_s, \alpha_t \}$ is linearly independent, and so is $\{ \beta_s, \beta_t \}$.

Proof. Suppose for a contradiction that $\alpha_s = \sum_{t \in S \setminus \{ s \}} \lambda_t \alpha_t$, where $\lambda_t \geq 0$ for all $t \in S \setminus \{ s \}$. By (C2) and (C3) of Definition 1.1.1,

$$1 = \langle \alpha_s, \beta_s \rangle = \sum_{t \in S \setminus \{ s \}} \lambda_t \langle \alpha_t, \beta_s \rangle \leq 0$$
which is absurd. Therefore $\alpha_s \notin PLC(\Pi_1 \setminus \{\alpha_s\})$. Now suppose that $s, t \in S$ with $s \neq t$. Obviously $\alpha_s$ and $\alpha_t$ are both nonzero (by Condition (C2)), and by what has just been proved, $\alpha_s$ is not a positive scalar multiple of $\alpha_t$. So to prove that $\{\alpha_s, \alpha_t\}$ is linearly independent, we only need to show that $\alpha_s$ is not a negative scalar multiple of $\alpha_t$.

But if $\alpha_s = -\lambda \alpha_t$ for some $\lambda > 0$ then $0 = \alpha_s + \lambda \alpha_t$, contradicting (C6) of Definition 1.1.1. Hence $\{\alpha_s, \alpha_t\}$ is linearly independent, as required.

Essentially the same argument can also be used to prove linear independence of $\{\beta_s, \beta_t\}$. □

Note that the above yields that for each $i \in \{1, 2\}$ and distinct $s, t \in S$ we have $\rho_{V_i}(s) \neq \rho_{V_i}(t)$.

**Lemma 1.1.6.** Suppose that $s, t \in S$ such that $m_{st} \notin \{1, 2, \infty\}$. Then for all $n \in \mathbb{N}$,

$$
(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = \frac{\sin(2n + 1)\theta}{\sin \theta} \alpha_s + \frac{-\cos \theta}{\langle \alpha_t, \beta_s \rangle} \frac{\sin(2n\theta)}{\sin \theta} \alpha_t
$$

and

$$
(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = \frac{\sin(2n + 1)\theta}{\sin \theta} \beta_s - \frac{-\cos \theta}{\langle \alpha_s, \beta_t \rangle} \frac{\sin(2n\theta)}{\sin \theta} \beta_t
$$

where $\theta = \pi/m_{st}$.

**Proof.** Recall from Definition 1.1.2 that $\rho_{V_1}(s)\alpha_t = \alpha_t - 2\langle \alpha_t, \beta_s \rangle \alpha_s$, and $\rho_{V_1}(s)\alpha_s = \alpha_s - 2\langle \alpha_s, \beta_t \rangle \alpha_s = -\alpha_s$. Similar formulas apply to $\rho_{V_1}(t)$. Thus the matrix of $\rho_{V_1}(s)\rho_{V_1}(t)$ in its action on the subspace with basis $\{\alpha_s, \alpha_t\}$ is

$$
\begin{pmatrix}
-1 & -2\langle \alpha_t, \beta_s \rangle \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-2\langle \alpha_s, \beta_t \rangle & -1
\end{pmatrix}
= \begin{pmatrix}
4\cos^2 \theta - 1 & 2\langle \alpha_t, \beta_s \rangle \\
-2\langle \alpha_s, \beta_t \rangle & -1
\end{pmatrix}
$$

since $\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle = \cos^2 \theta$. To prove the desired result, we only need to compute the $n$th power of the above matrix. Since $m_{st} \neq 2$ it follows
that \( \langle \alpha_s, \beta_t \rangle \neq 0 \neq \langle \alpha_t, \beta_s \rangle \), and we observe that
\[
\begin{pmatrix}
4 \cos^2 \theta - 1 & 2 \langle \alpha_t, \beta_s \rangle \\
-2 \langle \alpha_s, \beta_t \rangle & -1
\end{pmatrix}
= \begin{pmatrix}
\frac{-\cos \theta}{\langle \alpha_s, \beta_t \rangle} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
4 \cos^2 \theta - 1 & -2 \cos \theta \\
2 \cos \theta & -1
\end{pmatrix}
\begin{pmatrix}
\frac{-\cos \theta}{\langle \alpha_t, \beta_s \rangle} & 0 \\
0 & 1
\end{pmatrix}
= \frac{1}{\sin \theta}
\begin{pmatrix}
\frac{-\cos \theta}{\langle \alpha_s, \beta_t \rangle} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\sin(3\theta) & -\sin 2\theta \\
\sin 2\theta & -\sin \theta
\end{pmatrix}
\begin{pmatrix}
\frac{-\cos \theta}{\langle \alpha_t, \beta_s \rangle} & 0 \\
0 & 1
\end{pmatrix}.
\]

Now an induction on \( n \) yields that for all \( n \in \mathbb{N} \)
\[
\begin{pmatrix}
4 \cos^2 \theta - 1 & 2 \langle \alpha_t, \beta_s \rangle \\
-2 \langle \alpha_s, \beta_t \rangle & 1
\end{pmatrix}
^n
= \frac{1}{\sin \theta}
\begin{pmatrix}
\frac{-\cos \theta}{\langle \alpha_s, \beta_t \rangle} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\sin(2n + 1)\theta & -\sin 2n\theta \\
\sin 2n\theta & -\sin(2n - 1)\theta
\end{pmatrix}
\begin{pmatrix}
\frac{-\cos \theta}{\langle \alpha_t, \beta_s \rangle} & 0 \\
0 & 1
\end{pmatrix}.
\]

Expanding the right hand side and examining the coefficients in the first column yields the formula
\[
(\rho V^i(s) \rho V^i(t))^n(\alpha_s) = \frac{\sin(2n + 1)\theta}{\sin \theta} \alpha_s + \frac{-\cos \theta}{\langle \alpha_t, \beta_s \rangle} \frac{\sin 2n\theta}{\sin \theta} \alpha_t.
\]

The other formula follows by a similar calculation. \( \square \)

**Remark 1.1.7.** From Lemmas 1.1.6 and 1.1.5 we can see that if \( i \in \{1, 2\} \) and \( m_{st} \notin \{2, \infty\} \) then \( \text{ord}(\rho V^i(s) \rho V^i(t)) \geq m_{st} \). Indeed, in the subspace with basis \( \{\alpha_s, \alpha_t\} \) the elements
\[
\alpha_s, (\rho V^i(s) \rho V^i(t))(\alpha_s), (\rho V^i(s) \rho V^i(t))^2(\alpha_s), \ldots, (\rho V^i(s) \rho V^i(t))^{m_{st}-1}(\alpha_s)
\]
are all distinct, and in the subspace with basis \( \{\beta_s, \beta_t\} \) the elements
\[
\beta_s, (\rho V^2(s) \rho V^2(t))(\beta_s), (\rho V^2(s) \rho V^2(t))^2(\beta_s), \ldots, (\rho V^2(s) \rho V^2(t))^{m_{st}-1}(\beta_s)
\]
are all distinct.

**Lemma 1.1.8.** Suppose that \( s, t \in S \) such that \( m_{st} = \infty \). If we set \( \theta = \ln(\gamma + \sqrt{\gamma^2 - 1}) = \cosh^{-1}(\gamma) \), where \( \gamma = \sqrt{\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle} \), then
1. Non-orthogonal Realizations

(i) For all \(n \in \mathbb{N}\)

\[
(\rho V_1(s)\rho V_1(t))^n(\alpha_s) = \frac{\sinh(2n+1)\theta}{\sinh \theta} \alpha_s + \frac{-\gamma}{\langle \alpha_t, \beta_s \rangle} \frac{\sinh(2n\theta)}{\sinh \theta} \alpha_t
\]

And

\[
(\rho V_2(s)\rho V_2(t))^n(\beta_s) = \frac{\sinh(2n+1)\theta}{\sinh \theta} \beta_s + \frac{-\gamma}{\langle \alpha_s, \beta_t \rangle} \frac{\sinh(2n\theta)}{\sinh \theta} \beta_t,
\]

(ii) For each \(i \in \{1, 2\}\), \(\rho V_i(s)\rho V_i(t)\) has infinite order in \(\text{GL}(V_i)\).

Proof. (i) The matrix of \(\rho V_1(s)\rho V_1(t)\) in its action on the subspace with basis \(\{\alpha_s, \alpha_t\}\) is

\[
\begin{pmatrix}
-1 & -2\langle \alpha_t, \beta_s \rangle \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-2\langle \alpha_s, \beta_t \rangle & -1
\end{pmatrix}
= \begin{pmatrix}
4\gamma^2 - 1 & 2\langle \alpha_t, \beta_s \rangle \\
-2\langle \alpha_s, \beta_t \rangle & -1
\end{pmatrix}
\]

since \(\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle = \gamma^2\) (by (C4) of Definition 1.1.1). As in the proof of Lemma 1.1.6, to prove the required result we only need to compute the \(n\)th power of the above matrix. Since \(m_{st} = \infty\), it follows that \(\langle \alpha_s, \beta_t \rangle \neq 0 \neq \langle \alpha_t, \beta_s \rangle\), and we have

\[
\begin{pmatrix}
4\gamma^2 - 1 & 2\langle \alpha_t, \beta_s \rangle \\
-2\langle \alpha_s, \beta_t \rangle & -1
\end{pmatrix}
= \frac{1}{\sinh \theta} \begin{pmatrix}
-\gamma \langle \alpha_s, \beta_t \rangle & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
4\gamma^2 - 1 & -2\gamma \\
2\gamma & -1
\end{pmatrix}
\begin{pmatrix}
\frac{-\gamma}{\langle \alpha_s, \beta_t \rangle} & 0 \\
0 & 1
\end{pmatrix}
\]

Now an induction yields that, for all \(n \in \mathbb{N}\),

\[
\begin{pmatrix}
4\gamma^2 - 1 & 2\langle \alpha_t, \beta_s \rangle \\
-2\langle \alpha_s, \beta_t \rangle & -1
\end{pmatrix}
^n
= \begin{pmatrix}
\frac{-\gamma}{\langle \alpha_s, \beta_t \rangle} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{\sinh(2n+1)\theta}{\sinh \theta} & -\frac{\sinh(2n\theta)}{\sinh \theta} \\
\frac{-\sinh(2n\theta)}{\sinh \theta} & -\frac{\sinh(2n+1)\theta}{\sinh \theta}
\end{pmatrix}
\begin{pmatrix}
\frac{-\gamma}{\langle \alpha_t, \beta_s \rangle} & 0 \\
0 & 1
\end{pmatrix}
\]

Therefore, for each \(n \in \mathbb{N}\),

\[
(\rho V_1(s)\rho V_1(t))^n(\alpha_s) = \frac{\sinh(2n+1)\theta}{\sinh \theta} \alpha_s + \frac{-\gamma}{\langle \alpha_t, \beta_s \rangle} \frac{\sinh(2n\theta)}{\sinh \theta} \alpha_t.
\]
A similar argument shows that
\[(\rho_{V_1}(s)\rho_{V_2}(t))^n(\beta_s) = \frac{\sinh(2n+1)\theta}{\sinh \theta} \beta_s + \frac{-\gamma}{\langle \alpha_s, \beta_t \rangle} \sinh(2n\theta) \beta_t,\]
completing the proof of (i).

(ii) In the subspace with basis \{\alpha_s, \alpha_t\}, it is clear from (i) above that the elements \alpha_s, (\rho_{V_1}(s)\rho_{V_2}(t))(\alpha_s), (\rho_{V_1}(s)\rho_{V_2}(t))^2(\alpha_s), \cdots, are all distinct, and therefore \rho_{V_1}(s)\rho_{V_2}(t) has infinite order in GL(V_1). In the same way, \rho_{V_2}(s)\rho_{V_2}(t) has infinite order in GL(V_2). \qed

**Proposition 1.1.9.** Suppose that \(s, t\) are distinct elements of \(S\) such that \(m_{st} \neq \infty\). Then

(i) \(\rho_{V_1}(s)\rho_{V_1}(t)\) has order \(m_{st}\) in GL\((V_1)\);
(ii) \(\rho_{V_2}(s)\rho_{V_2}(t)\) has order \(m_{st}\) in GL\((V_2)\).

**Proof.** Recall that \(m_{st} = 1\) if and only if \(s = t\). So in this case the statement is simply that \(\rho_{V_i}(s)^2 = 1\) for each \(i\) and all \(s \in S\). We have already noted that this is true.

Let \(\alpha \in V_1\) be arbitrary and let \(s, t \in S\) be distinct. We see that

\[(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha) = (\rho_{V_1}(s)(\alpha - 2\langle \alpha, \beta_t \rangle \alpha_t) = \alpha - 2\langle \alpha, \beta_s \rangle \alpha_s - 2\langle \alpha, \beta_t \rangle (\alpha_t - 2\langle \alpha_t, \beta_s \rangle \alpha_s) = \alpha + 4\langle \alpha, \beta_t \rangle \langle \alpha_t, \beta_s \rangle - 2\langle \alpha, \beta_s \rangle \alpha_s - 2\langle \alpha, \beta_t \rangle \alpha_t.\]

In the case that \(m_{st} = 2\) we see that

\[(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha) = (\rho_{V_1}(t)\rho_{V_1}(s))(\alpha) = \alpha - 2\langle \alpha, \beta_s \rangle \alpha_s - 2\langle \alpha, \beta_t \rangle \alpha_t,\]
so that \(\rho_{V_1}(s)\) and \(\rho_{V_1}(t)\) commute. Hence \(\text{ord}(\rho_{V_1}(s)\rho_{V_1}(t)) = 2\).

It remains to consider the case when \(m_{st} > 2\). In the cases \(\alpha = \alpha_s\) and \(\alpha = \alpha_t\) the formula above gives

\[(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha_s) = (4 \cos^2\left(\frac{\pi}{m_{st}}\right) - 1)\alpha_s - 2\langle \alpha_s, \beta_t \rangle \alpha_t\]

and

\[(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha_t) = 2\langle \alpha_t, \beta_s \rangle \alpha_s - \alpha_t,\]
and therefore the action of $\rho_{V_1}(s)\rho_{V_1}(t)$ on $\mathbb{R}\{\alpha_s, \alpha_t, \alpha\}$ may be represented by the following matrix $M$:

$$M = \begin{pmatrix}
4 \cos^2(\pi/m_{st}) - 1 & 2 \langle \alpha_t, \beta_s \rangle & 4 \langle \alpha, \beta_t \rangle \langle \alpha_t, \beta_s \rangle - 2 \langle \alpha, \beta_s \rangle \\
-2 \langle \alpha_s, \beta_t \rangle & -1 & -2 \langle \alpha, \beta_t \rangle \\
0 & 0 & 1
\end{pmatrix}. $$

It is readily checked that $M$ has distinct eigenvalues $e^{i\frac{2\pi}{m_{st}}}, e^{-i\frac{2\pi}{m_{st}}}$ and 1, and hence has order $m_{st}$. Consequently $(\rho_{V_1}(s)\rho_{V_1}(t))^{m_{st}}(\alpha) = \alpha$. Since $\alpha$ is arbitrary, it follows that $(\rho_{V_1}(s)\rho_{V_1}(t))^{m_{st}} = 1$ in $\text{GL}(V_1)$. Thus we have $\text{ord}(\rho_{V_1}(s)\rho_{V_1}(t)) \leq m_{st}$, and in view of Remark 1.1.7 above, it follows that $\text{ord}(\rho_{V_1}(s)\rho_{V_1}(t))$ is precisely $m_{st}$.

The proof that $\text{ord}(\rho_{V_2}(s)\rho_{V_2}(t)) = m_{st}$ is entirely similar. □

Combining Lemma 1.1.8(ii) and Proposition 1.1.9, we have

**Corollary 1.1.10.** For each $i \in \{1, 2\}$, and for all $s, t \in S$,

$$\text{ord}(\rho_{V_i}(s)\rho_{V_i}(t)) = m_{st}. $$

□

Utilizing the formulas in Lemma 1.1.6 and Lemma 1.1.8 we may deduce the following:

**Lemma 1.1.11.** Suppose that $s, t \in S$, and let $n$ be an integer such that $0 \leq n < m_{st}$. Write

$$\underbrace{\cdots \rho_{V_1}(t)\rho_{V_1}(s)\rho_{V_1}(t)}_{n \text{ factors}} \alpha_s = \lambda_n \alpha_s + \mu_n \alpha_t$$

and

$$\underbrace{\cdots \rho_{V_1}(s)\rho_{V_1}(t)\rho_{V_1}(s)}_{n \text{ factors}} \alpha_t = \lambda'_n \alpha_s + \mu'_n \alpha_t.$$ 

Then $\lambda_n \geq 0$, $\mu_n \geq 0$, $\lambda'_n \geq 0$ and $\mu'_n \geq 0$.

**Proof.** If $m_{st} = 2$ then $\langle \alpha_s, \beta_t \rangle = 0 = \langle \alpha_t, \beta_s \rangle$, giving $\rho_{V_1}(t)\alpha_s = \alpha_s$ and $\rho_{V_1}(s)\alpha_t = \alpha_t$. So $\lambda_1 = \mu'_1 = 1$ and $\mu_1 = \lambda'_1 = 0$. Since obviously also $\lambda_0 = \mu'_0 = 1$ and $\mu_0 = \lambda'_0 = 0$, the statement in the Lemma holds
when $m_{st} = 2$. Thus we may assume that $m_{st} \geq 3$. Note in particular that this gives $\langle \alpha_s, \beta_t \rangle \neq 0 \neq \langle \alpha_t, \beta_s \rangle$.

Now suppose that $n$ is even, so that

$$
\cdots \rho V_1(t)\rho V_1(s)\rho V_1(t) \alpha_s = (\rho V_1(s)\rho V_1(t))^{n/2} \alpha_s.
$$

If $m_{st} < \infty$ then Lemma 1.1.6 gives

$$
\lambda_n = \frac{\sin(n+1)\theta}{\sin \theta} \quad \text{and} \quad \mu_n = \frac{-\cos \theta}{\langle \alpha_t, \beta_s \rangle} \frac{\sin n\theta}{\sin \theta},
$$

where $\theta = \pi/m_{st}$, while if $m_{st} = \infty$ then Lemma 1.1.8 gives

$$
\lambda_n = \frac{\sinh(n+1)\theta}{\sinh \theta} \quad \text{and} \quad \mu_n = \frac{-\gamma}{\langle \alpha_t, \beta_s \rangle} \frac{\sinh n\theta}{\sinh \theta},
$$

where $\gamma = \sqrt{\langle \alpha_t, \beta_s \rangle\langle \alpha_s, \beta_t \rangle} \geq 1$ and $\theta = \ln(\gamma + \sqrt{\gamma^2 - 1})$. Observe that if $0 \leq n < m_{st} < \infty$ then $\theta$, $n\theta$ and $(n+1)\theta$ all lie in the interval $[0, \pi]$, and since also $\langle \alpha_t, \beta_s \rangle < 0$ it follows that $\lambda_n \geq 0$ and $\mu_n \geq 0$ whenever $n$ is even and $m_{st}$ is finite. The same conclusion follows when $n$ is even and $m_{st} = \infty$, since in this case $\theta > 0$ (and $\langle \alpha_t, \beta_s \rangle < 0$ is still satisfied).

Next suppose that $n$ is odd. Then

$$
\cdots \rho V_1(t)\rho V_1(s)\rho V_1(t) \alpha_s = (\rho V_1(s)\rho V_1(t))^{(n-1)/2} \alpha_s
$$

and

$$
= \rho V_1(s)\rho V_1(t)^{(n+1)/2} \alpha_s.
$$

Observe that applying $\rho V_1(t)$ will not change the coefficient of $\alpha_s$ and applying $\rho V_1(s)$ will not change the coefficient of $\alpha_t$. Hence the coefficient of $\alpha_s$ in $\rho V_1(t)(\rho V_1(s)\rho V_1(t))^{(n-1)/2} \alpha_s$ is $\lambda_{n-1}$ and the coefficient of $\alpha_t$ in $\rho V_1(s)(\rho V_1(s)\rho V_1(t))^{(n+1)/2} \alpha_s$ is $\mu_{n+1}$; that is, $\lambda_n = \lambda_{n-1}$ and $\mu_n = \mu_{n+1}$. Applying Lemma 1.1.6 and Lemma 1.1.8 yields

$$
\lambda_n = \lambda_{n-1} = \begin{cases} 
\frac{\sin n\theta}{\sin \theta} & \text{if } m_{st} < \infty \\
\frac{\sinh n\theta}{\sinh \theta} & \text{if } m_{st} = \infty
\end{cases}
$$

and
\[ \mu_n = \mu_{n+1} = \begin{cases} 
- \cos \theta \frac{\sin(n+1)\theta}{\langle \alpha_t, \beta_s \rangle} & \text{if } m_{st} < \infty \\
- \gamma \frac{\sinh(n+1)\theta}{\sinh \theta} & \text{if } m_{st} = \infty 
\end{cases} \]

where \( \theta \) and \( \gamma \) are as previously, and it follows that \( \lambda_n \geq 0 \) and \( \mu_n \geq 0 \) when \( n \) is odd and \( n < m_{st} \). Finally by symmetry, \( \lambda'_n \geq 0 \) and \( \mu'_n \geq 0 \) whenever \( n < m_{st} \).

\[\square\]

Remark 1.1.12. The same argument applies equally well if we replace, respectively, \( \rho_{V_1}(s), \rho_{V_1}(t), \alpha_s \) and \( \alpha_t \) by \( \rho_{V_2}(s), \rho_{V_2}(t), \beta_s \) and \( \beta_t \).

Definition 1.1.13. Let \((W, R)\) be a Coxeter system. The length function of \(W\) with respect to \(R\) is the function \(l: W \rightarrow \mathbb{N}\) defined by

\[ l(w) = \min \{ n \in \mathbb{N} \mid w = r_1r_2 \cdots r_n, \text{ where } r_1, r_2, \ldots, r_n \in R \}, \]

for all \( w \in W \). We say that \( w = r_1r_2 \cdots r_n \) with \( r_1, r_2, \ldots, r_n \in R \) is a reduced expression for \( w \) if \( l(w) = n \).

Let \( W \) and \( f \) be as in the statement of Theorem 1.1.4. The homomorphism \( f: W \rightarrow W_1 \) permits us to define an action of \( W \) on \( V_1 \) by \( wx := (f(w))x \) for all \( w \in W \) and \( x \in V_1 \).

Proposition 1.1.14. Let \( w \in W \) and \( s \in S \). If \( l(wr_s) \geq l(w) \), then \( w\alpha_s \in \text{PLC}(\Pi_1) \).

Proof. Choose \( w \in W \) of minimal length such that the assertion fails for some \( \alpha_s \in \Pi_1 \), and choose such an \( \alpha_s \). Certainly \( w \neq 1 \), since \( 1\alpha_s = \alpha_s \) is trivially a positive linear combination of \( \Pi_1 \). Thus \( l(w) > 1 \), and we may choose \( t \in S \) such that \( w_1 = wr_t \) has length \( l(w) - 1 \). If \( l(w_1r_s) \geq l(w_i) \), then \( l(w_1r) \geq l(w_1) \) for both \( r = r_s \) and \( r = r_t \). Alternatively, if \( l(w_1r_s) < l(w_1) \), we define \( w_2 = w_1r_s \), and note that \( l(w_2r) \geq l(w_2) \) will hold for both \( r = r_s \) and \( r = r_t \) if \( l(w_2r_t) \geq l(w_2) \). If this latter condition is not satisfied then we define \( w_3 = w_2r_t \). Continuing in this way we find, for some positive integer
1.1. Paired reflection representations

$k$, a sequence of elements $w_0 = w, w_1, w_2, \cdots, w_k$ with $l(w_i) = l(w) - i$ for all $i = 0, 1, 2, \cdots, k$, and, when $i < k$,

$$w_{i+1} = \begin{cases} w_ir_s & \text{if } i \text{ is odd} \\ w_ir_t & \text{if } i \text{ is even} \end{cases}$$

Now since $0 \leq l(w_k) = l(w) - k$, we conclude that $l(w)$ is an upper bound for the possible values of $k$. Choosing $k$ to be as large as possible, we deduce that $l(w_k)$ is an upper bound for the possible values of $k$. By the minimality of our original counterexample it follows that $w\alpha_s$ and $w\alpha_t$ are both in PLC($\Pi_1$).

We have $w = w_kv$, where $v$ is an alternating product of $r_s$’s and $r_t$’s, ending in $r_t$, and with $k$ factors altogether. Obviously this means that $l(v) \leq k$. But $w = w_kv$ gives $l(w) \leq l(w_k) + l(v)$, so it follows that $l(v) \geq l(w) - l(w_k) = k$, and hence $l(v) = k$. Furthermore, in view of the hypothesis that $l(wr_s) \geq l(w)$, and since $w_kvr_s = wr_s$, we have

$$l(w_k) + l(vr_s) \geq l(wr_s) \geq l(w) = l(w_k) + k = l(w_k) + l(v),$$

and hence $l(vr_s) \geq l(v)$. In particular, $v$ cannot have a reduced expression in which the final factor is $r_s$, for if so $vr_s$ would have a strictly shorter expression.

Since $r_s$ and $r_t$ satisfy the defining relations of the dihedral group of order $2m_{st}$, it follows that every element of the subgroup generated by $r_s$ and $r_t$ has an expression of length less than $m_{st} + 1$ as an alternating product of $r_s$ and $r_t$. Thus $l(v) \leq m_{st}$. Moreover, if $m_{st}$ is finite then the two alternating products of length $m_{st}$ define the same element; so $l(v)$ cannot equal $m_{st}$, as $v$ has no reduced expression ending with $r_s$. Thus Lemma 1.1.11 above yields $v\alpha_s = \lambda_1\alpha_s + \mu_1\alpha_t$ for some non-negative coefficients $\lambda_1$ and $\mu_1$. Hence

$$w\alpha_s = w_kv\alpha_s = w_k(\lambda_1\alpha_s + \mu_1\alpha_t) = \lambda_1w_k\alpha_s + \mu_1w_k\alpha_t \in PLC(\Pi_1)$$
since $w_k\alpha_s, w_k\alpha_t \in \text{PLC}(\Pi_1)$. This contradicts our original choice of $w$ and $\alpha_s$ as a counterexample to the statement of the proposition; so if $w \in W$, $s \in S$, with $l(wr_s) \geq l(w)$, then $w\alpha_s \in \text{PLC}(\Pi_1)$. □

Now we are ready to complete the proof of Theorem 1.1.4:

**Proof of Theorem 1.1.4.** Suppose, for a contradiction, that the kernel of $f$ is nontrivial, and choose $w$ in the kernel of $f$ with $w \neq 1$. Then $l(w) > 0$, and we may write $w = w'r_s$ for some $s \in S$ and $w' \in W$ with $l(w') = l(w) - 1$. Since $l(w'r_s) > l(w')$, Proposition 1.1.14 yields $w'\alpha_s \in \text{PLC}(\Pi_1)$. But

$$\alpha_s = w\alpha_s = (w'r_s)\alpha_s = w'r_s\alpha_s = w'(-\alpha_s) = -w'\alpha_s$$

and hence $0 = \alpha_s + w'\alpha_s \in \text{PLC}(\Pi_1)$, contradicting our original assumption that $0 \notin \text{PLC}(\Pi_1)$ (condition (C6)). Therefore the kernel of $f$ is trivial, as required. □

By Proposition 1.1.9 we may define a homomorphism $g: W \rightarrow W_2$ satisfying $g(r_s) = \rho_{V_2}(s)$ for all $s \in S$ and obtain an action of $W$ on $V_2$. Applying exactly the same arguments that lead to Theorem 1.1.4 gives the analogous result for $W_2$ and $V_2$:

**Theorem 1.1.15.** Suppose that $W$ is any group generated by a set $R := \{r_s \mid s \in S\}$ satisfying the relations $(r_s r_t)^m_{st} = 1$ for all $s, t \in S$ such that $m_{st} < \infty$; and suppose that $g: W \rightarrow W_2$ is a group homomorphism satisfying $g(r_s) = \rho_{V_2}(s)$ for all $s \in S$. Then $g$ is necessarily injective. □

Combining Theorem 1.1.4 and Theorem 1.1.15, we have now shown that

$$W \cong W_1 \cong W_2,$$

where $W$ is the abstract Coxeter group determined by the Coxeter parameters of our Coxeter datum $\mathcal{C}$. We refer to $W_1$ and $W_2$ as the realizations of $W$ in $V_1$ and $V_2$ respectively.
1.2. Root Systems and Canonical Coefficients

Having established in Section 1.1 that $W_1$ and $W_2$ are Coxeter groups, we now wish to develop a theory of root systems for $W_1$ and $W_2$.

As above, let $W$ be the abstract Coxeter group determined by the Coxeter datum $\mathcal{C}$, and let $R = \{ r_s \mid s \in S \}$ be its distinguished generating set. Also as above we use the isomorphisms $W \to W_1$ and $W \to W_2$ to define an action of $W$ on $V_1$ and an action of $W$ on $V_2$ (so that $r_s x = x - 2 \langle x, \beta_s \rangle \alpha_s$ and $r_s y = y - 2 \langle \alpha_s, y \rangle \beta_s$ for all $x \in V_1$ and $y \in V_2$, and all $s \in S$). Observe that these actions are faithful, in the sense that if $wx = x$ for all $x$ (in $V_1$ or $V_2$) then $w = 1$.

An easy induction on $l(w)$ yields the following extension to Proposition 1.1.3:

**Lemma 1.2.1.** Let $x \in V_1$, $y \in V_2$. Then $\langle x, y \rangle = \langle wx, wy \rangle$, for all $w \in W$.

The following definition is a natural generalization of the concept of the root system of a Coxeter group, as defined in [10] and [12], for example.

**Definition 1.2.2.** (i) Define $\Phi_1 := W(\Pi_1) = \{ w\alpha_s \mid w \in W, s \in S \}$, and $\Phi_2 := W(\Pi_2) = \{ w\beta_s \mid w \in W, s \in S \}$. For each $i \in \{1, 2\}$, we call $\Phi_i$ the root system for $V_i$, and its elements the roots of $W$ in $V_i$.

We call $\Pi_i$ the set of simple roots in $\Phi_i$, and also refer to $\Pi_i$ as the root basis for $\Phi_i$.

(ii) For each $i \in \{1, 2\}$, set $\Phi_i^+ := \Phi_i \cap \text{PLC}(\Pi)$, and $\Phi_i^- = -\Phi_i^+$. We call $\Phi_i^+$ the set of positive roots in $\Phi_i$, and $\Phi_i^-$ the set of negative roots in $\Phi_i$.

For each $i \in \{1, 2\}$, we adopt the traditional diagrammatic description of simple roots $\Pi_i$: draw a graph that has one vertex for each $s \in S$, and join the vertices corresponding to $s, t \in S$ by an edge labelled by $m_{st}$ if $m_{st} > 2$. The label $m_{st}$ is often omitted if $m_{st} = 3$. Thus the diagram $\begin{array}{c} r \\ s \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} t \end{array}$ corresponds to $\Pi_1 = \{ \alpha_r, \alpha_s, \alpha_t \}$.
and \( \Pi_2 = \{ \beta_r, \beta_s, \beta_t \} \). Unlike in the case of the standard geometric realization of Coxeter groups, this diagram does not uniquely determine the individual values of \( \langle \alpha_r, \beta_s \rangle, \langle \alpha_s, \beta_r \rangle, \langle \alpha_s, \beta_t \rangle, \langle \alpha_t, \beta_s \rangle, \langle \alpha_r, \beta_t \rangle \) and \( \langle \alpha_t, \beta_r \rangle \), instead we can only tell that

\[
\langle \alpha_r, \beta_s \rangle \langle \alpha_s, \beta_r \rangle = \langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle = \langle \alpha_t, \beta_r \rangle \langle \alpha_r, \beta_t \rangle = \frac{1}{4}.
\]

Suppose that we know

\[
\langle \alpha_r, \beta_s \rangle = -\frac{1}{4}, \quad \langle \alpha_s, \beta_r \rangle = -1, \\
\langle \alpha_s, \beta_t \rangle = -\frac{1}{6}, \quad \langle \alpha_t, \beta_s \rangle = -\frac{3}{2}, \\
\langle \alpha_t, \beta_r \rangle = -\frac{1}{10}, \quad \langle \alpha_r, \beta_t \rangle = -\frac{5}{2}.
\]

Then

\[
r_s \alpha_r = \alpha_r - 2 \langle \alpha_r, \beta_s \rangle \alpha_s = \alpha_r + \frac{1}{2} \alpha_s;
\]

\[
(r_r r_s) \alpha_r = r_r (\alpha_r + \frac{1}{2} \alpha_s) = \frac{1}{2} \alpha_s;
\]

\[
(r_t r_r r_s) \alpha_r = r_t (\frac{1}{2} \alpha_s) = \frac{1}{2} \alpha_s + \frac{1}{6} \alpha_t;
\]

\[
(r_s r_t r_s r_r) \alpha_r = r_s (\frac{1}{2} \alpha_s + \frac{1}{6} \alpha_t) = \frac{1}{6} \alpha_t;
\]

\[
(r_r r_s r_t r_r r_s) \alpha_r = r_r (\frac{1}{6} \alpha_t) = \frac{1}{6} \alpha_t + \frac{1}{30} \alpha_r;
\]

\[
(r_t r_s r_t r_s r_r r_s) \alpha_r = r_t (\frac{1}{6} \alpha_t + \frac{1}{30} \alpha_r) = \frac{1}{30} \alpha_r.
\]

**Remark 1.2.3.** We see from the above example that it is possible for a non-trivial positive scalar multiple of a root to also be a root, lying in the same \( W \)-orbit as the root itself. Clearly if \( w \alpha = \lambda \alpha \), where \( 1 \neq \lambda \in \mathbb{R} \), then \( w^n \alpha = \lambda^n \alpha \), showing that there are infinitely many scalar multiples of \( \alpha \) in \( \Phi_1 \). And of course, all roots in the \( W \)-orbit of \( \alpha \) will possess this same property. Despite the fact that in this regard \( \Phi_1 \) and \( \Phi_2 \) are different from root systems defined in orthogonal geometric realizations, it nevertheless turns out that all major properties of root systems can be generalized to the non-orthogonal setting. We begin with the observation that any root in \( \Phi_i \) can be expressed as a linear combination of simple roots from \( \Pi_i \) with coefficients being all non-positive or all non-negative:
Lemma 1.2.4. For each $i \in \{1, 2\}$, $\Phi_i = \Phi_i^+ \cup \Phi_i^-$, where $\cup$ denotes disjoint union.

Proof. The condition (C6), which says that $0 \notin \text{PLC}(\Pi_i)$ (for each $i \in \{1, 2\}$), ensures that $\Phi_i^+ \cap \Phi_i^- = \emptyset$.

Let $x \in \Phi_i$. By Definition 1.2.2(i) there exists $w \in W$ and $\alpha_s \in \Pi_i$ with $x = w\alpha_s$. Let $w' = wr_s$, noting that the lengths of $w$ and $w'$ differ by at most 1. Now if $l(w') \geq l(w)$ then Proposition 1.1.14 yields that $x = w\alpha_s \in \Phi_i^+$. On the other hand if $l(w) \geq l(w')$, then Proposition 1.1.14 yields that $w'\alpha_s \in \Phi_i^+$, and this in turn gives $x = wr_s(\alpha_s) = w(-\alpha_s) = -w'\alpha_s \in \Phi_i^-$. This yields that $\Phi_i = \Phi_i^+ \cup \Phi_i^-$. The same reasoning also shows that $\Phi_2 = \Phi_2^+ \cup \Phi_2^-$. □

The preceding proof shows, incidentally, that $l(wr_s) = l(w)$ is not a possibility, since a root $z \in \Phi_i$ ($i = 1, 2$) cannot simultaneously be positive and negative. This observation naturally leads to the following key combinatorial fact of the action of $W$ on $\Phi_i$, $i = 1, 2$:

Corollary 1.2.5. If $w \in W$ and $s \in S$, then

$$l(wr_s) = \begin{cases} 
  l(w) + 1 & \text{if } w\alpha_s \in \Phi_i^+ \text{, and } w\beta_s \in \Phi_i^+, \\
  l(w) - 1 & \text{if } w\alpha_s \in \Phi_i^- \text{, and } w\beta_s \in \Phi_i^-.
\end{cases}$$

□

Remark 1.2.6. From the above, we observe that $w\alpha_s \in \Phi_i^+$ if and only if $w\beta_s \in \Phi_i^+$, and $w\alpha_s \in \Phi_i^-$ if and only if $w\beta_s \in \Phi_i^-$, for all $w \in W$ and $s \in S$.

Remark 1.2.7. Since $\Pi_1$ and $\Pi_2$ need not be linearly independent, although we know from Lemma 1.2.4 that each root in $\Phi_i$ is expressible as a linear combination of simple roots from $\Pi_i$ with coefficients all being of the same sign, that expression need not be unique. Thus the concept of the coefficient of a simple root in a given root is potentially ambiguous. The rest of this section is devoted to finding a canonical way of expressing any given root as a linear combination of simple roots.
roots, by specifying the canonical coefficients of the simple roots in a given root.

Let $V'_1$ be a vector space over $\mathbb{R}$ with basis $\Pi'_1 := \{ \alpha'_s \mid s \in S \}$ in bijective correspondence with $S$, and let $V'_2$ be a vector space over $\mathbb{R}$ with basis $\Pi'_2 := \{ \beta'_s \mid s \in S \}$, also in bijective correspondence with $S$. Define linear maps $\pi_1: V'_1 \rightarrow V_1$ and $\pi_2: V'_2 \rightarrow V_2$ by requiring that

$$\pi_1(\sum_{s \in S} \lambda_s \alpha'_s) = \sum_{s \in S} \lambda_s \alpha_s$$

and

$$\pi_2(\sum_{s \in S} \mu_s \beta'_s) = \sum_{s \in S} \mu_s \beta_s,$$

for all $\lambda_s, \mu_s \in \mathbb{R}$, and define a bilinear map $\langle \cdot, \cdot \rangle: V'_1 \times V'_2 \rightarrow \mathbb{R}$ by requiring that

$$\langle \alpha'_s, \beta'_t \rangle = \langle \alpha_s, \beta_t \rangle$$

for all $s, t \in S$. Observe that

$$\langle x', y' \rangle = \langle \pi_1(x'), \pi_2(y') \rangle$$

for all $x' \in V'_1$ and $y' \in V'_2$.

With these definitions $\mathcal{C}' = (S, V'_1, V'_2, \Pi'_1, \Pi'_2, \langle \cdot, \cdot \rangle)$ is clearly a Coxeter datum having the same parameters as our original Coxeter datum $\mathcal{C}$, and thus corresponding to the same abstract Coxeter system.

For each $s \in S$, define linear transformations $\rho_{V'_1}(s): V'_1 \rightarrow V'_1$ and $\rho_{V'_2}(s): V'_2 \rightarrow V'_2$ by

$$\rho_{V'_1}(s)(x') = x' - 2 \langle x', \beta'_s \rangle \alpha'_s,$$

for all $x' \in V'_1$, and

$$\rho_{V'_2}(s)(y') = y' - 2 \langle \alpha'_s, y' \rangle \beta'_s,$$

for all $y' \in V'_2$, noting that $\pi_i \rho_{V'_i}(s) = \rho_{V'_i}(s) \pi_i$ for all $s \in S$ and $i \in \{1, 2\}$. For each $i \in \{1, 2\}$, let $R'_i = \{ \rho_{V'_i}(s) \mid s \in S \}$, and let $W'_i$ be the subgroup of $\text{GL}(V'_i)$ generated by $R'_i$.

Defining $(W, R)$ to be the abstract Coxeter system corresponding to the Coxeter datum $\mathcal{C}'$ (or $\mathcal{C}$), Theorems 1.1.4 and 1.1.15 yield isomorphisms $f'_1: W \rightarrow W'_1$ and $f'_2: W \rightarrow W'_2$, and we use these to define actions of $W$ on $V'_1$ and $V'_2$ via $wz' = (f'_1(w))(z')$ for all $w \in W$ and
z' \in V'_i$, for each $i \in \{1, 2\}$. Note that since by definition $f'_i(r_s) = \rho_{V'_i}(s)$ for all $s \in S$, we have

$$
\pi_i f'_i(r_s) = f_i(r_s) \pi_i \quad \text{for all } s \in S \text{ and } i \in \{1, 2\}
$$

where $f_1: W \to W_1$ is the $f$ of Theorem 1.1.4 and $f_2$ is the analogous isomorphism $W \to W_2$. Since $W$ is generated by $\{r_s \mid s \in S\}$ it follows that $\pi_i f'_i(w) = f_i(w) \pi_i$ for all $w \in W$ and $i \in \{1, 2\}$. That is,

$$(1.2.1) \quad \pi_i(wz') = w\pi_i(z') \quad \text{for all } w \in W \text{ and } z' \in V'_i$$

for each $i \in \{1, 2\}$. In other words, $\pi_1$ and $\pi_2$ are $W$-module homomorphisms.

**Definition 1.2.8.** Define $\Phi'_1 := \{w\alpha'_s \mid w \in W \text{ and } s \in S\}$ (the root system for $W$ in $V'_1$) and $\Phi'_2 := \{w\beta'_s \mid w \in W \text{ and } s \in S\}$ (the root system for $W$ in $V'_2$).

Since $\Pi'_1$ and $\Pi'_2$ are linearly independent, the expressions of roots in $\Phi'_1$ and $\Phi'_2$ in terms of $\Pi'_1$ and $\Pi'_2$ are unique. In particular, for each $i \in \{1, 2\}$, the coefficient of a given simple root in $\Pi'_i$ in any root of $\Phi'_i$ is uniquely determined. We will utilize this fact to specify a preferred way of expressing roots in $\Phi_1$ and $\Phi_2$ as linear combinations of elements from $\Pi_1$ and $\Pi_2$. The following proposition will be a key step:

**Proposition 1.2.9.** For each $i \in \{1, 2\}$, the restriction of $\pi_i$ defines a $W$-equivariant bijection $\Phi'_i \to \Phi_i$.

**Notation:** Define $\phi_1: \Phi_1 \to \Phi'_1$ and $\phi_2: \Phi_2 \to \Phi'_2$ to be the inverses of the bijections guaranteed by Proposition 1.2.9.

To prove Proposition 1.2.9, we need a few elementary results and some further notation.

**Definition 1.2.10.** For each $i \in \{1, 2\}$, define an equivalence relation $\sim_i$ on $\Phi_i$ as follows: if $z_1, z_2 \in \Phi_i$ then $z_1 \sim_i z_2$ if and only if $z_1$ and $z_2$ are (nonzero) scalar multiples of each other. For each $z \in \Phi_i$, write $\widehat{z}$ for the equivalence class containing $z$, and set $\widehat{\Phi}_i = \{\widehat{z} \mid z \in \Phi_i\}$. 
Observe that the action of $W$ on $\Phi_i$ (for $i = 1, 2$) gives rise to a well-defined action of $W$ on $\hat{\Phi}_i$ satisfying $w\hat{z} = \hat{wz}$ for all $w \in W$ and all $z \in \Phi_i$.

**Definition 1.2.11.** For each $w \in W$, define

$$N_1(w) := \{ \hat{\alpha} | \alpha \in \Phi_1^+ \text{ and } w(\alpha) \in \Phi_1^- \}$$

and

$$N_2(w) := \{ \hat{\beta} | \beta \in \Phi_2^+ \text{ and } w(\beta) \in \Phi_2^- \}.$$  

Observe that $N_1(w)$ and $N_2(w)$ can alternatively be characterised as follows:

$$N_1(w) := \{ \hat{\alpha} | \alpha \in \Phi_1^- \text{ and } w(\alpha) \in \Phi_1^+ \}$$

and

$$N_2(w) := \{ \hat{\beta} | \beta \in \Phi_2^- \text{ and } w(\beta) \in \Phi_2^+ \}.$$  

Thus if $\hat{z} \in \hat{\Phi}_i$, then $\hat{z} \in N_i(w)$ if and only if one of $z$ and $wz$ is positive and the other negative.

**Lemma 1.2.12.** (i) If $s \in S$, then $N_1(r_s) = \{ \hat{\alpha}_s \}$ and $N_2(r_s) = \{ \hat{\beta}_s \}$.

(ii) Let $w \in W$. Then $N_1(w)$ and $N_2(w)$ both have cardinality $l(w)$.

(iii) Let $w_1, w_2 \in W$ and let $\Delta$ denote set symmetric difference. Then

$$N_i(w_1 w_2) = w_2^{-1} N_i(w_1) \Delta N_i(w_2) \quad \text{for each } i \in \{1, 2\}.$$  

(iv) Let $w_1, w_2 \in W$. Then for each $i \in \{1, 2\}$,

$$l(w_1 w_2) = l(w_1) + l(w_2) \text{ if and only if } N_i(w_2) \subseteq N_i(w_1 w_2).$$

**Proof.** (i) Let $s \in S$. Clearly $r_s(\alpha_s) = \alpha_s - 2\langle \alpha_s, \beta_s \rangle \alpha_s = -\alpha_s$, and so $\hat{\alpha}_s \in N_1(r_s)$. Now let $\alpha \in \Phi_1^+$, and write $\alpha = \sum_{r \in S} \lambda_r \alpha_r$ with $\lambda_r \geq 0$ for all $r \in S$. Suppose that $r_s(\alpha) \in \Phi_1^-$; that is, $\alpha - 2\langle \alpha, \beta_s \rangle \alpha_s = -\sum_{r \in S} \mu_r \alpha_r$, for some $\mu_r$ with $\mu_r \geq 0$ for all $r \in S$. Then

$$0 = (\lambda_s + \mu_s - 2\langle \alpha, \beta_s \rangle) \alpha_s + \sum_{r \in S \setminus \{s\}} (\lambda_r + \mu_r) \alpha_r.$$  

0.0cm
Now apply $\langle \cdot, \beta_s \rangle$ to this expression. Since $\langle \alpha_s, \beta_s \rangle = 1$ and $\langle \alpha_r, \beta_s \rangle \leq 0$ whenever $r$ is in $S \setminus \{s\}$, we conclude that $\lambda_s + \mu_s - 2\langle \alpha, \beta_s \rangle$ must be nonnegative. But this gives $0 \in \text{PLC}(\Pi_1)$, contradicting (C6) of Definition 1.1.1, unless all coefficients in (1.2.2) are zero. Thus $\lambda_r = \mu_r = 0$ for all $r \in S \setminus \{s\}$, forcing $\alpha$ to be a positive scalar multiple of $\alpha_s$.

Hence the only positive roots in $\Phi_1$ made negative by applying $r_s$ are of the form $\lambda \alpha_s$, where $\lambda$ is positive, that is, $N_1(r_s) = \{\hat{\alpha}_s\}$.

Exactly the same reasoning gives that $N_2(r_s) = \{\hat{\beta}_s\}$, and this completes the proof of (i).

(ii) If $w \in W$, define $n_1(w) = |N_1(w)|$ and $n_2(w) = |N_2(w)|$. We shall use induction on $l(w)$ to prove that $n_1(w) = l(w)$ for all $w \in W$. Exactly similar arguments can be used to prove that $n_2(w) = l(w)$ for all $w \in W$.

If $l(w) = 0$ then $w = 1$ and clearly $n_1(1) = l(1) = 0$. Now assume that $l(w) > 0$. Then there exist $s \in S$ and $w' \in W$ such that $w = w'r_s$ and $l(w') = l(w) - 1$. If we can prove that

\[(1.2.3) \quad n_1(w') = n_1(w) - 1 \]

then it will follow from the inductive hypothesis that

$$n_1(w) = n_1(w') + 1 = l(w') + 1 = l(w)$$

as required. Observe that (1.2.3) will follow if we can prove that

$$r_s(N_1(w) \setminus \{\hat{\alpha}_s\}) = N_1(w').$$

Let $\hat{x} \in N_1(w')$, choosing the representative root $x$ to be positive. Observe that $\hat{x} \neq \hat{\alpha}_s$, since $l(w'r_s) > l(w')$ and hence $w'\alpha_s \in \Phi_1^+$ by Corollary 1.2.5. So part (i) yields that $r_s x \in \Phi_1^+$. Moreover, $w(r_s x) = w'r_s r_s x = w' x \in \Phi_1^-$ (since $\hat{x} \in N_1(w')$). Hence $r_s \hat{x} \in N_1(w)$, and $\hat{x} \in r_s N_1(w)$. But clearly $\hat{x} \neq r_s \hat{\alpha}_s$, and so it follows that

$$N_1(w') \subseteq r_s(N_1(w) \setminus \{\hat{\alpha}_s\}).$$
Conversely, let \( \hat{x} \in N_1(w) \setminus \{ \hat{\alpha}_s \} \) with \( x \in \Phi_1^+ \). By part (i) \( r_s x \in \Phi_1^+ \); moreover, \( w' r_s x = w r_s x = wx \in \Phi_1^- \) (since \( \hat{x} \in N_1(w) \)). Hence
\[
 r_s (N_1(w) \setminus \{ \hat{\alpha}_s \}) \subseteq N_1(w').
\]
Therefore \( r_s (N_1(w) \setminus \{ \hat{\alpha}_s \}) = N_1(w') \), as required. This completes the proof of (ii).

(iii) Suppose that \( \hat{x} \in N_1(w_1 w_2) \), where we may assume that \( x \in \Phi_1^+ \) without loss of generality. Observe that this gives \( w_1 w_2 x \in \Phi_1^- \).

If \( w_2 x \in \Phi_1^+ \) then \( \hat{x} \notin N_1(w_2) \), since \( x \) and \( w_2 x \) are both positive; furthermore, \( w_2 \hat{x} \in N_1(w_1) \), since \( w_2 x \) is positive and \( w_1 w_2 x \) is negative. So \( \hat{x} \in w_2^{-1} N_1(w_1) + N_1(w_2) \). On the other hand, if \( w_2 x \in \Phi_1^- \) then \( \hat{x} \in N_1(w_2) \) and \( w_2 \hat{x} \notin N_1(w_1) \), again giving \( \hat{x} \in w_2^{-1} N_1(w_1) + N_1(w_2) \).

Since \( \hat{x} \) was chosen arbitrarily it follows that
\[
N_1(w_1 w_2) \subseteq w_2^{-1} N_1(w_1) + N_1(w_2).
\]

Conversely, suppose that \( \hat{x} \in w_2^{-1} N_1(w_1) + N_1(w_2) \), choosing \( x \in \Phi_1^+ \).

Note that either \( \hat{x} \in N_1(w_2) \) and \( \hat{x} \notin w_2^{-1} N_1(w_1) \), giving \( \hat{x} \in N_1(w_2) \) and \( w_2 \hat{x} \notin N_1(w_1) \), or else \( \hat{x} \notin N_1(w_2) \) and \( \hat{x} \in w_2^{-1} N_1(w_1) \), in which case \( \hat{x} \notin N_1(w_2) \) and \( w_2 \hat{x} \in N_1(w_1) \).

If \( \hat{x} \in N_1(w_2) \) and \( w_2 \hat{x} \notin N_1(w_1) \), then \( w_2 x \) must be opposite in sign to \( x \), and \( w_1 w_2 x \) must be of the same sign as \( w_2 x \). Thus \( w_2 x \in \Phi_1^- \) and \( w_1 w_2 x \in \Phi_1^- \). On the other hand, if \( \hat{x} \notin N_1(w_2) \) and \( w_2 \hat{x} \in N_1(w_1) \) then \( w_2 x \) must be of the same sign as \( x \) and \( w_1 w_2 x \) must be opposite in sign to \( w_2 x \). Thus \( w_2 x \in \Phi_1^+ \) and \( w_1 w_2 x \in \Phi_1^- \). So \( w_1 w_2 x \in \Phi_1^- \) in either case, and since \( \hat{x} \in w_2^{-1} N_1(w_1) + N_1(w_2) \) was arbitrary it follows that
\[
w_2^{-1} N_1(w_1) + N_1(w_2) \subseteq N_1(w_1 w_2).
\]

Since this reverses the inclusion proved above we conclude that equality holds, and since exactly similar arguments apply for \( N_2 \), this completes the proof of (iii).

(iv) By part (ii) above \( l(w_1 w_2) = l(w_1) + l(w_2) \) if and only if \( |N_1(w_1 w_2)| = |N_1(w_1)| + |N_1(w_2)| = |w_2^{-1} N_1(w_1)| + |N_1(w_2)| \). By
part (iii) this happens if and only if \( w_2^{-1}N_1(w_1) \cap N_1(w_2) = \emptyset \), and this in turn happens if and only if \( N_1(w_2) \subseteq N_1(w_1w_2) \). As usual, the same reasoning applies for \( N_2 \).

If \( w = r_{s_1}r_{s_2} \cdots r_{s_l} (s_1, \ldots, s_l \in S) \) is such that \( l(w) = l \), then it is easy to see that

\[
N_1(w) = \{ \hat{\alpha}_{s_l}, r_{s_l} \hat{\alpha}_{s_{l-1}}, r_{s_l}r_{s_{l-1}} \hat{\alpha}_{s_{l-2}}, \ldots, r_{s_l}r_{s_{l-1}} \cdots r_{s_2} \hat{\alpha}_{s_1} \}
\]

and

\[
N_2(w) = \{ \hat{\beta}_{s_l}, r_{s_l} \hat{\beta}_{s_{l-1}}, r_{s_l}r_{s_{l-1}} \hat{\beta}_{s_{l-2}}, \ldots, r_{s_l}r_{s_{l-1}} \cdots r_{s_2} \hat{\beta}_{s_1} \}.
\]

If \( w_1, w_2 \in W \) such that \( l(w_1w_2) = l(w_1) + l(w_2) \), then we call \( w_2 \) a right hand segment of \( w_1w_2 \).

**Lemma 1.2.13.** \( W \) is finite if and only if \( \Phi_1 \) is finite, and if and only if \( \Phi_2 \) is finite.

**Proof.** Since \( |\Pi_1| = |S| = |\{ r_s \mid s \in S \}| \leq |W| \), it follows that \( |\Pi_1| < \infty \) whenever \( |W| < \infty \). Then

\[
|\Phi_1| = |\{ w\tilde{x} \mid w \in W \text{ and } x \in \Pi_1 \}|
\leq |\{ wx \mid w \in W \text{ and } x \in \Pi_1 \}|
\leq |W||\Pi_1|
< \infty.
\]

Conversely, assume that \( |\Phi_1| < \infty \). Define an equivalence relation \( \sim \) on \( \Phi_1 \) as follows: for \( x_1, x_2 \in \Phi_1, x_1 \sim x_2 \) if there is a positive \( \lambda \) such that \( x_1 = \lambda x_2 \). For each \( x \in \Phi_1 \), write \( \tilde{x} \) for the equivalence class containing \( x \) and set \( \tilde{\Phi}_1 := \{ \tilde{x} \mid x \in \Phi_1 \} \). Since \( -x \) is a root whenever \( x \) is a root, \( |\tilde{\Phi}_1| = 2|\Phi_1| < \infty \). The action of \( W \) on \( \Phi_1 \) naturally induces a well-defined action of \( W \) on \( \tilde{\Phi}_1 \) satisfying \( w\tilde{x} := \tilde{wx} \). Now for each \( w \in W \) define a map \( \sigma_w : \tilde{\Phi}_1 \to \tilde{\Phi}_1 \) by \( \sigma_w(\tilde{x}) := \tilde{wx} \) for all \( \tilde{x} \in \tilde{\Phi}_1 \). Then \( \sigma_w \) is a permutation of \( \tilde{\Phi}_1 \), and furthermore, \( w \mapsto \sigma_w \) is a homomorphism \( \sigma : W \to \text{Sym}(\tilde{\Phi}_1) \) (the symmetric group on \( \tilde{\Phi}_1 \)). Now if \( w \) is in the kernel of \( \sigma \) then \( w\tilde{x} = \tilde{x} \) for all \( x \in \Phi_1 \), and in particular, \( w\tilde{x} = \tilde{x} \) for
all \( x \in \Pi_1 \). But by Corollary 1.2.5 this means that \( l(wr_s) > l(w) \) for all \( s \in S \), and therefore \( w = 1 \). Thus \( \sigma \) is injective, and

\[ |W| \leq |\text{Sym}(\tilde{\Phi}_1)| = |\tilde{\Phi}_1|! < \infty \]

as required.

And as usual, exactly similar arguments yield that \( W \) is finite if and only if \( \tilde{\Phi}_2 \) is finite. \( \square \)

Let \( K \subseteq S \). If we define \( V_{1K} \) to be the subspace of \( V_1 \) spanned by \( \Pi_{1K} = \{ \alpha_s \mid s \in K \} \) and \( V_{2K} \) to be the subspace of \( V_2 \) spanned by \( \Pi_{2K} = \{ \beta_s \mid s \in K \} \), and let \( \langle , \rangle_K \) be the restriction of \( \langle , \rangle \) to \( V_{1K} \times V_{2K} \), then clearly \( (K, V_{1K}, V_{2K}, \Pi_{1K}, \Pi_{2K}, \langle , \rangle_K) \) is a Coxeter datum with parameters \( (m_{st} \mid s, t \in K) \). Write \( W_K = \langle r'_s \mid s \in K \rangle \) for the corresponding abstract Coxeter group, and let \( \eta: W_K \to W \) be the homomorphism defined by \( r'_s \mapsto r_s \) for all \( s \in K \). It follows immediately from the formulas for the actions of \( W \) on \( V_1 \) and \( W_K \) on \( V_{1K} \) that \( r'_sv = r_s v \) for all \( s \in K \) and \( v \in V_{1K} \), and therefore \( wv = \eta(w)v \) for all \( w \in W_K \) and \( v \in V_{1K} \). Since the action of \( W_{1K} \) on \( V_{1K} \) is faithful, it follows that \( \eta \) is injective. Thus \( W_K \) can be identified with the standard parabolic subgroup of \( W \) generated by the set \( \{ r_s \mid s \in K \} \).

**Definition 1.2.14.** Given \( K \subseteq S \), we define \( \Phi_{1K} \) and \( \Phi_{2K} \) to be the root systems of \( W_K \) in \( V_{1K} \) and \( V_{2K} \) respectively, and \( \Phi_{1K}^+, \Phi_{2K}^+ \) to be the corresponding sets of positive roots.

In other words,

\[ \Phi_{1K} = \{ w\alpha_r \mid w \in W_K \text{ and } r \in K \} \]

and

\[ \Phi_{1K}^+ = \Phi_{1K} \cap \text{PLC}(\Pi_{1K}), \]

and similarly for \( \Phi_{2K} \) and \( \Phi_{2K}^+ \).

**Remark 1.2.15.** It is a consequence of Lemma 1.2.13 that \( W_K \) is finite if and only if \( \tilde{\Phi}_{1K} \) and \( \tilde{\Phi}_{2K} \) are finite.
1.2. Root Systems and Canonical Coefficients

It is well known (and in fact follows easily from Corollary 1.2.5 and Lemma 1.2.12) that if \( W_K \) is finite then there is a unique longest element \( w_K \in W_K \), satisfying \( N_1(w_K) = \{ \hat{\alpha} | \alpha \in \Phi_{1K} \} \). However, for our present purposes we require this only in the special case that \( K \) has cardinality 2.

**Notation:** Let \( r, s \in S \) be such such \( m_{rs} < \infty \). Define \( w_{\{r,s\}} \in \langle r, r_s \rangle \) to be the element \( r_{rs} r_{rs} \cdots = r_{rs} r_{rs} \cdots \), where there are \( m_{rs} \) factors on each side, the factors being alternately \( r \) and \( r_s \).

**Lemma 1.2.16.** Let \( w \in W \) and \( s \in S \) be such that \( w \alpha_s = \nu \alpha_t \) for some positive scalar \( \nu \), and suppose that \( r \in S \) such that \( l(w r) < l(w) \).

Then

(i) \( \langle r, r_s \rangle \) is finite.

(ii) \( N_1(w_{\{r,s\}} r_s) = \Phi_{1\{r,s\}} \setminus \{ \hat{\alpha}_s \} \).

(iii) \( l(w_{\{r,s\}}^{-1} r_s) = l(w) - l(w_{\{r,s\}} r_s) \).

**Proof.** (i) Since \( l(w r) < l(w) \), Corollary 1.2.5 yields that \( w \alpha_r \in \Phi_{1}^- \). Furthermore, \( w \hat{\alpha}_r \neq \hat{\alpha}_t \), for otherwise \( \hat{\alpha}_r = \hat{\alpha}_s \), forcing \( r = s \) by Lemma 1.1.5, contrary to the fact that \( w \alpha_s \) is positive and \( w \alpha_r \) is negative. Now observe that

\[
wr_s(\alpha_r) = w(\alpha_r - 2(\alpha_r, \beta_s) \alpha_s) = w(\alpha_r) - 2(\alpha_r, \beta_s)w(\alpha_s)
\]

\[
= w(\alpha_r) - 2(\alpha_r, \beta_s)\alpha_t \in \Phi_1^- \setminus R \{ \alpha_t \} 
\]

a scalar multiple of \( \alpha_t \)

and assume, for a contradiction, that \( wr_s(\alpha_r) \in \text{PLC}(\Pi_1) \). Rearranging the above equation gives

\[
wr_s(\alpha_r) - w(\alpha_r) = -2(\alpha_r, \beta_s)\alpha_t \in \text{PLC}(\Pi_1) 
\]

a positive scalar multiple of \( \alpha_t \)

and, since \( -w(\alpha_r) \in (\Phi_1^+ \setminus R \{ \alpha_t \}) \), the left hand side is not a scalar multiple of \( \alpha_t \). So, moving the \( \alpha_t \) term from the left side to the right, this expresses \( \lambda \alpha_t \), for some scalar \( \lambda \), as a positive linear combination of \( \Pi_1 \setminus \{ \alpha_t \} \). But if \( \lambda \leq 0 \) this implies that \( 0 \in \text{PLC}(\Pi_1) \), contradicting condition (C6) of a Coxeter datum, while if \( \lambda > 0 \) it
implies that \( \alpha_t \in \text{PLC}(\Pi_1 \setminus \{\alpha_t\}) \) contradicting Lemma 1.1.5. Therefore \( wr_s(\alpha_r) \in \Phi_1^- \). Since both \( wr_s(\alpha_r) \) and \( wr_s(\alpha_s) = -\alpha_t \) are negative, it follows that \( wr_s(\lambda\alpha_r + \mu\alpha_s) \) is a negative linear combination of \( \Pi_1 \) whenever \( \lambda, \mu \geq 0 \). Hence \( wr_s(\lambda\alpha_r + \mu\alpha_s) \in \Phi_1^- \) whenever \( \lambda\alpha_r + \mu\alpha_s \in \Phi_1^+ \{r,s\} \). This says precisely that \( \Phi_1^+ \{r,s\} \subseteq N_1(wr_s) \). Since \( N_1(wr_s) \) is a finite set of size \( l(wr_s) \) by Lemma 1.2.12(ii), it follows from Remark 1.2.15 above that \( \langle r_r, r_s \rangle \) must be finite. Therefore \( w_{\{r,s\}} \) exists.

(ii) Since \( w_{\{r,s\}} \) is the longest element in \( W_{\{r,s\}} \) it follows that \( l(w_{\{r,s\}}r_r) < l(w_{\{r,s\}}r_s) \) and \( l(w_{\{r,s\}}r_r) < l(w_{\{r,s\}}r_s) \). So Corollary 1.2.5 yields that \( w_{\{r,s\}}r_r \in \Phi_1^- \) and \( w_{\{r,s\}}r_s \in \Phi_1^- \). This then implies that \( \Phi_1^+ \{r,s\} \subseteq N_1(w_{\{r,s\}}) \). On the other hand \( w_{\{r,s\}} = \underbrace{r_r r_r \cdots}_{m_{rs}} r_r r_r r_s r_s \cdots, \underbrace{m_{rs}}_{m_{rs}} \text{ factors} \underbrace{m_{rs}}_{m_{rs}} \text{ factors} \)

so it follows from Lemma 1.2.12 (ii) and the repeated application of Lemma 1.2.12 (iii) that \( N_1(w_{\{r,s\}}) \subseteq \Phi_1^+ \{r,s\} \) whence \( \Phi_1^+ \{r,s\} = \Phi_1^\{r,s\} \). Observe that \( w_{\{r,s\}}r_s \alpha_s = -w_{\{r,s\}}r_s \alpha_s \in \Phi_1^+ \), and so

\[
\Phi_1^+ \{r,s\} \setminus \{\hat{\alpha}\} \subseteq N_1(w_{\{r,s\}}r_s).
\]

Thus to show the desired result, we only need to establish that

\[
|N_1(w_{\{r,s\}}r_s)| = |\Phi_1^+ \{r,s\}| - 1.
\]

Indeed, Lemma 1.2.12 (ii) yields that \( |N_1(w_{\{r,s\}}r_s)| = l(w_{\{r,s\}}r_s) \), and we have already checked that \( l(w_{\{r,s\}}r_s) = l(w_{\{r,s\}}r_s) - 1 \), thus

\[
|N_1(w_{\{r,s\}}r_s)| = l(w_{\{r,s\}}r_s) = l(w_{\{r,s\}}r_s) - 1 = |\Phi_1^+ \{r,s\}| - 1.
\]

(iii) To show that \( w_{\{r,s\}}r_s \) is a right hand segment of \( w \), it is enough to show that \( N_1(w_{\{r,s\}}r_s) \subseteq N_1(w) \) by Lemma 1.2.12(iv). So let \( \alpha \) be an arbitrary positive root such that \( \hat{\alpha} \in N_1(w_{\{r,s\}}r_s) \), and observe by part (ii) that \( \alpha = \lambda\alpha_r + \mu\alpha_s \) for some \( \lambda > 0 \) and \( \mu \geq 0 \). Thus

\[
w\alpha = \lambda w\alpha_r + \mu w\alpha_s = \lambda w\alpha_r + \mu w\alpha_s.
\]
Suppose for a contradiction that \( w\alpha \in \Phi_1^+ \). Then

\[
\frac{w\alpha - \lambda w(\alpha_r)}{\mu \nu \alpha_t} = \text{a positive scalar multiple of } \alpha_t,
\]

and, since \(-w(\alpha_r) \in (\Phi_1^+ \setminus \mathbb{R}\{\alpha_t\})\), the left hand side is not a scalar multiple of \( \alpha_t \). So, moving the \( \alpha_t \) term from the left side to the right, this expresses \( \lambda' \alpha_t \), for some scalar \( \lambda' \), as a positive linear combination of \( \Pi_1 \{ \alpha_t \} \). But if \( \lambda' \leq 0 \) this implies that \( 0 \in \text{PLC}(\Pi_1) \), contradicting condition (C6) of a Coxeter datum, while if \( \lambda' > 0 \) it implies that \( \alpha_t \in \text{PLC}(\Pi_1 \setminus \{ \alpha_t \}) \) contradicting Lemma 1.1.5. Therefore \( w\alpha \in \Phi_1^- \), and therefore \( \hat{\alpha} \in N_1(w) \). Hence all elements of \( N_1(w_{\{r,s\}r_s}) \) lie in \( N_1(w) \), as required. \( \square \)

Now we are ready to prove Proposition 1.2.9:

**Proof of Proposition 1.2.9.** Since \( \Phi_1' = \{ w\alpha'_s \mid w \in W \text{ and } s \in S \} \), to prove that the restriction of \( \pi_1 \) to \( \Phi_1' \) is bijective it suffices to show that if \( \pi_1(w\alpha'_s) = \pi_1(v\alpha'_t) \) for some \( w, v \in W \) and \( s, t \in S \), then \( w\alpha'_s = v\alpha'_t \). Since \( \pi_1 \) is a \( W \)-homomorphism and \( \pi_1(\alpha'_s) = \alpha_s \) for all \( s \in S \) we see that \( \pi_1(w\alpha'_s) = w\alpha_s \) and \( \pi_1(v\alpha'_t) = v\alpha_t \), and deduce that it suffices to prove the following statement: if \( w\alpha_s = \alpha_t \) for some \( w \in W \) and \( s, t \in S \), then \( w\alpha'_s = \alpha'_t \).

We assume that \( w(\alpha_s) = \alpha_t \), and proceed by an induction on \( l(w) \).

The case \( l(w) = 0 \) reduces to the statement that if \( \alpha_s = \alpha_t \) for some \( s, t \in S \) then \( \alpha'_s = \alpha'_t \), which is trivially true since \( \Pi_1 \) and \( \Pi_1' \) are both assumed to be in bijective correspondence with \( S \). So we may assume that \( l(w) > 0 \), and choose \( r \in S \) such that \( l(wr_r) < l(w) \). Lemma 1.2.16 yields that \( \langle r_r, r_s \rangle \) is a finite dihedral group (hence \( m_{rs} \) is finite), and

\[
l(w(w_{\{r,s\}r_s}^{-1})) = l(w) - l(w_{\{r,s\}r_s}).
\]

We treat separately the cases \( m_{rs} \) even and \( m_{rs} \) odd.
If $m_{rs} = 2k$ is even, then $w_{(r,s)} = (r_r r_s)^k = (r_s r_r)^k$, and then the formulas in Lemma 1.1.6 yield

\begin{equation}
(w_{(r,s)} r_s)\alpha_s = -w_{(r,s)} \alpha_s = -(r_s r_r)^k \alpha_s
\end{equation}

\begin{equation}
\begin{aligned}
&= -\frac{\sin((m_{st} + 1)\frac{\pi}{m_{st}})}{\sin(\frac{\pi}{m_{st}})} \alpha_s
- \frac{\cos(\frac{\pi}{m_{st}})}{\langle \alpha_r, \beta_s \rangle} \sin(\frac{\pi}{m_{st}}) \alpha_r \\
&= \alpha_s,
\end{aligned}
\end{equation}

and by exactly the same calculation in $V'_1$,

\begin{equation}
(w_{(r,s)} r_s)(\alpha'_s) = \alpha'_s.
\end{equation}

Now since

\begin{equation}
\alpha_t = w\alpha_s = w(w_{(r,s)} r_s)^{-1}(w_{(r,s)} r_s)(\alpha_s) = w(w_{(r,s)} r_s)^{-1}(\alpha_s)
\end{equation}

as required.

If $m_{st} = 2k + 1$ is odd, then $w_{(r,s)} r_s = \ldots r_r r_s r_r = (r_s r_r)^k$. Now the formulas in Lemma 1.1.6 yield that

\begin{equation}
(w_{(r,s)} r_s)\alpha_s = (r_s r_r)^k \alpha_s
\end{equation}

\begin{equation}
\begin{aligned}
&= \frac{\sin(\pi)}{\sin(\frac{\pi}{m_{st}})} \alpha_s
+ \frac{-\cos(\frac{\pi}{m_{st}})}{\langle \alpha_r, \beta_s \rangle} \sin(2k\frac{\pi}{m_{st}}) \alpha_r \\
&= \frac{-\cos(\frac{\pi}{m_{st}})}{\langle \alpha_r, \beta_s \rangle} \alpha_r,
\end{aligned}
\end{equation}
and by exactly the same calculation in $V'_1$,

\[(w_{\{r,s\}}r_s)\alpha'_s = \frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s \rangle} \alpha'_r.\]  

(1.2.10)

Now since

\[\alpha_t = w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha_s\]

(1.2.11)

and $l(w(w_{\{r,s\}}r_s)^{-1}) < l(w)$, the inductive hypothesis yields that

\[\alpha'_t = w(w_{\{r,s\}}r_s)^{-1}\left(\frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s \rangle} \alpha'_r \right).\]  

(1.2.12)

It follows that

\[w\alpha'_s = w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha'_s\]

\[= w(w_{\{r,s\}}r_s)^{-1}\left(\frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s \rangle} \alpha'_r \right)\]  

(by (1.2.10))

\[= \alpha'_t\]  

(by (1.2.12))

as required.

The other half of the proposition follows by similar arguments applied in $V_2$ and $V'_2$. □

While we are at it, we can also prove the following:

**Proposition 1.2.17.** Suppose that $s, t \in S$. If $w(\alpha_s) = \nu \alpha_t$ for some $w \in W$ and some non-zero $\nu$, then $w(\beta_s) = \frac{1}{\nu} \beta_t$.

**Proof.** If $l(w) = 0$, then $s = t$ and $\nu = 1$, and there is nothing to prove. Thus we may assume that $l(w) \geq 1$ and proceed by an induction on $l(w)$. Choose $r \in S$, such that $l(wr) = l(w) - 1$. Again by Lemma 1.2.16 $\langle r, r_s \rangle$ is finite (and $m_{rs}$ is finite too), and hence $w_{\{r,s\}}$ exists. The same rank 2 calculation as used in the proof of Proposition 1.2.9 above yields that

\[(w_{\{r,s\}}r_s)(\alpha_s) = \begin{cases} 
\alpha_s & \text{if } m_{rs} \text{ is even}, \\
-\frac{\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s \rangle} \alpha_r & \text{if } m_{rs} \text{ is odd}.
\end{cases}\]  

(1.2.13)
Similarly

\[(1.2.14) \quad (w_{rs}r_s)(\beta_s) = \begin{cases} 
\beta_s & \text{if } m_{rs} \text{ is even}, \\
-\cos(\pi/m_{rs}) \beta_r \langle \alpha_s, \beta_r \rangle & \text{if } m_{rs} \text{ is odd}.
\end{cases}\]

First suppose that \(m_{rs}\) is even. Since

\[\nu \alpha_t = w \alpha_s = w(w_{rs}r_s)^{-1}(w_{rs}r_s)(\alpha_s) = w(w_{rs}r_s)^{-1}(\alpha_s)\]

by (1.2.13) and \(l(w(w_{rs}r_s)^{-1}) < l(w)\), the inductive hypothesis yields that

\[(1.2.15) \quad \frac{1}{\nu} \beta_t = w(w_{rs}r_s)^{-1}(\beta_s).\]

It follows that

\[w \beta_s = w(w_{rs}r_s)^{-1}(w_{rs}r_s)(\beta_s) = w(w_{rs}r_s)^{-1}(\beta_s) = \frac{1}{\nu} \beta_t\]

by (1.2.14)

and hence the desired result follows by induction.

Next suppose that \(m_{rs}\) is odd. Since

\[\nu \alpha_t = w \alpha_s = w(w_{rs}r_s)^{-1}(w_{rs}r_s)(\alpha_s) = w(w_{rs}r_s)^{-1}(\alpha_s)\]

by (1.2.14)

or, equivalently,

\[-\nu \frac{\cos(\pi/m_{rs})}{\langle \alpha_s, \beta_r \rangle} \alpha_t = w(w_{rs}r_s)^{-1} \alpha_r,\]

and since \(l(w(w_{rs}r_s)^{-1}) < l(w)\), the inductive hypothesis yields that

\[-\frac{\cos(\pi/m_{rs})}{\nu \langle \alpha_s, \beta_r \rangle} \beta_t = w(w_{rs}r_s)^{-1} \beta_r.\]

That is

\[(1.2.16) \quad \frac{1}{\nu} \beta_t = w(w_{rs}r_s)^{-1}(-\frac{\cos(\pi/m_{rs})}{\langle \alpha_s, \beta_r \rangle} \beta_r),\]

and it follows that

\[w \beta_s = w(w_{rs}r_s)^{-1}(w_{rs}r_s)\beta_s\]

\[= w(w_{rs}r_s)^{-1}(-\frac{\cos(\pi/m_{rs})}{\langle \alpha_s, \beta_r \rangle} \beta_r) = \frac{1}{\nu} \beta_t\]

by (1.2.14) and (1.2.16).
whence the desired result follows by induction.

It follows immediately from Proposition 1.2.17 that there is a well-defined map \( \Phi_1 \rightarrow \Phi_2 \) such that \( w\alpha_s \mapsto w\beta_s \) for all \( s \in S \) and \( w \in W \). This is clearly the unique \( W \)-equivariant map \( \Phi_1 \rightarrow \Phi_2 \) satisfying \( \alpha_s \mapsto \beta_s \) for all \( s \in S \).

**Definition 1.2.18.** Let \( \phi : \Phi_1 \rightarrow \Phi_2 \) be the \( W \)-equivariant map satisfying \( \phi(\alpha_s) = \beta_s \) for all \( s \in S \).

It follows readily from Proposition 1.2.17 and \( W \)-equivariance that \( \phi \) is a bijection and the following holds:

**Corollary 1.2.19.** Given \( t \in S \) and a nonzero scalar \( \nu \), then \( \nu\alpha_t \in \Phi_1 \) if and only if \( \frac{1}{\nu}\beta_t \in \Phi_2 \). Furthermore, if \( \nu\alpha_t \in \Phi_1 \) then \( \phi(\nu\alpha_t) = \frac{1}{\nu}\beta_t \).

In fact we can generalize the above, as follows.

**Lemma 1.2.20.** Suppose that \( \alpha \in \Phi_1 \), and \( \nu \neq 0 \) such that \( \nu\alpha \in \Phi_1 \). Then

\[ \phi(\nu\alpha) = \frac{1}{\nu}\phi(\alpha). \]

**Proof.** Since \( \alpha \in \Phi_1 \), we may write \( \alpha = w\alpha_t \) for some \( w \in W \) and \( t \in S \), and since \( \nu\alpha \in \Phi_1 \) it follows that \( \nu\alpha_t = \nu w^{-1}\alpha = w^{-1}(\nu\alpha) \) is in \( \Phi_1 \). Hence by Corollary 1.2.19 and the \( W \)-equivariance of \( \phi \),

\[
\phi(\nu\alpha) = \phi(\nu(w\alpha_t)) = \phi(w(\nu\alpha_t)) = w\phi(\nu\alpha_t) = w(\frac{1}{\nu}\phi(\alpha_t)) = \frac{1}{\nu}\phi(w\alpha_t) = \frac{1}{\nu}\phi(\alpha)
\]

as required.

Equipped with Proposition 1.2.9, we may define the canonical coefficient of any simple root of \( \Pi_1 \) (respectively, of \( \Pi_2 \)) in a given root of \( \Phi_1 \) (respectively, of \( \Phi_2 \)). Let \( \alpha \in \Phi_1 \) and \( \beta \in \Phi_2 \), and let \( \alpha' = \phi_1(\alpha) \) and \( \beta' = \phi_2(\beta) \) be the corresponding elements of \( \Phi_1' \) and \( \Phi_2' \). Since
Π′_1 and Π′_2 are linearly independent, for each \( t \in S \) there are uniquely determined real numbers \( \text{coeff}_{\alpha_t}(\alpha) \) and \( \text{coeff}_{\beta_t}(\beta) \) satisfying

\[
\alpha' = \sum_{t \in S} \text{coeff}_{\alpha_t}(\alpha) \alpha'_t,
\]

and

\[
\beta' = \sum_{t \in S} \text{coeff}_{\beta_t}(\beta) \beta'_t.
\]

Then Proposition 1.2.9 yields that

\[
\alpha = \pi_1(\alpha') = \pi_1(\sum_{t \in S} \text{coeff}_{\alpha_t}(\alpha) \alpha'_t) = \sum_{t \in S} \text{coeff}_{\alpha_t}(\alpha) \alpha_t
\]

and

\[
\beta = \pi_2(\beta') = \pi_2(\sum_{t \in S} \text{coeff}_{\beta_t}(\beta) \beta'_t) = \sum_{t \in S} \text{coeff}_{\beta_t}(\beta) \beta_t.
\]

**Definition 1.2.21.** Suppose that \( \alpha \in \Phi_1 \) and \( \beta \in \Phi_2 \). For each \( t \in S \), we call \( \text{coeff}_{\alpha_t}(\alpha) \) the *canonical coefficient* of \( \alpha_t \) in the root \( \alpha \) and \( \text{coeff}_{\beta_t}(\beta) \) the *canonical coefficient* of \( \beta_t \) in \( \beta \).

### 1.3. The canonical coefficients and the depth of roots

In this section we study the canonical coefficients defined in section 1.2. It turns out that for each \( s \in S \) and \( \alpha \in \Phi_1 \), the coefficient \( \text{coeff}_{\alpha_s}(\alpha) \) is closely related to the coefficient \( \text{coeff}_{\phi_s}(\phi(\beta)) \), where \( \phi \) is the \( W \)-equivariant map defined in Definition 1.2.18. Also we introduce the concept of *depth* of a root following the conventions of [6] and [5], and we give a characterisation of the depth of a root in terms of the length of the reflection corresponding to that root.

**Definition 1.3.1.** Given \( \alpha \in \Phi_1 \) and \( \beta \in \Phi_2 \), define the *support of \( \alpha \)*, written \( \text{supp}(\alpha) \), and the *support of \( \beta \)*, written \( \text{supp}(\beta) \), to be the sets:

\[
\text{supp}(\alpha) = \{ \alpha_s \mid s \in S \text{ and } \text{coeff}_{\alpha_s}(\alpha) \neq 0 \}
\]

and

\[
\text{supp}(\beta) = \{ \beta_s \mid s \in S \text{ and } \text{coeff}_{\beta_s}(\beta) \neq 0 \}.
\]
The following group-theoretic result is purely about Coxeter groups and is well known (for example, Section 2 of [14]). It is essential for Proposition 1.3.3 below and it is going to be used repeatedly in Section 2.3 later. For completeness, a proof is included here.

**Lemma 1.3.2.** Let $I \subseteq S$ and $w \in W$. Choose $w' \in wW_I$ to be of minimal length. Then $l(w'v) = l(w') + l(v)$ for all $v \in W_I$. In particular, $w'\alpha_s \in \Phi_1^+$ whenever $s \in I$.

**Proof.** We prove this result by an induction on $l(v)$, the case $l(v) = 0$ being trivial. So assume that $l(v) > 0$, and write $v = v'r_s$ for some $v' \in W_I$ and $s \in I$ such that $l(v') = l(v) - 1$. If we can show that $l(w'v) = l(w'v') + 1$ then the desired result will follow by induction. By Corollary 1.2.5, to prove that $l(w'v) = l(w'v') + 1$ it suffices to show that $(w'v')(\alpha_s) \in \Phi_1^+$. Observe that, for any $t \in I$, the minimality of $l(w')$ in $wW_I$ together with Corollary 1.2.5 implies that $l(w'r_t) = l(w') + 1$, and $w'\alpha_t \in \Phi_1^+$. Moreover, since $l(v'r_s) = l(v) = l(v') + 1$, Corollary 1.2.5 yields that $v'\alpha_s \in \Phi_1^+$, and so we may write $v'\alpha_s = \sum \lambda_t \alpha_t$, where $\lambda_t \geq 0$ for all $t \in I$. Since $w'\alpha_t \in \Phi_1^+ \subseteq PLC(\Pi_1)$ for all $t \in I$, it follows that $(w'v')(\alpha_s) = \sum \lambda_t w'\alpha_t \in \Phi_1^+ \cap PLC(\Pi_1) = \Phi_1^+$, as required.

Corollary 1.2.19 says that if $t \in S$ and $\lambda$ is any nonzero scalar then $\lambda \alpha_t$ is a root if and only if $\frac{1}{\lambda} \beta_t$ is root. Thus in the case of singleton support roots $\alpha = \lambda \alpha_t$ and $\phi(\alpha) = \frac{1}{\lambda} \beta_t$ ($t \in S$), for any $s \in S$ we have either $\text{coeff}_{\alpha_s}(\phi(\alpha)) = 0$ (if $s \neq t$) or $\text{coeff}_{\alpha_s}(\phi(\alpha)) = 1$ (if $s = t$). An extension to this fact is given in the next proposition.

**Proposition 1.3.3.** Let $\alpha \in \Phi_1^+$. The for each $t \in S$ we have

$$\text{coeff}_{\alpha_t}(\alpha) > 0 \text{ if and only if } \text{coeff}_{\beta_t}(\phi(\alpha)) > 0$$

and in this case $\text{coeff}_{\alpha_t}(\alpha) \text{ coeff}_{\beta_t}(\phi(\alpha)) \geq 1$.

**Proof.** We use induction on the length of $w \in W$ to prove that the statement holds whenever $\alpha = w\alpha_r$ for some $r \in S$. Note that $\alpha = w\alpha_r$
implies that $\phi(\alpha) = w\phi(\alpha_r) = w\beta_r$. If $l(w) = 0$ then the assertion is trivially true.

Suppose then that $l(w) > 0$, and let $s \in S$ and $w' \in W$ be such that $w = w'r_s$ and $l(w') = l(w) - 1$. Choose $w''$ to be a minimal length element of $w'(r_r, r_s)$. Then $l(w'') \leq l(w'r_s) < l(w)$, and hence the inductive hypothesis applies to $w''\alpha_r$ as well as to $w''\alpha_s$, so that for all $t \in S$ either $\text{coeff}_{\alpha_t}(w''\alpha_r) \text{coeff}_{\beta_t}(w''\beta_r) \geq 1$ or else $\text{coeff}_{\alpha_t}(w''\alpha_r)$ and $\text{coeff}_{\beta_t}(w''\beta_r)$ are both zero, and either $\text{coeff}_{\alpha_t}(w''\alpha_s) \text{coeff}_{\beta_t}(w''\beta_s) \geq 1$ or else $\text{coeff}_{\alpha_t}(w''\alpha_s)$ and $\text{coeff}_{\beta_t}(w''\beta_s)$ are both zero.

Since $l(w''r_r), l(w''r_s) \geq l(w'')$ by the minimality of $l(w'')$ in the coset $w'(r_r, r_s)$, it follows from Corollary 1.2.5 that $w''\alpha_r, w''\beta_r, w''\alpha_s$ and $w\beta_s$ are all positive. Thus for all $t \in S$,

$$(1.3.1) \quad \text{coeff}_{\alpha_t}(w''\alpha_r) \geq 0 \quad \text{and} \quad \text{coeff}_{\beta_t}(w''\beta_r) \geq 0$$

and

$$(1.3.2) \quad \text{coeff}_{\alpha_t}(w''\alpha_s) \geq 0 \quad \text{and} \quad \text{coeff}_{\beta_t}(w''\beta_s) \geq 0.$$ 

Let $u \in \langle r_r, r_s \rangle$ such that $w = w''u$. Since $w''$ is of minimal length in the coset $w'(r_r, r_s)$, it follows from Lemma 1.3.2 that $l(w) = l(w'') + l(u)$, and hence $N_1(u) \subseteq N_1(w)$ by Lemma 1.2.12. Since $w\alpha_r \in \Phi_1^+$, it follows that $u\alpha_r \in \Phi_1^+$ too. Hence Corollary 1.2.5 yields that $l(ur_r) > l(u)$, and, in particular, $u$ has no reduced expression ending in $r_r$. This implies that $l(u) < m_{rs}$, and so Lemma 1.1.11 yields that

$$u\alpha_r = \lambda_1 \alpha_r + \mu_1 \alpha_s \quad \text{and} \quad u\beta_r = \lambda_2 \beta_r + \mu_2 \beta_s$$

for some some nonnegative scalars $\lambda_1, \lambda_2, \mu_1$ and $\mu_2$. Observe that this implies

$$u\phi_1(\alpha_r) = \lambda_1 \phi_1(\alpha_r) + \mu_1 \phi_1(\alpha_s) \quad \text{and} \quad u\phi_2(\beta_r) = \lambda_2 \phi_2(\beta_r) + \mu_2 \phi_2(\beta_s)$$

Hence

$$\phi_1(w\alpha_r) = w''\phi_1(u\alpha_r) = \lambda_1 w''\phi_1(\alpha_r) + \mu_1 w''\phi_1(\alpha_s)$$

and
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\[ \phi_2(w\beta_r) = w''\phi_2(u\beta_r) = \lambda_2 w'' \phi_2(\beta_r) + \mu_2 w'' \phi_2(\beta_s). \]

It follows that for all \( t \in S \),

\[
\text{coeff}_{\alpha_t}(w\alpha_r) = \text{coeff}_{\alpha_t}(w''(w\alpha_r)) = \lambda_1 \text{coeff}_{\alpha_t}(w'' \alpha_r) + \mu_1 \text{coeff}_{\alpha_t}(w'' \alpha_s),
\]

(1.3.3)

and likewise \( \text{coeff}_{\beta_t}(w\beta_r) = \lambda_2 \text{coeff}_{\beta_t}(w'' \beta_r) + \mu_2 \text{coeff}_{\beta_t}(w'' \beta_s). \)

A direct rank 2 calculation (as in Lemmas 1.1.6 and 1.1.8) yields explicit formulas for the coefficients \( \lambda_1, \mu_1, \lambda_2, \) and \( \mu_2 \), and we find that \( \lambda_1 > 0 \) if and only if \( \lambda_2 > 0 \) and \( \mu_1 > 0 \) if and only if \( \mu_2 > 0 \).

Moreover, if \( m_{rs} = m < \infty \) then \( \lambda_1 \lambda_2 \) and \( \mu_1 \mu_2 \) belong to the set \( \{(\sin k\theta)^2 \mid 1 \leq k \leq m - 1\} \cup \{0\} \), where \( \theta = \pi/m \), while if \( m_{rs} = \infty \) then \( \lambda_1 \lambda_2 \) and \( \mu_1 \mu_2 \) belong to \( \{(\sinh k\theta)^2 \mid k \in \mathbb{N}\} \cup \{0\} \), where \( \theta = \cosh^{-1}(\sqrt{\langle \alpha_r, \beta_s \rangle \langle \alpha_s, \beta_r \rangle}) \). It follows in either case that if \( \lambda_1 \lambda_2 \) or \( \mu_1 \mu_2 \) is nonzero then it is at least 1.

Suppose that \( \text{coeff}_{\alpha_t}(w\alpha_r) > 0 \). Then (1.3.3) shows that one of \( \lambda_1 \text{coeff}_{\alpha_t}(w'' \alpha_r) \) or \( \mu_1 \text{coeff}_{\alpha_t}(w'' \alpha_s) \) must be strictly positive. Now if \( \lambda_1 \text{coeff}_{\alpha_t}(w'' \alpha_r) > 0 \) then \( \lambda_1 > 0 \) and \( \text{coeff}_{\alpha_t}(w'' \alpha_r) > 0 \), and since \( l(w'') < l(w) \) the inductive hypothesis yields that \( \text{coeff}_{\beta_t}(w'' \beta_r) > 0 \). Thus in this case (1.3.4) yields that for all \( t \in S \),

\[
\text{coeff}_{\beta_t}(w\beta_r) = \lambda_2 \text{coeff}_{\beta_t}(w'' \beta_r) + \mu_2 \text{coeff}_{\beta_t}(w'' \beta_s) \\
\geq \lambda_2 \text{coeff}_{\beta_t}(w'' \beta_r) \\
> 0
\]

since \( \mu_2 \) and \( \text{coeff}_{\beta_t}(w'' \beta_s) \) are both nonnegative. On the other hand, if \( \mu_1 \text{coeff}_{\alpha_t}(w'' \alpha_r) > 0 \) a similar argument yields that

\[
\text{coeff}_{\beta_t}(w\beta_r) \geq \mu_2 \text{coeff}_{\beta_t}(w'' \beta_s) > 0,
\]

so that \( \text{coeff}_{\alpha_t}(w\alpha_r) > 0 \) implies \( \text{coeff}_{\beta_t}(w\beta_r) > 0 \) in either case.

By symmetry, we can also deduce that \( \text{coeff}_{\beta_t}(w\beta_r) > 0 \) implies \( \text{coeff}_{\alpha_t}(w\alpha_r) > 0 \), completing the proof of the first part.
Suppose that \( t \in S \) and \( \text{coeff}_{\alpha}(w\alpha_r) \) and \( \text{coeff}_{\beta}(w\beta_r) \) are positive. Observe from (1.3.3) that either \( \lambda_1 \text{coeff}_{\alpha}(w''\alpha_r) \) or \( \mu_1 \text{coeff}_{\alpha}(w''\alpha_s) \) must be positive, and consider first the case \( \lambda_1 \text{coeff}_{\alpha}(w''\alpha_r) > 0 \). Then \( \lambda_1 > 0 \), which implies \( \lambda_1 \lambda_2 \geq 1 \) by the rank 2 calculation, and \( \text{coeff}_{\alpha}(w''\alpha_r) > 0 \), which implies that \( \text{coeff}_{\alpha}(w''\alpha_r) \text{coeff}_{\beta}(w''\beta_r) \geq 1 \) by the inductive hypothesis. Hence
\[
\text{coeff}_{\alpha}(w\alpha_r) \text{coeff}_{\beta}(w\beta_r) \geq (\lambda_1 \text{coeff}_{\alpha}(w''\alpha_r))(\lambda_2 \text{coeff}_{\beta}(w''\beta_r)) \]
\[
= (\lambda_1 \lambda_2)(\text{coeff}_{\alpha}(w''\alpha_r) \text{coeff}_{\beta}(w''\beta_r)) \]
\[
\geq 1.
\]
On the other hand, if \( \mu_1 \text{coeff}_{\alpha}(w''\alpha_s) > 0 \) the same conclusion follows, since
\[
\text{coeff}_{\alpha}(w\alpha_r) \text{coeff}_{\beta}(w\beta_r) \geq (\mu_1 \text{coeff}_{\alpha}(w''\alpha_s))(\mu_2 \text{coeff}_{\beta}(w''\beta_s)) \]
\[
= (\mu_1 \mu_2)(\text{coeff}_{\alpha}(w''\alpha_s) \text{coeff}_{\beta}(w''\beta_s)) \]
\[
\geq 1.
\]
by the rank 2 calculation and the inductive hypothesis, and this completes the proof of the second assertion of the proposition. \( \square \)

It is readily seen from Proposition 1.3.3 that:

**Corollary 1.3.4.** Let \( \alpha \in \Phi_1 \), then
\[
\alpha \in \Phi_1^+ \text{ if and only if } \phi(\alpha) \in \Phi_2^+,
\]
and
\[
\alpha \in \Phi_1^- \text{ if and only if } \phi(\alpha) \in \Phi_2^-.
\]
Furthermore, \( \text{supp}(\phi(\alpha)) = \phi(\text{supp}(\alpha)) \).

**Proof.** Observe the following if and only if statements: \( \alpha \in \Phi_1^+ \) if and only if for some \( t \in S \), \( \text{coeff}_{\alpha}(\alpha) > 0 \), and by Proposition 1.3.3 this happens if and only \( \text{coeff}_{\beta}(\phi(\alpha)) > 0 \) for some \( t \in S \), and this in turn happens if and only if \( \phi(\alpha) \in \Phi_2^+ \).
Replace $\alpha$ by $-\alpha$, then the above yields that $\alpha \in \Phi^-_1$ if and only if $\phi(\alpha) \in \Phi^-_2$.

Finally, Proposition 1.3.3 yields that $\text{supp}(\phi(\alpha)) = \phi(\text{supp}(\alpha))$. \hfill $\Box$

**Definition 1.3.5.** (i) For each $\alpha \in \Phi^+_1$ and $\beta \in \Phi^+_2$ define the depth of $\alpha$ (written $dp_1(\alpha)$) and the depth of $\beta$ (written $dp_2(\beta)$) to be

$$dp_1(\alpha) = \min\{l(w) \mid w \in W \text{ and } w\alpha \in \Phi^-_1\},$$

and

$$dp_2(\beta) = \min\{l(w) \mid w \in W \text{ and } w\beta \in \Phi^-_2\}.$$

(ii) For $\alpha_1, \alpha_2 \in \Phi^+_1$ (respectively $\beta_1, \beta_2 \in \Phi^+_2$), write $\alpha_1 \preceq_1 \alpha_2$ (resp. $\beta_1 \preceq_2 \beta_2$) if and only if there exists $w \in W$ (resp. $w' \in W$) such that $\alpha_2 = w\alpha_1$, and $dp_1(\alpha_2) = dp_1(\alpha_1) + l(w)$ (respectively $\beta_2 = w'\beta_1$, and $dp_2(\beta_2) = dp_2(\beta_1) + l(w')$). Further we write $\alpha_1 \prec_1 \alpha_2$ if $\alpha_1 \preceq_1 \alpha_2$ but $\alpha_1 \neq \alpha_2$, and we write $\beta_1 \prec_2 \beta_2$ if $\beta_1 \preceq_2 \beta_2$ but $\beta_1 \neq \beta_2$.

**Lemma 1.3.6.** (Lemma 1.6 of [6]) $\preceq_1$ and $\preceq_2$ are partial orderings on $\Phi^+_1$ and $\Phi^+_2$ respectively. \hfill $\Box$

The following result can be easily deduced from Definition 1.3.5.

**Lemma 1.3.7.** Suppose that $\alpha_1, \alpha_2 \in \Phi^+_1$ and $\beta_1, \beta_2 \in \Phi^+_2$ such that $\alpha_1 = w\alpha_2$ and $\beta_1 = w'\beta_2$ for some $w, w' \in W$. Then

$$|dp_1(\alpha_1) - dp_1(\alpha_2)| \leq l(w) \text{ and } |dp_2(\beta_1) - dp_2(\beta_2)| \leq l(w').$$

\hfill $\Box$

**Lemma 1.3.8.** Suppose that $\alpha_1, \alpha_2 \in \Phi^+_1$, with $\alpha_1 \prec_1 \alpha_2$. Let $w \in W$ be of minimal length such that $\alpha_2 = w\alpha_1$. Let $w = r_{s_1}r_{s_2} \cdots r_{s_l}$, $(s_1, \cdots, s_l \in S)$ with $l = l(w)$. Then for all $i \in \{1, 2, \cdots, l - 1\}$,

$$(r_{s_{i+1}} \cdots r_{s_l})\alpha_1 \prec_1 (r_{s_i}r_{s_{i+1}} \cdots r_{s_l})\alpha_1.$$

**Proof.** Given these conditions, $dp_1(\alpha_2) = dp_1(\alpha_1) + l(w)$. Observe that $r_{s_{i+1}} \cdots r_{s_l}$ is a right hand segment of $r_{s_i}r_{s_{i+1}} \cdots r_{s_l}$, which in turn
is a right hand segment of $w$, so Lemma 1.2.12 (iv) yields that

$$N_1(r_{s_{i+1}} \cdots r_{s_l}) \subset N_1(r_{s_{i}}r_{s_{i+1}} \cdots r_{s_l}) \subseteq N_1(w),$$

In particular, $(r_{s_{i}}r_{s_{i+1}} \cdots r_{s_l})\alpha_1 \in \Phi^+_1$ for all $i \in \{1, \cdots, l\}$ (since $w\alpha_1 \in \Phi^+_1$). We prove this lemma by way of contradiction. suppose the statement of this lemma is false, and choose $i \in \{1, \cdots, l-1\}$ such that $(r_{s_{i+1}} \cdots r_{s_l})\alpha_1 \not\prec (r_{s_{i}}r_{s_{i+1}} \cdots r_{s_l})\alpha_1$. Thus

(1.3.5) \[ dp_1((r_{s_{i}} \cdots r_{s_l})\alpha_1) \leq dp_1((r_{s_{i+1}} \cdots r_{s_l})\alpha_1). \]

Since $\alpha_2 = (r_{s_{i}} \cdots r_{s_{i-1}})(r_{s_{i}} \cdots r_{s_l}\alpha_1)$ it follows that

$$dp_1(\alpha_1) + l(w) = dp_1(\alpha_2)$$

$$\leq (i - 1) + dp_1((r_{s_{i}} \cdots r_{s_l})\alpha_1) \quad \text{(by Lemma 1.3.7)}$$

$$\leq (i - 1) + dp_1(r_{s_{i+1}} \cdots r_{s_l}\alpha_1) \quad \text{(by (1.3.5))}$$

$$\leq (i - 1) + (l - i) + dp_1(\alpha_1) \quad \text{(by Lemma 1.3.7)}$$

$$= l - 1 + dp_1(\alpha_1)$$

which is absurd. Hence for all $i \in \{1, \cdots, l-1\}$,

$$(r_{s_{i+1}} \cdots r_{s_l})\alpha_1 \prec_1 (r_{s_{i}} \cdots r_{s_l})\alpha_1,$$

as required. \qed

**Remark 1.3.9.** Lemma 1.3.7 has a natural analogue in $\Phi^+_2$.

The following result is a generalisation of Lemma 1.7 of [6]. Essentially the same reasoning as that used in [6] can be used to prove it, but for completeness, a proof is included here.

**Lemma 1.3.10.** Let $s \in S$, $\alpha \in \Phi^+_1 \setminus \mathbb{R}\{\alpha_s\}$, and $\beta \in \Phi^+_2 \setminus \mathbb{R}\{\beta_s\}$. Then

$$dp_1(r_s\alpha) = \begin{cases} 
  dp_1(\alpha) - 1 & \text{if } \langle \alpha, \beta_s \rangle > 0, \\
  dp_1(\alpha) & \text{if } \langle \alpha, \beta_s \rangle = 0, \\
  dp_1(\alpha) + 1 & \text{if } \langle \alpha, \beta_s \rangle < 0;
\end{cases}$$
and

\[
\text{dp}_2(r_s\beta) = \begin{cases} 
\text{dp}_2(\beta) - 1 & \text{if } \langle \alpha_s, \beta \rangle > 0, \\
\text{dp}_2(\beta) & \text{if } \langle \alpha_s, \beta \rangle = 0, \\
\text{dp}_2(\beta) + 1 & \text{if } \langle \alpha_s, \beta \rangle < 0. 
\end{cases}
\]

**Proof.** If \( \langle \alpha, \beta_s \rangle = 0 \) then \( r_s\alpha = \alpha - 2\langle \alpha, \beta_s \rangle \alpha_s = \alpha \); hence trivially \( \text{dp}_1(r_s\alpha) = \text{dp}_1(\alpha) \).

Suppose next that \( \langle \alpha, \beta_s \rangle > 0 \), By Lemma 1.3.7, it suffices to show that \( \text{dp}_1(r_s\alpha) < \text{dp}_1(\alpha) \). To do so we construct a \( w \in W \) with \( w(r_s\alpha) \in \Phi_1^- \) and \( l(w) < \text{dp}_1(\alpha) \). Choose \( v \in W \) such that \( v\alpha \in \Phi_1^- \) and \( l(v) = \text{dp}_1(\alpha) \). If \( v\alpha_s \in \Phi_1^- \) we set \( w = vr_s \); then \( l(w) = l(v) - 1 \) (by Corollary 1.2.5) and \( w(r_s\alpha) = v\alpha \in \Phi_1^- \), as required. Hence we may assume that \( v\alpha_s \in \Phi_1^+ \). Now

\[
v(r_s\alpha) = v(\alpha - 2\langle \alpha, \beta_s \rangle \alpha_s) = \underbrace{v\alpha}_{\in \Phi_1^+} - 2\langle \alpha, \beta_s \rangle v\alpha_s \in \Phi_1^-.
\]

Furthermore, the condition \( \alpha \in \Phi_1^+ \setminus \mathbb{R}\{\alpha_s\} \) implies that \( v\alpha \) and \( -2\langle \alpha, \beta_s \rangle v\alpha_s \) are not scalar multiples of each other, and hence it follows that there are at least two simple roots in the support of \( v(r_s\alpha) \). Now choose \( r \in S, w \in W \) with \( v = r_rw \) and \( l(v) = l(w) + 1 \). Since \( v(r_s\alpha) \) has at least two simple roots in its support, it follows that applying \( r_r \) will not change the sign of \( v(r_s\alpha) \) (recall that \( N_1(r_r) = \{\hat{\alpha}_r\} \)). Thus \( w(r_s\alpha) = r_r(vr_s\alpha) \in \Phi_1^- \) and \( l(w) < l(v) = \text{dp}_1(\alpha) \).

Finally, suppose that \( \langle \alpha, \beta_s \rangle < 0 \). Then \( \langle r_s\alpha, \beta_s \rangle = -\langle \alpha, \beta_s \rangle > 0 \); so the preceding paragraph shows that

\[
\text{dp}_1(\alpha) = \text{dp}_1(r_s(r_s\alpha)) = \text{dp}_1(r_s\alpha) - 1.
\]

As usual, the other half of this lemma follows by an exactly similar argument. \( \square \)

**Lemma 1.3.11.** Suppose that \( \alpha \in W \). Then \( \text{dp}_1(\alpha) = \text{dp}_2(\phi(\alpha)) \).

**Proof.** Let \( w \in W \) be such that \( wa \in \Phi_1^- \) and \( \text{dp}_1(\alpha) = l(w) \). Then \( \phi(wa) \in \Phi_2^- \) by Corollary 1.3.4. Since \( \phi \) is \( W \)-equivariant, it follows
that \( w(\phi(\alpha)) \in \Phi_\cap \). Hence \( dp_2(\phi(\alpha)) \leq l(w) = dp_1(\alpha) \). By symmetry we also have \( dp_1(\alpha) \leq dp_2(\phi(\alpha)) \), whence equality.

\[ \square \]

**Lemma 1.3.12.** Suppose that \( \alpha \in \Phi_1 \) and \( s \in S \). Then

\[ \langle \alpha, \beta_s \rangle > 0 \text{ if and only if } \langle \alpha_s, \phi(\alpha) \rangle > 0; \]

and

\[ \langle \alpha, \beta_s \rangle = 0 \text{ if and only if } \langle \alpha_s, \phi(\alpha) \rangle = 0; \]

and

\[ \langle \alpha, \beta_s \rangle < 0 \text{ if and only if } \langle \alpha_s, \phi(\alpha) \rangle < 0. \]

**Proof.** This follows from Lemma 1.3.10 and Lemma 1.3.11. \( \square \)

And we can generalize the above to:

**Corollary 1.3.13.** Suppose that \( \alpha_1, \alpha_2 \in \Phi_1 \). Then

\[ \langle \alpha_1, \phi(\alpha_2) \rangle > 0 \text{ if and only if } \langle \alpha_2, \phi(\alpha_1) \rangle > 0; \]

and

\[ \langle \alpha_1, \phi(\alpha_2) \rangle = 0 \text{ if and only if } \langle \alpha_2, \phi(\alpha_1) \rangle = 0; \]

and

\[ \langle \alpha_1, \phi(\alpha_2) \rangle < 0 \text{ if and only if } \langle \alpha_2, \phi(\alpha_1) \rangle < 0. \]

**Proof.** Write \( \alpha_2 = w\alpha_s \) for some \( w \in W \) and \( s \in S \). Since \( \langle , \rangle \) is \( W \)-invariant and \( \phi \) is \( W \)-equivariant, thus

\[ \langle \alpha_1, \phi(\alpha_2) \rangle = \langle \alpha_1, w\beta_s \rangle = \langle w^{-1}\alpha_1, \beta_s \rangle. \]

Observe that Lemma 1.3.12 yields that \( \langle w^{-1}\alpha_1, \beta_s \rangle > 0 \) if and only if \( \langle \alpha_s, \phi(w^{-1}\alpha_1) \rangle > 0 \) which in turn happens if and only if

\[ \langle \alpha_s, \phi(w^{-1}\alpha_1) \rangle = \langle w\alpha_s, \phi(\alpha_1) \rangle = \langle \alpha_2, \phi(\alpha_1) \rangle > 0. \]

The rest of the desired result follows in a similar way. \( \square \)

**Definition 1.3.14.** Suppose that \( \alpha = w\alpha_s, \beta = \phi(\alpha) \), where \( w \in W \) and \( s \in S \). Then define \( r_\alpha, r_\beta \in W \) by \( r_\alpha = r_\beta = wr_sw^{-1} \), and we call
$r_\alpha$ the reflection corresponding to $\alpha$ and $r_\beta$ the reflection corresponding to $\beta$.

**Remark 1.3.15.** It is clear from Definition 1.3.14 and Lemma 1.2.1 that for all $x \in \Phi_1$ and $y \in \Phi_2$,
\[
 r_\alpha(x) = x - 2\langle x, \phi(\alpha) \rangle \alpha \quad \text{and} \quad r_\beta(y) = y - 2\langle \phi^{-1}(\beta), y \rangle \beta.
\]
Observe that $r_x(x) = -x$ and $r_y(y) = -y$. Note that $r_{\alpha s} = r_s$, $r_{\beta t} = r_t$ for all $s, t \in S$. Clearly for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, $r_\alpha = r_{-\alpha}$, and $r_\beta = r_{-\beta}$. Furthermore, suppose that $z_1, z_2 \in \Phi_i$ ($i = 1, 2$). Then $\hat{z}_1 = \hat{z}_2$ if and only if $r_{z_1} = r_{z_2}$.

**Lemma 1.3.16.** Suppose that $x \in \Phi_1$, $y \in \Phi_2$ and $w, v \in W$. Then $r_{wx} = wr_xw^{-1}$ and $r_{vy} = vr_yv^{-1}$.

**Proof.** Observe that for any $v \in V_1$
\[
 wr_xw^{-1}(v) = w(w^{-1}v - 2\langle w^{-1}v, \phi(x) \rangle x) = v - 2\langle w^{-1}v, \phi(x) \rangle wx = v - 2\langle v, w\phi(x) \rangle wx \quad \text{(since $\langle , \rangle$ is $W$-invariant)}
\]
\[
 = v - 2\langle v, \phi(wx) \rangle wx \quad \text{(since $\phi$ is $W$-equivariant)}
\]
\[
 = r_{wx}v.
\]
Since $v \in V_1$ was arbitrary, it follows that $r_{wx} = wr_xw^{-1}$. Entirely similar argument shows that $r_{vy} = vr_yv^{-1}$. □

Set $T := \bigcup_{w \in W} wRw^{-1}$ and call it the set of reflections in $W$. For each $i \in \{1, 2\}$ since $\Phi_i = W\Pi_i$, it follows from Lemma 1.3.16 and the definition of $T$ that there is a bijection between $\hat{\Phi}_i$ and $T$ via $\hat{z} \leftrightarrow r_z$.

The following Proposition is a natural extension to Corollary 1.2.5:

**Proposition 1.3.17.** For each $i \in \{1, 2\}$, let $w \in W$ and $x \in \Phi_i^\pm$. If $l(wrx) > l(w)$ then $wx \in \Phi_i^\pm$. If $l(wrx) < l(w)$ then $wx \in \Phi_i^-$. 
Proof. We prove the statement that \( l(wr_x) > l(w) \) if and only if \( wx \) is positive in the case \( x \in \Phi_1 \) here, and again we stress that a similar argument also shows the desired result holds in \( \Phi_2 \).

Observe that the second statement follows from the first, applied to \( wr_x \) in place of \( w \): indeed if \( l(wr_x) < l(w) \) then \( l((wr_x)r_x) > l(wr_x) \) forcing \( (wr_x)x = w(r_x)x = -wx \in \Phi_1^+ \), that is, \( wx \in \Phi_1^- \).

Now we prove the first statement in \( \Phi_1 \). Proceed by induction on \( l(w) \), the case \( l(w) = 0 \) being trivial. If \( l(w) > 0 \), there exists \( s \in S \) with \( l(rs) = l(w) - 1 \). Then

\[
(l(rs)w) = l(rs(wr_x)) \geq l(wr_x) - 1 > l(w) - 1 = l(rs)w.
\]

Then the inductive hypothesis yields that \( (rs)x \in \Phi_1^+ \). Suppose for a contradiction that \( wx \in \Phi_1^- \). Then \( \hat{x}^- \in N_1(r_s) \) and Lemma 1.2.12 (i) yields that \( wx = -\lambda \alpha_s \) for some \( \lambda > 0 \). But then \( r_swx = \lambda \alpha_s \) would imply \( (rs)x(r_sw)^{-1} = r_s \) by Lemma 1.3.16 and Remark 1.3.15. But this yields that \( wr_x = r_s w \) contradicting \( l(wr_x) > l(w) > l(rs)w \). As a result, \( wx \) must be positive. \( \square \)

It is readily checked that for all \( w \in W \) and \( t \in T \), \( l(wt) \equiv l(w) \) modulo 2. In particular, the length of any such \( t \) must be odd, which in turn shows that \( l(wt) \neq l(w) \) Combining this observation with Proposition 1.3.17 we have the following:

**Corollary 1.3.18.** Let \( w \in W \) and \( x \in \Phi_1^+ \) \((i \in \{1, 2\})\). Then

\[
\begin{cases}
  l(wr_x) > l(w) & \text{if and only if } wx \in \Phi_1^+ \\
  l(wr_x) < l(w) & \text{if and only if } wx \in \Phi_1^-.
\end{cases}
\]

\( \square \)

The next result gives a connection between the depth of a root and the length of the corresponding reflection.

**Lemma 1.3.19.** Let \( \alpha \in \Phi_1^+ \) such that \( \alpha = (r_{s_1}r_{s_2} \cdots r_{s_l})\alpha_{s_l} \), where \( s, s_1, \cdots, s_l \in S \) and \( dp_1(\alpha) = l + 1 \). Then \( l(r_{\alpha}) = 2l + 1 \).
1.3. The canonical coefficients and the depth of roots

Proof. The proof is based on an induction on $l$. The case $l = 0$ ($\alpha = \alpha_s$) is trivial. Thus we may assume that $l > 1$. Observe that Lemma 1.3.7 and Lemma 1.3.8 yield that $dp_1(r_{s_2} \ldots r_{s_l} \alpha_s) = l$. Then the inductive hypothesis yields that

$$l(r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) = 2l - 1.$$  

That is, $(r_{s_2} \ldots r_{s_l}) \alpha_s \prec_1 (r_{s_1} r_{s_2} \ldots r_{s_l}) \alpha_s$, and hence Lemma 1.3.10 gives

$$\langle r_{s_2} \ldots r_{s_l} \alpha_s, \beta_{s_1} \rangle < 0.$$  

Lemma 1.3.12 then yields that $\langle \alpha_{s_1}, (r_{s_2} \ldots r_{s_l}) \beta_{s_1} \rangle < 0$. Observe that $dp_1(\alpha) = dp_1(r_{s_1} r_{s_2} \ldots r_{s_l} \alpha_s) = l + 1$ implies that $l((r_{s_1} r_{s_2} \ldots r_{s_l}) = l$, in particular, $r_{s_2} \ldots r_{s_l}$ is a right hand segment of $r_{s_1} r_{s_2} \ldots r_{s_l}$. Since $(r_{s_1} r_{s_2} \ldots r_{s_l}) \alpha_s \in \Phi^+_1$, it follows that $(r_{s_2} \ldots r_{s_l}) \alpha_s \in \Phi^+_1$ too. Thus

$$\langle r_{s_2} \ldots r_{s_l} \alpha_s, \beta_{s_1} \rangle < 0.$$  

Hence

$$\langle r_{s_1} (r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) \alpha_s \rangle = r_{(r_{s_2} \ldots r_{s_l})}\alpha_s$$

$$= \alpha_{s_1} - 2 \langle \alpha_{s_1}, (r_{s_2} \ldots r_{s_l}) \beta_{s_1} \rangle (r_{s_2} \ldots r_{s_l} \alpha_s)$$

$$< 0 \in \Phi^+_1$$

Now we claim that

$$l((r_{s_1} r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) \in \Phi^+_1.$$  

If (1.3.9) were true, then Corollary 1.2.5 would yield that

$$l(r_{s_1} r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) = l(r_{s_1} r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) + 1 \ (by \ Corollary \ 1.2.5).$$  

Now we claim that

$$l((r_{s_1} r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) \in \Phi^+_1.$$  

If (1.3.9) were true, then Corollary 1.2.5 would yield that

$$l(r_{s_1} r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) = l(r_{s_1} r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) + 1$$

$$= l(r_{s_2} \ldots r_{s_l} r_{s_1} \alpha_s) + 2.$$
this combining with (1.3.6) gives us $l(r_{s_1} \cdots r_{s_l} r_{s_l} r_{s_l} \cdots r_{s_1}) = 2l + 1$. Thus the result follows by induction provided we can prove (1.3.9), which we shall prove now. Suppose that the opposite to (1.3.9) holds, then Lemma 1.2.12(i) together with (1.3.8) yields that

\[(r_{s_2} \cdots r_{s_l} r_{s_l} r_{s_l} \cdots r_{s_2}) \alpha_{s_1} = \lambda \alpha_{s_1}\]

for some $\lambda > 0$. But this says precisely that

\[(r_{s_l} r_{s_l} \cdots r_{s_2}) \alpha_{s_1} = \lambda r_{s_l} r_{s_l} \cdots r_{s_2} \alpha_{s_1}.\]

Observe that (1.3.6) forces $r_{s_l} r_{s_l} \cdots r_{s_2}$ to be a right hand segment of $r_{s_2} \cdots r_{s_l} r_{s_l} r_{s_l} \cdots r_{s_2}$. Thus (1.3.8) above yields that $(r_{s_l} \cdots r_{s_2}) \alpha_{s_1} \in \Phi_1^+$. Hence $r_{s_l} (r_{s_l} \cdots r_{s_2}) \alpha_{s_1} = \lambda (r_{s_l} \cdots r_{s_2}) \alpha_{s_1} \in \Phi_1^+$ too. This yields that $(r_{s_l} \cdots r_{s_2}) \alpha_{s_1}$ is not a positive scalar multiple of $\alpha_s$ by Lemma 1.2.12(i).

Observe now

\[
r_{s_l} (r_{s_l} \cdots r_{s_2}) \alpha_{s_1} = (r_{s_l} \cdots r_{s_2}) \alpha_{s_1} - 2 \langle (r_{s_l} \cdots r_{s_2}) \alpha_{s_1}, \beta_s \rangle \alpha_s
\]

forces $\langle r_{s_l} \cdots r_{s_2} \alpha_{s_1}, \beta_s \rangle = 0$. Since $\langle \cdot, \cdot \rangle$ is $W$-invariant, it follows that $\langle \alpha_{s_1}, r_{s_2} \cdots r_{s_l} \beta_s \rangle = 0$. Lemma 1.3.12 then yields $\langle r_{s_2} \cdots r_{s_l} \alpha_s, \beta_{s_1} \rangle = 0$, contradicting (1.3.7). Thus (1.3.9) follows, as required. □
Chapter 2

Special Topics in Non-Orthogonal Setting

2.1. Decompositions of Generic Root Systems

Suppose that $U_1$ and $U_2$ are vector spaces over the real field $\mathbb{R}$ and suppose that there exist a bilinear map $\langle \ , \ \rangle' : U_1 \times U_2 \to \mathbb{R}$ and linearly independent sets $X_1 \subset U_1$ and $X_2 \subset U_2$ indexed by the same set $S''$ via $s \mapsto x_s \in X_1$ and $s \mapsto y_s \in X_2$ for all $s \in S''$. Furthermore, suppose that the following condition holds:

\[(D1) \quad \langle x_s, y_t \rangle' = 1 \quad \text{for all} \quad s \in S''.\]

For each $s \in S''$, define $\rho_1(x_s) : U_1 \to U_1$ and $\rho_2(y_s) : U_2 \to U_2$ by

\[\rho_1(x_s)(u) := u - 2\langle u, y_s \rangle' x_s\]
\[\rho_2(y_s)(v) := v - 2\langle x_s, v \rangle' y_s\]

for all $u \in U_1$ and $v \in U_2$, and for each $i \in \{1, 2\}$, define

\[R''_i := \{\rho_i(x) \mid x \in X_i\};\]
\[W''_i := \langle R''_i \rangle;\]
\[\Phi''_i := W''_i X_i;\]
\[\Phi''_i^+ := \Phi''_i \cap \text{PLC}(X_i) \quad \text{and}\]
\[\Phi''_i^- := -\Phi''_i^+.\]

Proposition 2.1.1. For each $i \in \{1, 2\}$, suppose that $\Phi''_i = \Phi''_i^+ \cup \Phi''_i^-$. Then the following conditions must be satisfied for all $s, t \in S''$:

\[(D2) \quad \langle x_s, y_t \rangle' \leq 0 \quad \text{and} \quad \langle x_t, y_s \rangle' \leq 0 \quad \text{whenever} \quad s \neq t;\]
\[(D3) \quad \langle x_s, y_t \rangle' = 0 \quad \text{if and only if} \quad \langle x_t, y_s \rangle' = 0;\]
\[(D4) \quad \text{either} \quad \langle x_s, y_t \rangle' \langle x_t, y_s \rangle' = \cos^2\left(\frac{\pi}{m_{st}}\right) \quad \text{for some integer} \quad m_{st} \geq 2, \quad \text{or}\]
\[\text{else} \quad \langle x_s, y_t \rangle' \langle x_t, y_s \rangle' \geq 1.\]
To prove this proposition we shall need a few technical results first. These results are essentially taken from [1], and for completeness, their proofs are included here.

Let \( \mathcal{A} \) be a commutative \( \mathbb{R} \)-algebra, let \( q^{1/2} \) and \( X \) be units of \( \mathcal{A} \), and let \( \gamma \in \mathbb{R} \). Define \( A, B \) to be 2 by 2 matrices over \( \mathcal{A} \) given by:

\[
A = \begin{pmatrix} -1 & 2\gamma q^{1/2}X \\ 0 & q \end{pmatrix} \quad B = \begin{pmatrix} q & 0 \\ 2\gamma q^{1/2}X^{-1} & -1 \end{pmatrix}.
\]

It is easily proved by induction on \( n \in \mathbb{N} \) that

\[
(BA)^n = \begin{pmatrix} -q^n p_{2n-1} & q^n p_{2n} \\ -q^{n+\frac{1}{2}} p_{2n+1} X^{-1} & q^{n+\frac{1}{2}} p_{2n+1} \end{pmatrix}
\]

\[
(AB)^n = \begin{pmatrix} q^n p_{2n+1} & -q^n \frac{1}{2} p_{2n} X \\ q^{n+\frac{1}{2}} p_{2n+1} X^{-1} & -q^n p_{2n+1} \end{pmatrix}
\]

and

\[
(AB)^n = \begin{pmatrix} q^n p_{2n+1} & -q^n \frac{1}{2} p_{2n} X \\ q^{n+\frac{1}{2}} p_{2n+1} X^{-1} & -q^n p_{2n+1} \end{pmatrix}
\]

where \( p_n \in \mathbb{R} \) (for \( n \in \{-1\} \cup \mathbb{N} \)) is defined recursively by

\[
p_{-1} = -1, \quad p_0 = 0, \quad p_{n+1} = 2\gamma p_n - p_{n-1} \quad (n \in \mathbb{N}).
\]

The solution of the recurrence (2.1.5) is

\[
p_n = \begin{cases} n & \text{if } \gamma = 1 \\ (-1)^{n+1} n & \text{if } \gamma = -1 \\ \frac{1}{2\sqrt{\gamma^2 - 1}}[(\gamma + \sqrt{\gamma^2 - 1})^n - (\gamma - \sqrt{\gamma^2 - 1})^n] & \text{if } |\gamma| > 1 \\ \frac{\sin n\theta}{\sin \theta} & \text{where } \theta = \cos^{-1} \gamma & \text{if } |\gamma| < 1. \end{cases}
\]
Note that in the case $|\gamma| \geq 1$ we may alternatively write
\[ p_n = \frac{\sinh n\theta}{\sinh \theta} \quad \text{where} \quad \theta = \cosh^{-1} \gamma \]
(as in Lemma 1.1.8).

**Lemma 2.1.2.** (Dyer, [1, Lemma 2.2])

(i) Conditions (1') and (2') below are equivalent:
- (1') $p_n p_{n+1} \geq 0$ for all $n \in \mathbb{N}$;
- (2') $\gamma \in \{\cos(\pi/m) \mid m \in \mathbb{N}, m \geq 2\} \cup [1, \infty)$.

(ii) If $\gamma = \cos \frac{k\pi}{m}$ for some $k, m \in \mathbb{N}$ with $0 < k < m$ then the matrices $A$ and $B$ satisfy the equation $ABA \cdots = BAB \cdots$, where there are $m$ factors on either side.

(iii) If $q = 1$ then the matrix $AB$ has order $m$ if $\gamma = \cos \frac{k\pi}{m}$ for some $k, m \in \mathbb{N}$ with $0 < k < m$ and $\gcd(m, k) = 1$, and has infinite order otherwise.

**Proof.** (i): Assume that (1') holds. Observe that (2.1.5) yields that $p_1 = 1$ and $p_2 = 2\gamma$; hence $\gamma \geq 0$. Since (2') obviously holds if $\gamma \geq 1$, we may assume that $0 \leq \gamma < 1$. Choose $\theta$ so that $0 < \theta \leq \frac{\pi}{2}$ and $\cos \theta = \gamma$, and let $m$ be the largest integer such that
\[ 0 < \theta < 2\theta < \cdots < m\theta \leq \pi, \]
noting that $m \geq 2$. Now if $m\theta \neq \pi$ then $\pi < (m+1)\theta < 2\pi$, and it follows that
\[ p_m = \frac{\sin m\theta}{\sin \theta} > 0 \]
whereas
\[ p_{m+1} = \frac{\sin (m+1)\theta}{\sin \theta} < 0, \]
contradicting (1'). Hence $m\theta = \pi$, so that $\gamma = \cos \frac{\pi}{m}$ for some $m \geq 2$, whence (2') holds. Thus (1') implies (2').

Conversely, if (2') holds then it follows from (2.1.6) that (1') holds. This completes the proof of (i).
(ii) Suppose first that \( m \) is even, and write \( m = 2r \), so that our task is to prove that \((AB)^r = (BA)^r\). We have that \( p_n = \frac{\sin(\pi nk/2)}{\sin(\pi k/2r)} \), which gives \( p_{2r+1} = \frac{\sin(\pi (k+1)/2r)}{\sin(\pi k/2r)} = (-1)^k \) and \( p_{2r-1} = (-1)^{k+1} \) by a similar calculation, while \( p_{2r} = 0 \). Hence the required result follows immediately from (2.1.3) and (2.1.4).

If \( m \) is odd our task is to prove that \( B(AB)^r = A(BA)^r \), where \( m = 2r + 1 \). In this case we find that \( p_{2r+1} \neq 0 \), while \( p_{2r+2} = (-1)^k \) and \( p_{2r} = (-1)^{k+1} \), and the required result follows immediately from (2.1.1) and (2.1.2).

(iii) If \( \gamma = \cos \frac{\pi}{m} \) then it follows immediately from Part (ii) that \((AB)^m = 1\), since \( A^2 = B^2 = 1 \) when \( q = 1 \). Furthermore, if \( 0 < n < m \) then \( p_n = \frac{\sin(\pi nk/m)}{\sin(\pi k/m)} \neq 0 \), and it follows from (2.1.4) that \((AB)^n \neq 1\). Thus \( AB \) has order \( m \). Similarly if \( |\gamma| \geq 1 \) then it follows from (2.1.6) that \( p_n \) is nonzero for all integers \( n > 0 \), giving \((AB)^n \neq 1\) for all such \( n \), so that \( AB \) has infinite order.

Now we are ready to prove Proposition 2.1.1:

**Proof of Proposition 2.1.1.** We present a proof that \( \Phi_1'' = \Phi_1''^+ \cup \Phi_1''^- \) implies conditions (D2), (D3) and (D4) and we stress that the same argument applies equally to \( \Phi_2'' \).

Suppose that \( \Phi_1'' = \Phi_1''^+ \cup \Phi_1''^- \). Let \( s, t \in S'' \) be distinct. Consider the actions of \( \rho_1(x_s)\rho_1(x_t) \) and \( \rho_1(x_t)\rho_1(x_s) \) in the \( \{\rho_1(x_s), \rho_1(x_t)\}\)-invariant subspace \( \mathbb{R}x_s + \mathbb{R}x_t \):

\[
(r_1(x_t)\rho_1(x_s))(x_s) = \rho_1(x_t)(x_s - 2\langle x_s, y_t \rangle'x_s) = \rho_1(x_t)(-x_s) = -x_s + 2\langle x_s, y_t \rangle'x_t.
\]

Since \( \Phi_1'' = \Phi_1''^+ \cup \Phi_1''^- \) and \( X_1 \) is linearly independent, it follows from above that \( \langle x_s, y_t \rangle' \leq 0 \). Similarly, by looking at \( (\rho_1(x_s)\rho_1(x_t))(x_t) \) we may deduce that \( \langle x_t, y_s \rangle' \leq 0 \), whence (D2) holds.
Next, suppose that $s, t \in S''$ are distinct with $\langle x_s, y_t \rangle' = 0$. Consider

\[
\rho_1(x_t)\rho_1(x_s)(x_t) = \rho_1(x_t)(x_t - 2\langle x_t, y_s \rangle' x_s)
\]

\[
= -x_t - 2\langle x_t, y_s \rangle' x_s + 4\langle x_t, y_s \rangle'\langle x_s, y_t \rangle' x_t
\]

\[
= -x_t - 2\langle x_t, y_s \rangle' x_s.
\]

Since $\Phi''_1 = \Phi''_1^+ \cup \Phi''_1^-$ and $X_1$ is linearly independent, it follows from above that $\langle x_t, y_s \rangle' \geq 0$ and upon combining with (D2), this in turn yields that $\langle x_t, y_s \rangle' = 0$. In a similar way, we may deduce that $\langle x_t, y_s \rangle' = 0$ implies that $\langle x_s, y_t \rangle' = 0$, thus proving (D3).

Observe that if $\langle x_s, y_t \rangle' = 0$ and $\langle x_t, y_s \rangle' = 0$ then

\[
\langle x_s, y_t \rangle'\langle x_t, y_s \rangle' = \cos^2(\pi/2),
\]

satisfying (D4). Thus to prove that (D4) holds, we may assume that $\langle x_s, y_t \rangle' \neq 0$ and $\langle x_t, y_s \rangle' \neq 0$. Now let $\mathcal{A}, \gamma, q, X, A$ and $B$ be as defined above. We set

\[
\mathcal{A} = \mathbb{R};
\]

\[
q = 1;
\]

\[
\gamma = \sqrt{\langle x_s, y_t \rangle'\langle x_t, y_s \rangle'};
\]

and

\[
X = \frac{-\langle x_t, y_s \rangle'}{\sqrt{\langle x_s, y_t \rangle'\langle x_t, y_s \rangle'}}.
\]

Compared with the proofs of Lemma 1.1.6 and Lemma 1.1.8 we see that $A$ and $B$ are the matrices representing the actions of $\rho_1(x_s)$ and $\rho_1(x_t)$ respectively on the $\langle \{\rho_1(x_s), \rho_1(x_t)\}\rangle$-invariant subspace $\mathbb{R}x_s + \mathbb{R}x_t$. By (2.1.1) and (2.1.4) the condition

\[
\langle \{\rho_1(x_s), \rho_1(x_t)\}\rangle x_s \cup \langle \{\rho_1(x_s), \rho_1(x_t)\}\rangle x_t \subseteq \Phi''_1^+ \cup \Phi''_1^-
\]

is equivalent to $p_n p_{n+1} \geq 0$ for all $n \in \mathbb{N}$. By Lemma 2.1.2 above, the condition $p_n p_{n+1} \geq 0$ for all $n \in \mathbb{N}$ is, in turn, equivalent to

\[
\langle x_s, y_t \rangle'\langle x_t, y_s \rangle' \in \{\cos^2 \frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\} \cup [1, \infty),
\]

whence (D4) holds.
Now we are ready for the main result of this section:

**Theorem 2.1.3.** The following are equivalent:

1. For each \( i \in \{1, 2\} \), \( \Phi''_i = \Phi''_i^+ \cup \Phi''_i^- \);
2. \((S'', \text{span } X_1, \text{span } X_2, X_1, X_2, \langle \, , \, \rangle')\) is a Coxeter datum.

**Proof.** Suppose that (1) holds. It follows from the definition of \( \langle \, , \, \rangle' \) and Proposition 2.1.1 above that conditions (C1) to (C5) of the definition for a Coxeter datum are all satisfied in our present situation. Since \( X_1 \) and \( X_2 \) are linearly independent, it is clear that \( 0 \notin \text{PLC}(X_1) \) and \( 0 \notin \text{PLC}(X_2) \), showing that (C6) of the definition of a Coxeter datum is also satisfied. Thus (1) implies (2).

Suppose that (2) holds. Then (1) simply follows from Lemma 1.2.4 applied to the Coxeter datum \((S'', \text{span } X_1, \text{span } X_2, X_1, X_2, \langle \, , \, \rangle')\). \( \square \)

**Proposition 2.1.4.** Suppose that conditions (D2), (D3) and (D4) of Proposition 2.1.1 are satisfied. Then \((W''_1, R''_1)\) and \((W''_2, R''_2)\) are isomorphic Coxeter systems. Furthermore if \( s, t \in S'' \), \( s \neq t \), then

\[
\text{ord}(\rho_1(x_s) \rho_1(x_t)) = \begin{cases} 
  m & \text{if } \langle x_s, y_t \rangle' \langle x_t, y_s \rangle' = \cos^2 \frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\
  \infty & \text{if } \langle x_s, y_t \rangle' \langle x_t, y_s \rangle' \geq 1.
\end{cases}
\]

**Proof.** Keep all notation as in Chapter 1. We have already observed that \((S'', \text{span } X_1, \text{span } X_2, X_1, X_2, \langle \, , \, \rangle')\) is a Coxeter datum; then Theorems 1.1.4 and 1.1.15 yields that \((W''_1, R''_1)\) and \((W''_2, R''_2)\) are isomorphic Coxeter systems. Furthermore, applying Lemma 1.1.8 (ii) and Proposition 1.1.9 to this Coxeter datum, we deduce that for distinct \( s, t \in S'' \),

\[
\text{ord}(\rho_1(x_s) \rho_1(x_t)) = \begin{cases} 
  m & \text{if } \langle x_s, y_t \rangle' \langle x_t, y_s \rangle' = \cos^2 \frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\
  \infty & \text{if } \langle x_s, y_t \rangle' \langle x_t, y_s \rangle' \geq 1.
\end{cases}
\]
2.2. Canonical Generators of Reflection Root Subsystems

Keep all notation as in Chapter 1. Recall that \( T = \bigcup_{w \in W} wRw^{-1} \) is the set of reflections in \( W \).

Suppose that \( W' \) is a subgroup of \( W \). We say that \( W' \) is a reflection subgroup of \( W \) whenever \( W' = \langle W \cap T \rangle \). For each \( i \in \{1, 2\} \), if \( \Phi'_i \) is a subset of \( \Phi_i \) such that \( r_xy \in \Phi'_i \) whenever \( x, y \in \Phi'_i \), then we call \( \Phi'_i \) a root subsystem of \( \Phi_i \).

If \( W' \) is a reflection subgroup of \( W \), set \( \Phi_i(W') := \{ x \in \Phi_i \mid r_x \in W' \} \).

Let \( x, y \in \Phi_i(W') \). Then \( r_x, r_y \in W' \cap T \). Since Lemma 1.3.16 yields that \( r_{(r_xy)} = r_xr_yp_x \), it follows that \( r_{(r_xy)} \in W' \cap T \) showing that \( r_xy \in \Phi_i(W') \). Therefore \( \Phi_i(W') \) is a root subsystem of \( \Phi_i \) and we call \( \Phi_i(W') \) the root subsystem corresponding to \( W' \). It can be seen (for example, (1.4.2) of [11]) that the above correspondence gives a bijection between reflection subgroups \( W' \subseteq W \) and root subsystem \( \Phi_i(W') \subseteq \Phi_i \). Observe that \( \Phi(W') \) is stable under the action of \( W' \), indeed:

**Lemma 2.2.1.** Let \( W' \) be a reflection subgroup of \( W \). Then for each \( i \in \{1, 2\} \)

\[ W'\Phi_i(W') = \Phi_i(W') \]

**Proof.** We prove that \( W'\Phi_1(W') = \Phi_1(W') \) here and we stress that the other half follows in the same way. Let \( w \in W' \). By definition, we have \( w = t_1t_2 \cdots t_n \) where \( t_1, t_2, \ldots, t_n \in W' \cap T \). The definition of \( T \) yields that, for all \( i \in \{1, 2, \ldots, n\} \),

\[ t_i = w_ir_{s_i}w_i^{-1} = \underbrace{r_{(w_i\alpha_{s_i})}}_{\text{by Lemma 1.3.16}} \]

for some \( w_i \in W \) and \( s_i \in S \). Since \( r_{(w_i\alpha_{s_i})} \in W' \), it follows that \( w_i\alpha_{s_i} \in \Phi_1(W') \). Now let \( x \in \Phi_1(W') \) be arbitrary. It follows from the definition of a root subsystem that \( t_nx = r_{w_n\alpha_{s_n}}x \in \Phi_1(W') \).
This in turn yields that $t_{n-1}t_n x \in \Phi_1(W')$ and so on. In particular, $wx = t_1 \cdots t_n x \in \Phi_1(W')$. Since $x \in \Phi_1(W')$ is arbitrary, it follows that $w\Phi_1(W') \subseteq \Phi_1(W')$. Finally, replacing $w$ by $w^{-1}$ we see that $\Phi_1(W') \subseteq w\Phi_1(W')$. \hfill \Box

**Definition 2.2.2.** Let $W'$ be a reflection subgroup of $W$. For each $i \in \{1, 2\}$, set

$$\Delta_i(W') := \{ x \in \Phi_i^+ | N_i(r_x) \cap \Phi_i(W') = \{ \hat{x} \} \}.$$  

We call $\Delta_i(W')$ the *canonical roots* or *canonical generators* of $\Phi_i(W')$.

The key result in this section is a criterion for a set of positive roots in $\Phi_i(W')$, for some reflection subgroup $W'$ of $W$, to be the set of canonical roots of $\Phi_i(W')$. Note that at this stage it is not entirely obvious that $\Phi_i(W')$ is generated by $\Delta_i(W')$, and we shall prove this fact in (iii) of Lemma 2.2.6 below. We acknowledge that the works presented here closely follow those in Chapter 3 of [1].

**Remark 2.2.3.**

1. Note that for each $\alpha \in \Phi_1$, $r_\alpha = r_{\phi(\alpha)}$ (where $\phi$ is as in Definition 1.2.18), and it follows that for any reflection subgroup $W'$ of $W$, $\phi(\Phi_1(W')) = \Phi_2(W')$.

2. For $\alpha_1, \alpha_2 \in \Phi_1^+$, Corollary 1.3.4 yields that $r_{\alpha_1} \alpha \in \Phi_1^-$ if and only if $r_{\phi(\alpha_1)} \phi(\alpha_2) \in \Phi_2^-$. Thus $N_2(r_{\phi(\alpha_1)}) = \phi(N_1(r_{\alpha_1}))$.

These observations lead to:

**Lemma 2.2.4.** For any reflection subgroup $W'$ of $W$, $\phi$ restricts to a bijection

$$\Delta_1(W') \leftrightarrow \Delta_2(W').$$ \hfill \Box

Immediately from this observation we deduce that, for any reflection subgroup $W'$ of $W$,

$$\{ r_\alpha \in T | \alpha \in \Delta_1(W') \} = \{ r_\beta \in T | \beta \in \Delta_2(W') \}$$

and we shall give a set like this a special name:
Definition 2.2.5. Let $W'$ be a reflection subgroup of $W$. Set

$$S(W') := \{ r_x \in T \mid x \in \Delta_i(W') \}$$

for any $i \in \{1, 2\}$.

Observe that the definitions of $N_1(w)$, $N_2(w)$ ($w \in W$) and Proposition 1.3.17 yield that

$$S(W') = \{ t \in W' \cap T \mid \text{if } t' \in W' \cap T \text{ with } l(tt') < l(t) \text{ then } t = t' \}.$$ 

Thus our $S(W')$ here defines the same set of reflections in $W'$ as the set denoted by $S(W')$ in $[1]$ (see 1.6 and Theorem 1.8 of $[1]$). Hence we may apply Theorem 1.8 of $[1]$ directly:

Lemma 2.2.6. Let $W'$ be a reflection subgroup of $W$.

(i) (Lemma (1.7) (ii) $[1]$) If $t \in W' \cap T$, then there exist $m \in \mathbb{N}$ and $t_0, \ldots, t_m \in S(W')$ such that $t = t_m \cdots t_1 t_0 t \cdots t_m$.

(ii) (Theorem (1.8) (i) $[1]$) $(W', S(W'))$ is a Coxeter system.

(iii) For each $i \in \{1, 2\}$, let $x \in \Pi_i \setminus \Phi_i(W')$. Then

$$\Delta_i(r_x W' r_x) = r_x \Delta_i(W').$$

(iv) For each $i \in \{1, 2\}$, $\Phi_i(W') = W' \Delta_i(W')$.

Proof. Only (iii) and (iv) need to be proved here.

(iii): It is readily checked that $r \Phi_i(W') = \Phi_i(r W' r)$ for all $r \in T$. Since $x \in \Pi_i \setminus \Phi_i(W')$, it follows that $r_x \in R \setminus W'$. Let $y \in \Delta_i(W')$ be
arbitrary. Then

\[ N_i(r_{xy}) \cap \Phi_i(r_xW'r_x) \]
\[ = N_i(r_xr_yr_x) \cap \Phi_i(r_xW'r_x) \]
\[ = (r_xN_i(r_xr_y) + N_i(r_x)) \cap \Phi_i(r_xW'r_x) \]  
(by Lemma 1.3.16)
\[ = (r_xN_i(r_xr_y) + N_i(r_x)) \cap \Phi_i(r_xW'r_x) \]  
(by Lemma 1.2.12 (iii))
\[ = (r_xN_i(r_xr_y) + r_xN_i(r_y) + N_i(r_x)) \cap \Phi_i(r_xW'r_x) \]  
(again by Lemma 1.2.12 (iii))
\[ = r_x((r_yN_i(r_x) + N_i(r_y) + N_i(r_x)) \cap \Phi_i(W')) \]
\[ = r_x((r_y\{\hat{x}\} + N_i(r_y) + \{\hat{x}\}) \cap \Phi_i(W')) \]  
(by Lemma 1.2.12 (i))
\[ = r_x(N_i(r_y) \cap \Phi_i(W')) \]
\[ = \{r_{xy}\}. \]

Hence \( r_{xy} \in \Delta_i(r_xW'r_x) \). This proves that \( r_x\Delta_i(W') \subseteq \Delta_i(r_xW'r_x) \).

But \( x \in \Pi_i \setminus r_x\Phi_i(W') \), so the above yields that \( r_x\Delta_i(r_xW'r_x) \subseteq \Delta_i(W') \)
proving the desired result.

(iv): Since \( \Delta_i(W') \subseteq \Phi_i(W') \) for each \( i \in \{1, 2\} \) and \( \Phi_i(W') \) is a root subsystem it follows that \( r\Delta_i(W') \subseteq \Phi_i(W') \) for all \( r \in S(W') \).

Then part (ii) above yields that \( W'\Delta_i(W') \subseteq \Phi_i(W') \).

Conversely if \( x \in \Phi_i(W') \) then \( r_x \in W' \cap T \). By (i) above there are \( x_0, x_1, \ldots, x_m \in \Delta_i(W') \) \((m \in \mathbb{N})\) such that

\[ r_x = r_{x_m} \cdots r_{x_1}r_{x_0}r_{x_1} \cdots r_{x_m}. \]

Thus Lemma 1.3.16 yields that \( \lambda x = r_{(r_{x_m} \cdots r_{x_1})}(x_0) \in W'\Phi_i(W') \)
for some (nonzero) scalar \( \lambda \). Since \( \frac{1}{\lambda}x_0 = (r_{(r_{x_m} \cdots r_{x_1})})^{-1}(x) \in \Phi_i \), it follows that \( \frac{1}{\lambda}x_0 \in \Delta_i(W') \)
and hence \( x = r_{(r_{x_m} \cdots r_{x_1})}(\frac{1}{\lambda}x_0) \in W'\Delta_i(W') \)
as required. \( \square \)
**Definition 2.2.7.** Let $W'$ be a reflection subgroup of $W$, and let $l': W' \to \mathbb{N}$ be the length function on $(W', S(W'))$ defined by

$$l'(w) = \min\{ n \in \mathbb{N} | w = r_1 \cdots r_n, \text{ where } r_i \in S(W') \}.$$ 

If $w = r_1 \cdots r_n \in W'$ ($r_i \in S(W')$) and $n = l'(w)$ then $r_1 \cdots r_n$ is called a reduced expression for $w$ (with respect to $S(W')$).

**Lemma 2.2.8.** Let $W'$ be a reflection subgroup. For each $i \in \{1, 2\}$,

(i) $N_i(r_x) \cap \Phi_i(W') = \{x\}$ for all $x \in \Delta_i(W')$;

(ii) for all $w_1 \in W'$ and $w_2 \in W'$

$$N_i(w_1w_2) \cap \Phi_i(W') = w_2^{-1}(N_i(w_1) \cap \Phi_i(W')) + (N_i(w_2) \cap \Phi_i(W')).$$

**Proof.** (i) is just the definition of $\Delta_i(W')$.

(ii) Lemma 1.2.12(iii) yields that $N_i(w_1w_2) = w_2^{-1}N_i(w_1) + N_i(w_2)$, and hence

$$N_i(w_1w_2) \cap \Phi_i(W') = (w_2^{-1}N_i(w_1) \cap \Phi_i(W')) + (N_i(w_2) \cap \Phi_i(W')).$$

Since $w_2 \in W'$ it follows from Lemma 2.2.1 that $w_2^{-1}\Phi_i(W') = \Phi_i(W')$. Thus $w_2^{-1}N_i(w_1) \cap \Phi_i(W') = w_2^{-1}(N_i(w_1) \cap \Phi_i(W'))$ giving us

$$N_i(w_1w_2) \cap \Phi_i(W') = w_2^{-1}(N_i(w_1) \cap \Phi_i(W')) + (N_i(w_2) \cap \Phi_i(W')).$$

□

**Lemma 2.2.9.** Let $W'$ be a reflection subgroup. For each $i \in \{1, 2\}$ and all $w \in W'$, we have

(i) $|N_i(w) \cap \Phi_i(W')| = l'(w)$. Furthermore, if $w = r_{x_1} \cdots r_{x_n}$ is reduced with respect to $(W', S(W'))$ then

$$N_i(w) \cap \Phi_i(W') = \{\hat{y}_1, \cdots, \hat{y}_n\}$$

where $y_j = (r_{x_{n}} \cdots r_{x_{j+1}})x_j$ for all $j = 1, \cdots, n$.

(i) $N_i(w) \cap \Phi_i(W') = \{\hat{x} \in \Phi_i(W') | l'(wr_x) < l'(w)\}$. 
Proof. (i): For each \( j \in \{1, \cdots, n\} \), set \( t_j = r_{x_j} \cdots r_{x_{j+i}} r_{x_{j+1}} \cdots r_{x_n} \), that is, \( t_j = r_{y_j} \). If \( t_j = t_k \) where \( j > k \) then

\[
w = r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_n} t_k
\]

\[
= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_n} t_j
\]

\[
= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_{j-1}} r_{x_{j+1}} \cdots r_{x_n}
\]

contradicting \( l'(w) = n \). Hence the \( t_j \)'s are all distinct and consequently all the \( \hat{y}_j \)'s are all distinct. Now by repeated application of Lemma 2.2.8 (ii), for each \( i \in \{1, 2\} \) we have

\[
N_i(w) \cap \Phi_i(W')
\]

\[
= (N_i(r_{x_1} \cap \Phi_i(W')) + r_{x_1} (N_i(r_{x_1-1} \cap \Phi_i(W')) + \cdots + r_{x_1-1} \cdots r_x (N_i(r_{x_1} \cap \Phi_i(W'))
\]

\[
= \{\hat{y}_n\} + \{\hat{y}_n-1\} + \cdots + \{\hat{y}_1\}
\]

\[
= \{\hat{y}_1, \cdots, \hat{y}_n\}
\]

and consequently \( |N_i(w) \cap \Phi_i(W')| = l'(w) \).

(ii): Let \( w = r_{x_1} \cdots r_{x_n} \) be a reduced expression for \( w \in W' \) with respect to \( S(W') (x_j \in \Delta_i(W')) \). Then (i) yields that for each \( i \in \{1, 2\} \)

\[
N_i(w) \cap \Phi_i(W') = \{\hat{y}_1, \cdots, \hat{y}_n\}
\]

where \( y_j = (r_{x_1} \cdots r_{x_{j+i-1}}) x_j \), for all \( j \in \{1, \cdots, n\} \). Now for each such \( j \),

\[
wr_{y_j} = wr_{x_1} \cdots r_{x_{j+i-1}} x_j r_{x_{j+1}} \cdots r_{x_n} = r_{x_1} \cdots r_{x_{j-1}} r_{x_{j+1}} \cdots r_{x_n}
\]

and so \( l'(wr_{y_j}) \leq n - 1 < l'(w) \). Hence if \( \hat{x} \in N_i(w) \cap \Phi_i(W') \), then \( l'(wr_x) < l'(w) \).

Conversely suppose that \( x \in \Phi_i(W') \cap \Phi_i^+ \) and \( \hat{x} \notin N_i(w) \). Then \( \hat{x} \notin r_x (N_i(w) \cap \Phi_i(W')) \). But \( \hat{x} \in N_i(r_x) \cap \Phi_i(W') \), so

\[
\hat{x} \in r_x (N_i(w) \cap \Phi_i(W')) + (N_i(r_x) \cap \Phi_i(W')) = N_i(wr_x) \cap \Phi_i(W')
\]

and by what has just been proved, this implies that

\[
l'(w) = l'(wr_x) < l'(wr_x).
\]
Therefore if \( x \in \Phi_i(W') \cap \Phi_i^+ \) such that \( l'(wr_x) < l'(w) \), then we must have \( \hat{x} \in N_i(w) \cap \Phi_i(W') \).

**Lemma 2.2.10.** ((Lemma 3.2) of [1]) Let \( W' \) be a reflection subgroup of \( W \). For each \( i \in \{1, 2\} \), let \( x, y \in \Delta_i(W') \) such that \( r_x \neq r_y \). Let \( n = \text{ord}(r_x r_y) \). Then for \( 0 \leq m < n \)

\[
\begin{align*}
\机能 r_y \机能 r_x \机能 r_y x & \in \Phi_i^+ \quad \text{and} \quad \机能 r_y \机能 r_y \机能 r_x y \in \Phi_i^+.
\end{align*}
\]

**Proof.** Since \( x, y \in \Delta_i(W') \) it follows that

\[
l'(\机能 r_y \机能 r_x r_y) x = m + 1 > m = l'(\机能 r_y \机能 r_y \机能 r_x r_y)
\]

whenever \( m < n \). Then Lemma 2.2.9 (ii) yields that

\[
\hat{x} \notin N_i(\机能 r_y \机能 r_x r_y) \cap \Phi_i(W').
\]

Therefore \( \机能 r_y \机能 r_x r_y x \in \Phi_i^+ \).

By symmetry \( \机能 r_y \机能 r_x r_y y \in \Phi_i^+ \) too.

**Lemma 2.2.11.** ((Lemma 3.3) of [1]) Let \( W' \) be a reflection subgroup of \( W \). For each \( i \in \{1, 2\} \), let \( x, y \in \Delta_i(W') \) with \( r_x \neq r_y \). Let \( n = \text{ord}(r_x r_y) \) and write

\[
\机能 r_y \机能 r_x r_y x = c_m x + d_m y \quad \text{and} \quad \机能 r_x r_y r_x y = c'_m x + d'_m y.
\]

Then \( c_m \geq 0, \ d_m \geq 0, \ c'_m \geq 0 \) and \( d'_m \geq 0 \) whenever \( m < n \).

**Proof.** By symmetry, it will suffice to prove that \( d_m \geq 0 \) and \( d'_m \geq 0 \). The proof of this will be based on an induction on \( l(r_x) \).

Suppose first that \( l(r_x) = 1 \). Then \( \lambda x \in \Pi_i \) for some \( \lambda > 0 \). Write \( y = \sum_{z \in \Pi_i} \lambda_z z \) where \( \lambda_z \geq 0 \) for all \( z \in \Pi_i \). In fact, \( \lambda_{z_0} > 0 \) for some \( z_0 \in \Pi_i \setminus \{x\} \), since otherwise we would have \( y \in \mathbb{R} x \) and so \( r_x = r_y \).
Now for $0 \leq m < n$, Lemma 2.2.10 yields that

$$\left(\cdots r_y r_x r_y\right)x = c_m x + \sum_{z \in \Pi_i} d_m \lambda_z z.$$  

Observe that the coefficient of $z_0$ in this is $d_m \lambda_{z_0} \geq 0$. Since $\lambda_{z_0} > 0$, it follows that $d_m \geq 0$. Similarly $d'_m \geq 0$.

Suppose inductively now that the result is true for reflection subgroups $W''$ of $W$ and $x', y' \in \Delta_i(W'')$ with $r_x \neq r_y$ and $l(r_x) < l(r_x)$ where $l(r_x) \geq 3$. It is well known that there exists $r_z \in R$ (so $z \in \Pi_i$) that $l(r_z r_x r_z) = l(r_x) - 2$. Then $l(r_z r_x) < l(r_x)$, and thus $\hat{z} \in N_i(r_x)$. But since $x \in \Delta_i(W')$ and $x \neq z$ (since $l(r_x) \geq 3$), it follows that $r_z \notin W'$. Let $W'' = r_z W' r_z$. Lemma 2.2.6 (iii) yields that $\Delta_i(W'') = r_z \Delta_i(W')$ and therefore $r_z x, r_z y \in \Delta_i(W'')$. Now

$$\left(\cdots r_{y r_x r_y}\right)(r_z x) = c_m(r_z x) + d_m(r_z y)$$

and hence $\text{ord}(r_{r_z x} r_{r_z y}) = \text{ord}(r_x r_y) = n$. Since $l(r_{r_z x}) = l(r_x) - 2$, the inductive hypothesis gives

$$\left(\cdots r_{r_z x} r_{r_z y}\right)(r_z x) = c'_m(r_z x) + d'_m(r_z y)$$

where $d_m, d'_m \geq 0$ for $0 \leq m < n$. By (2.2.1) the result follows on applying $r_z$ to both sides of the last two equations. \[\square\]

**Proposition 2.2.12.** Let $W'$ be a reflection subgroup of $W$. Suppose that $x, y \in \Delta_i(W')$ with $r_x \neq r_y$. Let $n = \text{ord}(r_x r_y) \in \{\infty\} \cup \mathbb{N}$. Then

$\langle x, \phi(y) \rangle \leq 0$

and

$$\begin{cases} 
\langle x, \phi(y) \rangle \langle y, \phi(x) \rangle = \cos^2 \frac{\pi}{n} & (n \in \mathbb{N}, n \geq 2) \\
\langle x, \phi(y) \rangle \langle y, \phi(x) \rangle \in [1, \infty) & (n = \infty)
\end{cases}$$
2.2. Canonical Roots

Proof. Observe that since \( r_{\phi(x)} = r_x \neq r_y = r_{\phi(y)} \), it follows that \( \{x, y\} \) and \( \{\phi(x), \phi(y)\} \) are both linearly independent. Therefore we may apply last section to the present situation, that is,

\[ X_1 = \{x, y\} \quad X_2 = \{\phi(x), \phi(y)\} \quad \psi = \phi \quad \langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle \]
and

\[ R''_1 = R''_2 = \{r_x, r_y\} \quad W''_1 = W''_2 = \langle \{r_x, r_y\} \rangle \]
Consequently \( \Phi''_1 = W''_1X_1 = \langle \{r_x, r_y\}\rangle \{x, y\} \). Observe that the elements of \( \Psi \) are \( \pm(\cdots r_yr_x r_y) x \) and \( \pm(\cdots r_xr_y r_x) y \) (0 \( \leq m < \text{ord}(r_x r_y)) \).

Lemma 2.2.11 then yields that

\[ \Psi = \Psi^+ \cup \Psi^- \].

Therefore Proposition 2.1.1 yields that

\[ \begin{cases} 
  \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle = \cos^2 \frac{\pi}{n} & (n \in \mathbb{N}, n \geq 2) \\
  \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle \in [1, \infty) & (n = \infty).
\end{cases} \]

Let \( \Delta_1 \) be a subset of \( \Phi_1^+ \) satisfying the following conditions:

(1) \( r_x \neq r_y \) for all \( x, y \in \Delta_1, x \neq y \).
(2) \( \langle x, \phi(y) \rangle \leq 0 \) for all \( x, y \in \Delta_1, x \neq y \).
(3) \( \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle \in \{ \cos^2 \frac{\pi}{m} \mid m \in \mathbb{N}, m \geq 2 \} \cup [1, \infty) \), for all \( x, y \in \Delta_1 \).

Let \( W' = \langle r_x \mid x \in \Delta_1 \rangle \). The rest of this section is devoted to showing that

\[ \{ \lambda x \in \Phi_1^+ \mid \lambda > 0, x \in \Delta_1 \} = \Delta_1(W') \].

Most of the arguments to be used in proving this are adapted from [1].

Let \( \Delta_1' \) be a subset of \( \Delta_1 \).

Let \( U \) be a vector space over \( \mathbb{R} \) on a basis \( \Pi := \{ e_x \mid x \in \Delta_1' \} \). Let \( U' \) be a vector space over \( \mathbb{R} \) on a basis \( \Pi' := \{ f_{\phi(x)} \mid x \in \Delta_1' \} \). 
Define a bilinear map $\langle \, , \, \rangle : U \times U' \rightarrow \mathbb{R}$ by requiring
\[
\langle e_x, f_{\phi(y)} \rangle = \langle x, \phi(y) \rangle \quad \text{for all } x, y \in \Delta_1'.
\]
For each $x \in \Delta_1'$, define a linear transformation $\rho_x : U \rightarrow U$ by
\[
\rho_x(u) := u - 2\langle u, f_{\phi(x)} \rangle e_x \quad \text{for all } u \in U.
\]
Let
\[
\begin{align*}
R'' &= \{ \rho_x \mid x \in \Delta_1' \}; \\
W'' &= \langle R'' \rangle; \\
\Psi &= W''\Pi; \\
\Psi^+ &= \Psi \cap \text{PLC}(\pi); \\
\Psi^- &= -\Psi^+.
\end{align*}
\]
Observe that $(\Delta_1', U, U', \Pi, \Pi', \langle \, , \, \rangle)$ is a Coxeter datum, and it follows by Theorem 1.1.4 (as well as Proposition 2.1.4 above) that $(W'', R'')$ is a Coxeter system. Let $l'' : W'' \rightarrow \mathbb{N}$ be the length function on $(W'', R'')$. Corollary 1.2.5 applied to this Coxeter datum yields that:

**Proposition 2.2.13.** Suppose that $w'' \in W''$ and $x \in \Delta_1'$. Then
\[
l''(w'' \rho_x) = \begin{cases} 
  l''(w'') - 1 & \text{if } w'' e_x \in \Psi^- \\
  l''(w'') + 1 & \text{if } w'' e_x \in \Psi^+.
\end{cases} \quad \square
\]

Note that by Proposition 2.1.4, for $x, y \in \Delta_1$ with $x \neq y$ we have:
\[
\begin{align*}
\text{ord}(\rho_x, \rho_y) &= \\
\quad = \begin{cases} 
  m & \langle e_x, f_{\phi(y)} \rangle \langle e_y, f_{\phi(x)} \rangle = \cos^2 \frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\
  \infty & \langle e_x, f_{\phi(y)} \rangle \langle e_y, f_{\phi(x)} \rangle \geq 1
\end{cases} \\
\quad = \begin{cases} 
  m & \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle = \cos^2 \frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\
  \infty & \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle \geq 1
\end{cases} \\
\quad = \text{ord}(r_x r_y) \quad \text{(by Lemma 1.1.8 (ii) and Proposition 1.1.9 )}
\end{align*}
\]
Since $W'' = \langle R'' \rangle$ the above yield that there exists a homomorphism
\[
\theta : W'' \to W' \quad \text{such that } \theta(\rho_x) = r_x, \quad x \in \Delta_1'.
\]

Next let $L : U \to V_1$ be the $\mathbb{R}$-linear map such that $L(e_x) = x$, for all $x \in \Delta_1'$. We claim that
\[
(2.2.2) \quad L(w''u) = \theta(w'')L(u) \quad w'' \in W'', u \in U.
\]
To prove (2.2.2) first observe that if $x, y \in \Delta_1'$, then
\[
L(\rho_x(e_y)) = L(e_y - 2\langle e_y, f_{\phi(x)} \rangle e_x) = y - 2\langle e_y, f_{\phi(x)} \rangle x = y - 2\langle y, \phi(x) \rangle x = r_x y = \theta(\rho_x)L(e_y).
\]
By linearity, this gives that for all $x \in \Delta_1'$,
\[
L(\rho_x(u)) = \theta(\rho_x)L(u) \quad \text{for all } u \in U.
\]
Since $W'' = \langle \rho_x \mid x \in \Delta_1' \rangle$ and $\theta$ is a homomorphism, the claim (2.2.2) therefore follows by an induction on the length of $w''$ in $(W'', R'')$.

The following is an adaptation of Lemma 3.6 of [1] into our present situation.

**Proposition 2.2.14.** With the above notation,
\[
\Delta_1(W') \subseteq \{ \lambda z \in \Phi_1^+ \mid \lambda > 0, z \in \Delta_1 \}.
\]

**Proof.** Take $\Delta_1' = \Delta_1$. Since
\[
\theta(R'') = \{ r_x \mid x \in \Delta_i \} \quad \text{and} \quad W' = \langle r_x \mid x \in \Delta_i \rangle
\]
it follows that $\theta$ is surjective.

Let $x \in \Delta_1(W')$. Choose $w \in W''$ with $\theta(w) = r_x \in W'$. Now let $\rho_{x_1} \cdots \rho_{x_n} (x_i \in \Delta_1')$ be a reduced expression for $w$ in $(W'', R'')$. Note
that \( n \geq 1 \). Now \( l''(w \rho_{x_n}) < l''(w) \), so by Proposition 2.2.13 we have \( w e_{x_n} \in \Psi^{-} \), say
\[
we_{x_n} = - \sum_{y \in \Delta'_1} c_y e_y \quad \text{where } c_y \geq 0 \text{ for all } y \in \Delta'_1.
\]
Hence
\[
r_x(x_n) = \theta(w) L(e_{x_n}) = L(w e_{x_n}) = L(- \sum_{y \in \Delta'_1} c_y e_y) = - \sum_{y \in \Delta'_1} c_y y.
\]
Clearly \( r_x(x_n) \in \Phi_1 \) and because each \( y \in \Delta_1 \) can be expressed as a nonnegative linear combination of elements of \( \Pi_1 \), so it follows that \( r_x(x_n) \in \Phi_1^+ \). Since \( x_n \in \Delta_1 \subseteq \Phi_1(W') \subseteq \Phi_1^+ \) so it follows that
\[
(2.2.3) \quad \widehat{x}_n \in \mathcal{N}_1(r_x) \cap \Phi_1(W').
\]
But \( x \in \Delta_1(W') \) and so (2.2.3) above yields that \( x = \lambda x_n \) for some positive scalar \( \lambda \). Since \( x \in \Delta_1(W') \) was arbitrary it follows that
\[
\Delta_1(W') \subseteq \{ \lambda z \in \Phi_1^+ | \lambda > 0, z \in \Delta_1 \}.
\]
\[\square\]

The following is a generalization of Proposition 3.7 of [1].

**Proposition 2.2.15.** With the above notation
\[
\Delta_1(W') = \{ \lambda z \in \Phi_1^+ | \lambda > 0, z \in \Delta_1 \}.
\]

**Proof.** Take \( \Delta_1' = \Delta_1(W') \); this is possible by Proposition 2.2.14 and Proposition 2.2.12.

Since \( (W', S(W')) \) is a Coxeter system we know that
\[
\theta : (W'', R'') \rightarrow (W', S(W'))
\]
is a Coxeter system isomorphism. In particular
\[
(2.2.4) \quad l'(\theta(w'')) = l''(w'') \quad \text{for all } w'' \in W''
\]
where \( l' \) is the length function on \( (W', S(W')) \).
Let \( x \in \Delta_1 \). Then \( r_x \in W' \cap T \). By Lemma 2.2.6 (i), there exist \( x_0, \ldots, x_n \in \Delta_1(W') = \Delta'_1 \) such that

\[
r_x = r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_n}.
\]

Since \( r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_n} = r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_n} \), there is no loss of generality in assuming that \( l'(r_{x_n} \cdots r_{x_1} r_{x_0}) > l'(r_{x_n} \cdots r_{x_1}) \).

Then (2.2.4) yields that

\[
l''(\rho_{x_n} \cdots \rho_{x_1}) > l''(\rho_{x_n} \cdots \rho_{x_1}),
\]

and thus by Proposition 2.2.13 \( y := \rho_{x_n} \cdots \rho_{x_1}(e_{x_0}) \in \Psi^+ \), say

\[
y = \sum_{z \in \Delta'_1} c_z e_z \quad \text{where} \quad c_z \geq 0 \text{ for all } z \in \Delta'_1.
\]

Now we have \( L(y) = r_{x_n} \cdots r_{x_1}(x_0) = \sum_{z \in \Delta'_1} c_z z \in \Phi^+_1 \). Write \( y' = L(y) \).

Then

\[
y' = r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_n} = r_x.
\]

Since \( x, y' \in \Phi^+_1 \) the above yields that \( x = \mu y' = \sum_{z \in \Delta'_1} (\mu c_z) z \) for some positive scalar \( \mu \).

Now suppose that \( x \notin \Delta'_1(W') = \Delta'_1 \). Then by the definition of \( \Delta_1 \),

\[
\langle x, \phi(z) \rangle \leq 0 \quad \text{for all } z \in \Delta'_1, \text{ since } x \neq z.
\]

By Corollary 1.3.13 this implies that \( \langle z, \phi(x) \rangle \leq 0 \). But now

\[
1 = \langle x, \phi(x) \rangle = \langle \sum_{z \in \Delta'_1} (\mu c_z) z, \phi(x) \rangle = \sum_{z \in \Delta'_1} (\mu c_z) \langle z, \phi(x) \rangle \leq 0.
\]

This contradiction shows that the assumption \( x \notin \Delta'_1(W') \) is false. Since \( x \in \Delta_1 \) was arbitrary it follows that \( \Delta_1 \subseteq \Delta_1(W') \). Observe that if \( \alpha \in \Delta_1(W') \) then \( \lambda \alpha \in \Delta_1(W') \) whenever \( \lambda \alpha \in \Phi^+_1 \). Thus

\[
(2.2.5) \quad \{ \lambda x \in \Phi^+_1 \mid \lambda > 0, x \in \Delta_1 \} \subseteq \Delta_1(W').
\]

Finally, (2.2.5) and Proposition 2.2.14 together yield that

\[
\{ \lambda x \in \Phi^+_1 \mid \lambda > 0, x \in \Delta_1 \} \subseteq \Delta_1(W').
\]
2.3. Comparison with the Standard Geometric Realisation of Coxeter Groups

In this section we study the connections between the non-orthogonal geometric realization studied in Chapter 1 with the classical Tits representation (the standard orthogonal geometric realisation, in the sense of [10] and [12]) of Coxeter groups. We pay special attention to a comparison between the canonical coefficients defined in section 1.2 and their natural counterparts in the classical theory.

**Definition 2.3.1.** Let $V$ be a vector space over the real field $\mathbb{R}$, and let $(\ ,\ ) : V \times V \rightarrow \mathbb{R}$ be a symmetric bilinear form. Let $\Pi := \{ \gamma_s \ | \ s \in S \}$ be a set of linearly independent vectors satisfying the following conditions:

(C1') $(\gamma_s, \gamma_s) = 1$, for all $s \in S$;

(C2') $(\gamma_s, \gamma_t) \leq 0$, for all $s, t \in S$, $s \neq t$;

(C3') $(\gamma_s, \gamma_t)^2 = (\alpha_s, \beta_t)(\alpha_t, \beta_s)$, for all $s, t \in S$.

Let $\rho : S \rightarrow \text{GL}(V)$ be defined by

$$\rho(s)(v) : v - 2(v, \gamma_s)\gamma_s$$

for all $v \in V$.

Again it is readily checked that $\mathcal{C}'' = (S, V, V, \Pi, (\ ,\ ))$ is a Coxeter datum with Coxeter parameters $m_{st}$, $s, t \in S$. Let $W$ be the associated abstract Coxeter group. Then by Theorem 1.1.4, there is an isomorphism

$$f : W \rightarrow \{ \rho(s) \ | \ s \in S \} \text{ satisfying } f(r_s) = \rho(s) \text{ for all } s \in S.$$

We refer to such $V$ as the associated (to the non-orthogonal geometric realization) Standard geometric realisation (Tits representation) of $W$.

**Definition 2.3.2.** For all $w \in W$ and $v \in V$, we write $wv$ in place of $f(w)v$, and let the root system of $W$ in $V$ be denoted by

$$\Phi := W\Pi = \{ w\gamma_s \ | \ w \in W, s \in S \},$$
and we call the elements of $\Phi$ the set of roots of $W$ in $V$. Given $\gamma \in \Phi$, and $s \in S$, define $\text{coeff}_{\gamma_s} \in \mathbb{R}$ by requiring that $\gamma = \sum_{s \in S} \text{coeff}_{\gamma_s}(\gamma) \gamma_s$. Define $\Phi^+ := \Phi \cap \text{PLC}(\Pi)$, and call elements of $\Phi^+$ the set of positive roots in $V$; define $\Phi^- = -\Phi^+$, and call the elements of $\Phi^-$ the set of negative roots in $V$.

For $\gamma \in \Phi^+$, define the depth of $\gamma$ (denoted by $\text{dp}(\gamma)$) to be

$$\text{dp}(\gamma) = \min\{ m \in \mathbb{N} \mid w \in W \text{ such that } w\gamma \in \Phi^- \text{ and } l(w) = m \}.$$  

For $\gamma_1, \gamma_2 \in \Phi^+$, write $\gamma_1 \preceq \gamma_2$ if there exists some $w \in W$ such that $\gamma_2 = w\gamma_1$, and $\text{dp}(\gamma_2) = \text{dp}(\gamma_1) + l(w)$. Write $\gamma_1 \prec \gamma_2$ if $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. If $\gamma_1 \preceq \gamma_2$, we say that $\gamma_1$ precedes $\gamma_2$.

For each $\gamma \in \Phi$, define the reflection corresponding to $\gamma$, $r_\gamma : V \to V$ to be the linear transformation given by

$$r_\gamma(v) = v - 2(v, \gamma)\gamma, \text{ for all } v \in V.$$  

Observe that the above definitions give a Standard Tits representation of $W$ in the sense of [12] and [10]. It can be readily checked that $(\cdot, \cdot)$ is $W$-invariant, that is, $(x, y) = (wx, wy)$ for all $x, y \in V$ and $w \in W$. Suppose that $x \in \Phi$ and $\lambda x \in \Phi$ for some scalar $\lambda$. Then $x = w\gamma_s$ for some $w \in W$ and $s \in S$, and it follows from what we have just noted that

$$(x, x) = (w\gamma_s, w\gamma_s) = (\gamma_s, \gamma_s) = 1.$$  

Then $\lambda^2 = \lambda^2(x, x) = (\lambda x, \lambda x) = (w\gamma_s, w\gamma_s) = (\gamma_s, \gamma_s) = 1$ that is, $\lambda = \pm 1$. Thus we conclude that in $\Phi$ the only non trivial scalar multiple of a root is its negative.

Adapting the results obtained in Section 1.1 and Section 1.3 into the associated standard realization, we have:

**Lemma 2.3.3.**

(i) (Lemma 1.2.4) $\Phi = \Phi^+ \uplus \Phi^-$.  

(ii) (Lemma 1.2.13) $W$ is finite if and only if $\Phi$ is finite.  

(iii) (Lemma 1.2.1) The symmetric bilinear form $(\cdot, \cdot)$ is $W$-invariant, that is, for all $u, v \in V$ and for all $w \in W$, $(u, v) = (wu, wv)$.  

(iv) (Lemma 1.2.5) If \( w \in W \) and \( s \in S \), then
\[
l(wr_s) = \begin{cases} 
  l(w) + 1 & \text{if } w\gamma_s \in \Phi^+, \\
  l(w) - 1 & \text{if } w\gamma_s \in \Phi^-.
\end{cases}
\]

(v) (Lemma 1.7 of \([6]\)) Let \( s \in S \) and \( \gamma \in \Phi^+ \setminus \gamma_s \). Then
\[
\text{dp}(r_s \gamma) = \begin{cases} 
  \text{dp}(\gamma) - 1 & \text{if } (\gamma, \gamma_s) > 0, \\
  \text{dp}(\gamma) & \text{if } (\gamma, \gamma_s) = 0, \\
  \text{dp}(\gamma) + 1 & \text{if } (\gamma, \gamma_s) < 0.
\end{cases}
\]

(vi) (Proposition 2.1 \([5]\)) Let \( \gamma \in \Phi \), and \( r \in S \). Then \( \text{coeff}_{r_s}(\gamma) > 0 \) implies that \( \text{coeff}_{r_s}(\gamma) \geq 1 \). Furthermore, if \( 0 < \text{coeff}_{r_s}(\gamma) < 2 \), then either \( \text{coeff}_{r_s}(\gamma) = 1 \) or \( \text{coeff}_{r_s}(\gamma) = 2 \cos\left(\frac{\pi}{m_{r_1 r_2}}\right) \), where \( r_1, r_2 \in S \) with \( 4 \leq m_{r_1 r_2} < \infty \).

By Lemma 2.3.3(i), for any given \( \gamma \in \Phi \), \( \text{coeff}_{\gamma_s}(\gamma) \) are of the same sign for all \( s \in S \) (either all non-negative or all non-positive). The support \( \text{supp}(\gamma) \) of \( \gamma \) is the set of all \( \gamma_s \in \Pi \) with \( \text{coeff}_{\gamma_s}(\gamma) \neq 0 \).

The following result establishes the connection between the non-orthogonal geometric realisation of \( W \) and the Tits representation defined above.

**Proposition 2.3.4.** There are \( W \)-equivariant maps \( f_1 : \Phi_1 \to \Phi \), and \( f_2 : \Phi_2 \to \Phi \), satisfying
\[
f_1(\alpha_s) = \gamma_s = f_2(\beta_s)
\]
for all \( s \in S \).

**Proof.** This result follows from similar arguments to those used to prove Proposition 1.2.9 and Proposition 1.2.17. \( \Box \)

**Remark 2.3.5.** Unlike in Proposition 1.2.9, we stress that these \( W \)-equivariant maps \( f_1 \) and \( f_2 \) are not injective in general.
Lemma 2.3.6. Suppose that $\alpha \in \Phi_1$. Then $\alpha \in \Phi_1^+$ implies that $f_1(\alpha) \in \Phi^+$, and $\alpha \in \Phi_1^-$ implies that $f_1(\alpha) \in \Phi^-$. 

Proof. Let $\alpha \in \Phi_1$. Then $\alpha = w_\alpha r$, for some $w \in W$, and $r \in S$. If $\alpha \in \Phi_1^+$, then Corollary 1.2.5 yields that $l(w_\alpha r) = l(w) + 1$, in which case Lemma 2.3.3 (iv) and Proposition 2.3.4 yield that

$$f_1(\alpha) = f_1(w_\alpha r) = w f_1(\alpha r) = w \gamma_r \in \Phi^+.$$ 

Likewise we see that $\alpha \in \Phi_1^-$ implies that $f_1(\alpha) \in \Phi^-$. □

Lemma 2.3.7. Suppose that $\alpha \in \Phi_1^+$. Then $dp_1(\alpha) = dp(f_1(\alpha))$, that is, depth is $W$-invariant.

Proof. Let $w \in W$ be such that $w \alpha \in \Phi_1^-$, and $dp_1(\alpha) = l(w)$. Then Lemma 2.3.6 and Proposition 2.3.4 yield $f_1(w \alpha) = w f_1(\alpha) \in \Phi^-$, and so $dp(f_1(\alpha)) \leq l(w) = dp_1(\alpha)$. By symmetry $dp_1(\alpha) \leq dp(f_1(\alpha))$ as well, whence equality. □

Corollary 2.3.8. Suppose that $\alpha \in \Phi_1$, and $s \in S$. Then

$$(f_1(\alpha), \gamma_s) > 0 \text{ if and only if } \langle \alpha, \beta_s \rangle > 0,$$

and

$$(f_1(\alpha), \gamma_s) = 0 \text{ if and only if } \langle \alpha, \beta_s \rangle = 0,$$

and

$$(f_1(\alpha), \gamma_s) < 0 \text{ if and only if } \langle \alpha, \beta_s \rangle < 0.$$

Proof. Follows from Lemma 2.3.7 and Lemma 2.3.3 (v). □

The above can be immediately generalized to the following:

Corollary 2.3.9. Suppose that $\alpha_1, \alpha_2 \in \Phi_1$. Then

$$(f_1(\alpha_1), f_1(\alpha_2)) > 0 \text{ if and only if } \langle \alpha_1, \phi(\alpha_2) \rangle > 0,$$

and

$$(f_1(\alpha_1), f_1(\alpha_2)) = 0 \text{ if and only if } \langle \alpha_1, \phi(\alpha_2) \rangle = 0,$$

and
\[(f_1(\alpha_1), f_1(\alpha_2)) < 0 \quad \text{if and only if} \quad \langle \alpha_1, \phi(\alpha_2) \rangle < 0.\]

**Proof.** Apply the same argument used in the proof of Corollary 1.3.13 to Corollary 2.3.8 and the desired result follows.

\[\square\]

**Proposition 2.3.10.** For each \( \alpha \in \Phi_1 \), and for each \( r \in S \),

\[ \text{coeff}_{\alpha_r}(\alpha) \text{coeff}_{\beta_r}(\phi(\alpha)) \geq (\text{coeff}_{\gamma_r}(f_1(\alpha)))^2. \]

**Proof.** Replace \( \alpha \) by \(-\alpha\) if necessary, we may assume that \( \alpha \in \Phi_1^+ \) and furthermore, we may write \( \alpha = w\alpha_s \), where \( w \in W \) and \( s \in S \). The proof is based on an induction on \( l(w) \). If \( l(w) = 0 \), then the result clearly holds. Thus we may assume that \( l(w) \geq 1 \), and choose \( t \in S \) such that \( l(wr_t) = l(w) - 1 \). Then we may write \( w = w_1w_2 \), where \( w_2 \) is an alternating product of \( r_s \) and \( r_t \), ending in \( r_t \), and \( w_1 \) is of minimal length in the coset \( w\langle r_s, r_t \rangle \). Thus Lemma 1.3.2 yields that \( l(w) = l(w_1) + l(w_2) \), \( l(w_1r_s) > (w_1) \), and \( l(w_1r_t) > l(w_1) \).

Consequently Corollary 1.2.5 yields that \( w_1\alpha_s \in \Phi_1^+ \) and \( w_1\alpha_t \in \Phi_1^+ \). Now a rank 2 calculation yields that \( w_2\alpha_s = p\alpha_s + \lambda q\alpha_t \), where \( \lambda \) is a positive constant and \( pq \geq 0 \). If \( p, q < 0 \) then

\[ \alpha = w\alpha_s = w_1w_2\alpha_s = w_1(p\alpha_s + \lambda q\alpha_t) = pw_1\alpha_s + \lambda qw_1\alpha_t \in \Phi_1^- \]

contradicting the assumption that \( \alpha \in \Phi_1^+ \). Therefore \( p, q \geq 0 \). Now a direct rank 2 calculation shows that

\[ w_2\gamma_s = p\gamma_s + q\gamma_t, \quad w_2\alpha_s = p\alpha_s + \lambda q\alpha_t, \quad \text{and} \quad w_2\beta_s = p\beta_s + \frac{q}{\lambda}\beta_t. \]

Recall the \( W \)-equivariant maps \( \phi_1 \) and \( \phi_2 \) from Proposition 1.2.9 and we see that the above yields

\[ w_2\alpha'_s = \phi_1(w_2\alpha_s) = p\alpha'_s + \lambda q\alpha'_t \quad \text{and} \quad w_1\beta'_s = \phi_2(w_2\beta_s) = p\beta'_s + \frac{q}{\lambda}\beta'_t. \]

Now we set

\[ x := \text{coeff}_{\alpha}(\alpha) , x' := \text{coeff}_{\beta}(\phi(\alpha)) , x'' := \text{coeff}_{\gamma}(f_1(\alpha)); \]

and
2.3. Comparison With Standard Geometric Realization

\[ y := \text{coeff}_{\alpha_r}(w_1 \alpha_s), \quad y' := \text{coeff}_{\beta_r}(w_1 \beta_s), \quad y'' := \text{coeff}_{\gamma_r}(w_1 \gamma_s); \]

and

\[ z := \text{coeff}_{\alpha_t}(w_1 \alpha_t), \quad z' := \text{coeff}_{\beta_t}(w_1 \beta_t), \quad z'' := \text{coeff}_{\gamma_t}(w_1 \gamma_t). \]

Since \( l(w_1) < l(w) \), it follows from the inductive hypothesis that

\[ yy' \geq (y'')^2 \quad \text{and} \quad zz' \geq (z'')^2. \]

Now

\[ xx' - (x'')^2 = (py + \lambda qz)(py' + \frac{1}{\lambda} qz') - (py'' + qz'')^2 \]

\[ = p^2(yy' - y''^2) + q^2(zz' - z''^2) + pq(\frac{1}{\lambda}yz' + \lambda yz' - 2y''z''). \]

From the inductive hypothesis the first two summands are nonnegative. It follows from the inductive hypothesis and the geometric mean and arithmetic mean inequality applied to the terms \( \frac{1}{\lambda}yz' + \lambda yz' \) (indeed \( \frac{1}{\lambda}yz' + \lambda yz' \geq 2\sqrt{yy'zz'} \geq 2y''z'' \) ) that the third summand \( pq(\frac{1}{\lambda}yz' + \lambda yz' - 2y''z'') \) is also nonnegative, whence \( xx' - x''^2 \geq 0 \) and the desired result follows by induction. \( \square \)

**Proposition 2.3.11.** Suppose that \( \alpha_1, \alpha_2 \in \Phi_1 \). Then

\[ \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \geq (f_1(\alpha_1), f_1(\alpha_2))^2. \]

**Proof.** Since both \( \langle , \rangle \) and \( ( , ) \) are \( W \)-invariant, and \( \phi, f_1 \) are \( W \)-equivariant, we may replace \( \alpha_1 \) and \( \alpha_2 \) by \( u\alpha_1 \) and \( u\alpha_2 \) for a suitable \( u \in W \) such that \( \alpha_2 = \alpha_s \) for some \( s \in S \). Furthermore, replace \( \alpha_1 \) by \( -\alpha_1 \) if needs be, we may assume that \( \alpha_1 \in \Phi_1^+ \). We proceed by an induction on the depth of \( \alpha_1 \).

If \( dp_1(\alpha_1) = 1 \), then \( \alpha_1 = \lambda \alpha_r \), where \( r \in S \) and \( \lambda \) is a positive constant. Then Corollary 1.2.19 yields that \( \phi(\alpha_1) = \frac{1}{\lambda} \beta_r \), and hence

\[ \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle = \lambda \langle \alpha_r, \beta_s \rangle \frac{1}{\lambda} \langle \alpha_s, \beta_r \rangle = \langle \alpha_r, \beta_s \rangle \langle \alpha_s, \beta_r \rangle \]

\[ = (\gamma_r, \gamma_s)^2 \quad \text{(by the definition of ( , )}) \]

\[ = (f_1(\alpha_1), f_1(\alpha_2)). \]
Thus we may assume that $dp_1(\alpha_1) > 1$. Next if $\langle \alpha_1, \beta_s \rangle > 0$, then Lemma 1.3.10 yields that $r_s \alpha_1 \prec_1 \alpha_1$, and hence

$$\langle \alpha_1, \beta_s \rangle \langle \alpha_s, \phi(\alpha_1) \rangle$$

$$= \langle r_s \alpha_1, r_s \beta_s \rangle \langle r_s \alpha_s, \phi(r_s \alpha_1) \rangle = (-\langle r_s \alpha_1, \beta_s \rangle)(-\langle \alpha_s, \phi(r_s \alpha_1) \rangle)$$

$$\geq (f_1(r_s \alpha_1), \gamma_s)^2 \quad \text{(by the inductive hypothesis)}$$

$$= (f_1(\alpha_1), -\gamma_s)^2$$

$$= (f_1(\alpha_1), \gamma_s)^2$$

as required. Thus we may further assume that $\langle \alpha_1, \beta_s \rangle \leq 0$.

Next choose $t \in S$ such that $r_t \alpha_1 \prec_1 \alpha_1$. Then Lemma 1.3.10 yields that $\langle \alpha_1, \beta_t \rangle > 0$ and, in particular, $s \neq t$. Let $w \in W$ be a maximal length alternating product of $r_s$ and $r_t$ ending in $r_t$ such that $dp_1(w \alpha_1) = dp_1(\alpha_1) - l(w)$. If $w = r_{s_1} \cdots r_{s_l}$ is a reduced expression for $w$, then Lemma 1.3.8 yields that $r_{s_l} \alpha_1 \prec_1 \alpha_1$, that is, $\langle \alpha_1, \beta_{s_l} \rangle > 0$. Therefore $s \neq s_l$, and $w$ has no reduced expression ending in $r_s$. Furthermore, we observe that $dp_1(w \alpha_1) \leq dp_1(\alpha_1)$, and so the inductive hypothesis yields that

$$(2.3.2) \quad \langle w \alpha_1, \beta_s \rangle \langle \alpha_s, \phi(w \alpha_1) \rangle \geq (\gamma_s, f_1(w \alpha_1))^2$$

and

$$(2.3.3) \quad \langle w \alpha_1, \beta_t \rangle \langle \alpha_t, \phi(w \alpha_1) \rangle \geq (\gamma_t, f_1(w \alpha_1))^2.$$
Thus
\[
\langle \alpha_1, \beta_s \rangle \langle \alpha_s, \phi(\alpha_1) \rangle - (\gamma_s, f_1(\alpha_1))^2 \\
= \langle w\alpha_1, w\beta_s \rangle \langle w\alpha_s, \phi(w\alpha_1) \rangle - (w\gamma_s, f_1(w\alpha_1))^2 \\
\text{ (since } \langle , \rangle \text{ and } ( , ) \text{ are } W\text{-invariant}) \\
= \langle w\alpha_1, p\beta_s + q\beta_t \rangle \langle p\alpha_s + \lambda q\alpha_t, \phi(w\alpha_1) \rangle - (f_1(w\alpha_1), p\gamma_s + q\gamma_t)^2 \\
= p^2(\langle w\alpha_1, \beta_s \rangle \langle \alpha_s, \phi(w\alpha_1) \rangle - (f_1(w\alpha_1), \gamma_s)^2) \\
\underline{A} + q^2(\langle w\alpha_1, \beta_t \rangle \langle \alpha_t, \phi(w\alpha_1) \rangle - (f_1(w\alpha_1), \gamma_s)^2) + C. \\
\underline{B}
\]

where
\[
C = pq(\frac{1}{\lambda} \langle w\alpha_1, \beta_t \rangle \langle \alpha_s, \phi(w\alpha_1) \rangle + \lambda \langle w\alpha_1, \beta_s \rangle \langle \alpha_t, \phi(w\alpha_1) \rangle \\
- 2(f_1(w\alpha_1), \gamma_s)(f_1(w\alpha_1), \gamma_t)).
\]

It follows from (2.3.2) and (2.3.3) that A and B are both nonnegative.

It follows from the geometric mean and arithmetic mean inequality that
\[
\frac{1}{\lambda} \langle w\alpha_1, \beta_t \rangle \langle \alpha_s, \phi(w\alpha_1) \rangle + \lambda \langle w\alpha_1, \beta_s \rangle \langle \alpha_t, \phi(w\alpha_1) \rangle \\
\geq 2\sqrt{\langle w\alpha_1, \beta_s \rangle \langle \alpha_s, \phi(w\alpha_1) \rangle \langle w\alpha_1, \beta_t \rangle \langle \alpha_t, \phi(w\alpha_1) \rangle} \\
\geq 2(f_1(w\alpha_1), \gamma_s)(f_1(w\alpha_1), \gamma_t) \quad \text{(by (2.3.2) and (2.3.3))},
\]
that is, \( C \geq 0 \) as well. Therefore \( \langle \alpha_1, \beta_s \rangle \langle \alpha_s, \phi(\alpha_1) \rangle \geq (\gamma_s, f_1(\alpha_1))^2 \),
and the desired result follows by induction. \( \square \)

**Lemma 2.3.12.** Suppose that \( \alpha \in \Phi_1 \), and \( t \in S \). Then \( \text{coeff}_{\alpha_t}(\alpha) = 0 \) if and only if \( \text{coeff}_{\gamma_t}(f_1(\alpha)) = 0 \).

**Proof.** By Proposition 2.3.10 we only need to show that the condition \( \text{coeff}_{\gamma_t}(f_1(\alpha)) = 0 \) implies that \( \text{coeff}_{\alpha_t}(\alpha) = 0 \). Replacing \( \alpha \) by \(-\alpha\) if needs be, we may assume that \( \alpha \in \Phi_1^+ \). We may write \( \alpha = w\alpha_r \) for some \( w \in W \) and \( r \in S \). If \( l(w) = 0 \) then there is nothing to prove. Thus we may assume that \( l(w) \geq 1 \) and proceed by an induction on \( l(w) \).

Choose \( s \in S \) such that \( l(wr_s) = l(w) - 1 \), and write \( w = w_1w_2 \) where
$w_1 \in w(r_r, r_s)$ is of minimal length, and $w_2$ is an alternating product of $r_r$ and $r_s$ ending in $r_s$. By Lemma 1.3.2, $l(w) = l(w_1) + l(w_2)$. Since $w_2$ is a right segment of $w$, and since $\alpha = w\alpha_r$ is positive, it follows that $w_2\alpha_r \in \Phi_1^+$. Again a rank 2 calculation shows that

$$w_2\gamma_r = p\gamma_r + q\gamma_s, \quad w_2\alpha_r = p\alpha_r + \lambda q\alpha_s$$

and

$$w_2\beta_r = p\beta_r + \frac{q}{\lambda}\beta_s$$

for some non-negative constants $p, q$ and positive constant $\lambda$. Again Proposition 1.2.9 yields that

$$w_2\alpha'_r = \phi_1(w_2\alpha_r) = p\alpha'_r + \lambda q\alpha'_s$$

and

$$w_2\beta'_r = \phi_2(w_2\beta_r) = p\beta'_r + \frac{q}{\lambda}\alpha'_s.$$  

Then

$$0 = \operatorname{coeff}_{\gamma_t}(f_1(\alpha)) = \operatorname{coeff}_{\gamma_t}(w\gamma_r) = \operatorname{coeff}_{\gamma_t}(w_1(p\gamma_r + q\gamma_s))$$

$$= p \operatorname{coeff}_{\gamma_t}(w_1\gamma_r) + q \operatorname{coeff}_{\gamma_t}(w_1\gamma_s).$$

By Lemma 1.3.2 and Lemma 2.3.3 (iv), $w_1\gamma_r$ and $w_1\gamma_s$ are both positive, and hence

$$p \operatorname{coeff}_{\gamma_t}(w_1\gamma_r) = q \operatorname{coeff}_{\gamma_t}(w_1\gamma_s) = 0.$$

Then the inductive hypothesis yields that

$$p \operatorname{coeff}_{\alpha_t}(w_1\alpha_r) = q \operatorname{coeff}_{\alpha_t}(w_1\alpha_s) = 0$$

and therefore

$$\operatorname{coeff}_{\alpha_t}(\alpha) = \operatorname{coeff}_{\alpha_t}(w\alpha_r) = p \operatorname{coeff}_{\alpha_t}(w_1\alpha_r) + \lambda q \operatorname{coeff}_{\alpha_t}(w_1\alpha_s) = 0$$

as required.

Combining the above with Lemma 2.3.6 we immediately have:

**Corollary 2.3.13.** Suppose that $\alpha \in \Phi_1$, and $t \in S$. Then

$$\operatorname{coeff}_{\alpha_t}(\alpha) = 0 \text{ if and only if } \operatorname{coeff}_{\gamma_t}(f_1(\alpha)) = 0,$$

and

$$\operatorname{coeff}_{\alpha_t}(\alpha) > 0 \text{ if and only if } \operatorname{coeff}_{\gamma_t}(f_1(\alpha)) > 0,$$

and

$$\operatorname{coeff}_{\alpha_t}(\alpha) < 0 \text{ if and only if } \operatorname{coeff}_{\gamma_t}(f_1(\alpha)) < 0.$$
Proposition 2.3.14. Suppose that $\alpha \in \Phi_1^+$ and $r \in S$. Then

$$\text{coeff}_{\alpha r}(\alpha) \text{coeff}_{\beta r}(\phi(\alpha)) = 1 \text{ if and only if } \text{coeff}_{\gamma r}(f_1(\alpha)) = 1.$$ 

Proof. Suppose that $\text{coeff}_{\alpha r}(\alpha) \text{coeff}_{\beta r}(\phi(\alpha)) = 1$. Proposition 2.3.10 and Lemma 2.3.3(vi) combined yield that either $\text{coeff}_{\gamma r}(f_1(\alpha)) = 1$ or $\text{coeff}_{\gamma r}(f_1(\alpha)) = 0$. Observe that we can rule out the latter for it contradicts Corollary 2.3.13.

Next for the converse implication, suppose that $\text{coeff}_{\gamma r}(f_1(\alpha)) = 1$. As usual we write $\alpha = w\alpha_s$, $w \in W$ and $s \in S$. If $l(w) = 0$, then there is nothing to prove. Thus we may assume that $l(w) > 0$ and proceed by induction. Again we choose $t \in S$ such that $l(wr_t) = l(w) - 1$, and write $w = w_1 w_2$, where $w_1 \in w\langle r_s, r_t \rangle$ is of minimal length, and $w_2$ is an alternating product of $r_s$ and $r_t$ ending in $r_t$. Lemma 1.3.2 then yields that $l(w) = l(w_1) + l(w_2)$. Since $w_2$ is a right hand segment of $w$, then the condition $w\alpha_s \in \Phi_1^+$ yields that $w_2\alpha_s \in \Phi_1^+$, which in turn yields that $w_2\gamma_s \in \Phi^+$ by Corollary 2.3.13. Hence, direct rank 2 calculations yield that there are nonnegative constants $p$, $q$ and positive constant $\lambda$ such that

$$w_2\gamma_s = p\gamma_s + q\gamma_t ; w_2\alpha_s = p\alpha_s + \lambda q\alpha_t ; w_2\beta_s = p\beta_s + \frac{1}{\lambda} q\beta_t.$$ 

Again Proposition 1.2.9 yields that

$$w_2\alpha’_s = \phi_1(w_2\alpha_s) = p\alpha’_s + \lambda q\alpha’_t \text{ and } w_2\beta’_s = \phi_2(w_2\beta_s) = p\beta’_s + \frac{q}{\lambda} \beta’_t.$$ 

And consequently

$$f_1(\alpha) = w\gamma_s = w_1(p\gamma_s + q\gamma_t) = pw_1\gamma_s + qw_1\gamma_t.$$ 

Hence

$$(2.3.4) \quad 1 = p \text{coeff}_{\gamma r}(w_1\gamma_s) + q \text{coeff}_{\gamma r}(w_1\gamma_t).$$
It follows from Lemma 1.3.2 and Lemma 2.3.3 (iv) combined that $w_1 \gamma_s \in \Phi^+$ and $w_1 \gamma_t \in \Phi^+_1$. Thus Lemma 2.3.3(vi) yields that precisely one of the following is the case:

$$\text{coeff}_{\gamma_s}(w_1 \gamma_s) = p = 1, \text{ and } q \text{ coeff}_{\gamma_t}(w_1 \gamma_t) = 0$$

or

$$\text{coeff}_{\gamma_t}(w_1 \gamma_t) = q = 1, \text{ and } p \text{ coeff}_{\gamma_s}(w_1 \gamma_s) = 0.$$ 

Since $l(w_1) < l(w)$, the inductive hypothesis and Corollary 2.3.13 above imply either

$$p^2 \text{ coeff}_{\alpha_s}(w_1 \alpha_s) \text{ coeff}_{\beta_s}(w_1 \beta_s) = 1, \text{ and } q \text{ coeff}_{\alpha_t}(w_1 \alpha_t) = 0$$

or

$$q^2 \text{ coeff}_{\alpha_t}(w_1 \alpha_t) \text{ coeff}_{\beta_t}(w_1 \beta_t) = 1, \text{ and } p \text{ coeff}_{\alpha_s}(w_1 \alpha_s) = 0.$$

Therefore

$$\text{coeff}_{\alpha_r}(\alpha) \text{ coeff}_{\beta_r}(\phi(\alpha))$$

$$= \text{coeff}_{\alpha_r}(pw_1 \alpha_s + \lambdaqw_1 \alpha_t) \text{ coeff}_{\beta_r}(pw_1 \beta_s + \frac{1}{\lambda}qw_1 \beta_t)$$

$$= (p \text{ coeff}_{\alpha_r}(w_1 \alpha_s) + \lambda q \text{ coeff}_{\alpha_r}(w_1 \alpha_t)) \cdot (p \text{ coeff}_{\beta_r}(w_1 \beta_s) + \frac{q}{\lambda} \text{ coeff}_{\beta_r}(w_1 \beta_t))$$

$$= p^2 \text{ coeff}_{\alpha_r}(w_1 \alpha_s) \text{ coeff}_{\beta_r}(w_1 \beta_s) + \frac{1}{\lambda}pq \text{ coeff}_{\alpha_r}(w_1 \alpha_s) \text{ coeff}_{\beta_r}(w_1 \beta_t)$$

$$+ \lambda pq \text{ coeff}_{\alpha_r}(w_1 \alpha_t) \text{ coeff}_{\beta_r}(w_1 \beta_s) + q^2 \text{ coeff}_{\alpha_r}(w_1 \alpha_t) \text{ coeff}_{\beta_r}(w_1 \beta_t)$$

$$= 1$$

because precisely one summand is nonzero and is equal to 1. \qed

**Proposition 2.3.15.** Suppose that $\alpha \in \Phi^+_1$ and $r \in S$, such that

$$1 \leq \text{coeff}_{\alpha_r}(\alpha) \text{ coeff}_{\beta_r}(\phi(\alpha)) < 4.$$ 

Then either

$$\text{coeff}_{\alpha_r}(\alpha) \text{ coeff}_{\beta_r}(\phi(\alpha)) = 1$$

or

$$\text{coeff}_{\alpha_r}(\alpha) \text{ coeff}_{\beta_r}(\phi(\alpha)) = 4 \cos^2\left(\frac{\pi}{m}\right),$$

where $m = m_{r_1r_2}$, for some $r_1, r_2 \in S$ and $4 \leq m < \infty$. 

**Proof.** First we write $\alpha = w\alpha_s$ for some $w \in W$ and $s \in S$. If $l(w) = 0$, then there is nothing to prove. Thus we may assume that $l(w) > 0$ and proceed by induction again. Similarly as in Proposition 2.3.14, we choose $t \in S$ such that $l(wr_t) = l(w) - 1$, and let $w_1 \in w\langle r_s, r_t \rangle$ be of minimal length, and $w_2 \in \langle r_s, r_t \rangle$ such that $w = w_1w_2$. Again as in the proof of Proposition 2.3.14, there are nonnegative constants $p, q$ and positive constant $\lambda$ such that

$$w_2\gamma_s = p\gamma_s + q\gamma_t; \quad w_2\alpha_s = p\alpha_s + \lambda q\alpha_t; \quad w_2\beta_s = p\beta_s + \frac{1}{\lambda}q\beta_t.$$

Again Proposition 1.2.9 yields that

$$w_2\alpha_s' = \phi_1(w_2\alpha_s) = p\alpha_s' + \lambda q\alpha_t' \quad \text{and} \quad w_2\beta_s' = \phi_2(w_2\beta_s) = p\beta_s' + \frac{q}{\lambda}\beta_t'.
$$

And consequently

$$\operatorname{coeff}_{\alpha_s'}(\alpha) \operatorname{coeff}_{\beta_s'}(\phi(\alpha)) = \operatorname{coeff}_{\alpha_s'}(w_1(p\alpha_s + \lambda q\alpha_t)) \operatorname{coeff}_{\beta_s'}(w_1(p\beta_s + \frac{q}{\lambda}\beta_t)) = (p \operatorname{coeff}_{\alpha_s'}(w_1\alpha_s) + \lambda q \operatorname{coeff}_{\alpha_s'}(w_1\alpha_t))$$

$$+ \left(p \operatorname{coeff}_{\beta_s'}(w_1\beta_s) + \frac{q}{\lambda} \operatorname{coeff}_{\beta_s'}(w_1\beta_t)\right) = p^2 \operatorname{coeff}_{\alpha_s'}(w_1\alpha_s) \operatorname{coeff}_{\beta_s'}(w_1\beta_s) + \frac{q}{\lambda}pq \operatorname{coeff}_{\alpha_s'}(w_1\alpha_s) \operatorname{coeff}_{\beta_s'}(w_1\beta_t) + \lambda pq \operatorname{coeff}_{\alpha_s'}(w_1\alpha_t) \operatorname{coeff}_{\beta_s'}(w_1\beta_s) + q^2 \operatorname{coeff}_{\alpha_s'}(w_1\alpha_t) \operatorname{coeff}_{\beta_s'}(w_1\beta_t).
$$

Suppose that $p, q > 0$, $\operatorname{coeff}_{\alpha_s'}(w_1\alpha_s) > 0$ and $\operatorname{coeff}_{\alpha_s'}(w_1\alpha_t) > 0$. Since $p = \operatorname{coeff}_{\gamma_s}(w_2\gamma_s)$ and $q = \operatorname{coeff}_{\gamma_t}(w_2\gamma_t)$, it follows from Lemma 2.3.3(vi) and Proposition 1.3.3 combined that

$$p^2 \operatorname{coeff}_{\alpha_s'}(w_1\alpha_s) \operatorname{coeff}_{\beta_s'}(w_1\beta_s) \geq 1$$

and

$$q^2 \operatorname{coeff}_{\alpha_s'}(w_1\alpha_t) \operatorname{coeff}_{\beta_s'}(w_1\beta_t) \geq 1$$

From these a geometric mean and arithmetic mean inequality argument yields that

$$\frac{1}{\lambda}pq \operatorname{coeff}_{\alpha_s'}(w_1\alpha_s) \operatorname{coeff}_{\beta_s'}(w_1\beta_t) + \lambda pq \operatorname{coeff}_{\alpha_s'}(w_1\alpha_t) \operatorname{coeff}_{\beta_s'}(w_1\beta_s) \geq 2.$$
But (2.3.5), (2.3.6) and (2.3.7) together will give that
\[
\text{coeff}_{\alpha_r}(\alpha) \text{coeff}_{\beta_r}(\phi(\alpha)) \geq 4.
\]
contradicting our assumption. Therefore at least one of the terms \(p\), \(q\), \(\text{coeff}_{\alpha_r}(w_1\alpha_s)\) or \(\text{coeff}_{\alpha_r}(w_1\alpha_t)\) must be zero. Then Proposition 1.3.3 yields that either
\[
(2.3.8)
1 \leq \text{coeff}_{\alpha_r}(\alpha) \text{coeff}_{\beta_r}(\phi(\alpha)) = p^2 \text{coeff}_{\alpha_r}(w_1\alpha_s) \text{coeff}_{\beta_r}(w_1\beta_s) < 4,
\]
or
\[
(2.3.9)
1 \leq \text{coeff}_{\alpha_r}(\alpha) \text{coeff}_{\beta_r}(\phi(\alpha)) = q^2 \text{coeff}_{\alpha_r}(w_1\alpha_t) \text{coeff}_{\beta_r}(w_1\beta_t) < 4.
\]
If (2.3.8) is the case, then Proposition 2.3.10 yields that
\[
(2.3.10)
1 \leq p \text{coeff}_{\gamma_r}(w_1\gamma_s) < 2;
\]
whereas if (2.3.9) is the case, then Proposition 2.3.10 yields that
\[
(2.3.11)
1 \leq q \text{coeff}_{\gamma_r}(w_1\gamma_t) < 2.
\]
Suppose that (2.3.8) is the case. Then Lemma 2.3.3(vi) yields that at least one of \(p\) or \(\text{coeff}_{\gamma_r}(w_1\gamma_s)\) must be 1. If \(p = 1\), since \(l(w_1) < l(w)\), it follows from the inductive hypothesis that
\[
\text{coeff}_{\alpha_r}(\alpha) \text{coeff}_{\beta_r}(\phi(\alpha)) = \text{coeff}_{\alpha_r}(w_1\alpha_s) \text{coeff}_{\beta_r}(w_1\beta_s)
\]
\[
= \left\{ \begin{array}{ll}
1 & \\
\text{or} & \\
4 \cos^2\left(\frac{\pi}{m}\right), &
\end{array} \right.
\]
where \(m\) is an integer of the required form. On the other hand if \(\text{coeff}_{\gamma_r}(w_1\gamma_s) = 1\), then by the last Proposition
\[
\text{coeff}_{\alpha_r}(w_1\alpha_s) \text{coeff}_{\beta_r}(w_1\beta_s) = 1.
\]
Hence \(\text{coeff}_{\alpha_r}(\alpha) \text{coeff}_{\beta_r}(\phi(\alpha)) = p^2\). By Lemma 2.3.3(vi), \(p^2\) is either \(1\) or \(4 \cos^2\left(\frac{\pi}{m'}\right)\), where \(m'\) is an integer of the required form.
Similarly, if (2.3.9) is the case, then
\[
\text{coeff}_{r}(\alpha) \text{ coeff}_{\phi(\alpha)} = \begin{cases} 
1 \\
\text{or} \\
4 \cos^2\left(\frac{\pi}{m''}\right),
\end{cases}
\]
where \(m''\) is an integer of the required form, and this completes our proof.

The following is a well known result due to Dyer. A proof of this result can be found in [6] (Proposition 1.4) and [13] (Proposition 4.5.4).

**Proposition 2.3.16.** Let \(\gamma_1, \gamma_2 \in \Phi^+\). Then the (dihedral) subgroup of \(W\) generated by \(r_{\gamma_1}\) and \(r_{\gamma_2}\) is finite if and only if \(|(\gamma_1, \gamma_2)| < 1\). □

Combining Proposition 2.3.11 and the above, we can immediately deduce the following:

**Corollary 2.3.17.** Let \(\alpha_1, \alpha_2 \in \Phi_1\) with \(r_{\alpha_1} \neq r_{\alpha_2}\). The subgroup of \(W\) generated by \(r_{\alpha_1}\) and \(r_{\alpha_2}\) is finite if \(\langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle < 1\). □

We prove the converse to Corollary 2.3.17 at the end of Chapter 3 Section 3.5.

**Lemma 2.3.18.** (Lemma 2.4 (i) of [1]) Suppose that \(\gamma_1, \gamma_2 \in \Phi^+\) are distinct. Then
\[
\langle \{r_{\gamma_1}, r_{\gamma_2}\}\{\gamma_1, \gamma_2\} \subseteq \text{PLC}(\{\gamma_1, \gamma_2\}) \cup -\text{PLC}(\{\gamma_1, \gamma_2\})
\]
if and only if
\[
(\gamma_1, \gamma_2) \in (-\infty, -1] \cup \{-\cos \frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\}.
\]
□

**Proof.** We apply the results obtained in Section 2.2 here. We set \(\{\gamma_1, \gamma_2\} = X_1 = X_2\) and let \(\psi\) to be the identity map on \(\{\gamma_1, \gamma_2\}\). Consequently \((, ) = \langle , \rangle'\) and hence Proposition 2.1.1 yields that
\[
\langle \{r_{\gamma_1}, r_{\gamma_2}\}\{\gamma_1, \gamma_2\} \subseteq \text{PLC}(\{\gamma_1, \gamma_2\}) \cup -\text{PLC}(\{\gamma_1, \gamma_2\})
\]
if and only if
\[(\gamma_1, \gamma_2) \in (-\infty, -1] \cup \{-\cos \frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\},\]
as required. \[\square\]

**Proposition 2.3.19.** Suppose that \(\alpha_1, \alpha_2 \in \Phi_1^+\) such that \(r_{\alpha_1} \neq r_{\alpha_2}\). Then

\[
\begin{cases}
\langle \alpha_1, \phi(\alpha_2) \rangle \leq 0 \quad \text{and} \\
\langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \in \{\cos^2 \frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\} \cup [1, \infty)
\end{cases}
\]

if and only if

\[
(f_1(\alpha_1), f_1(\alpha_2)) \in (-\infty, -1] \cup \{-\cos \frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\}.
\]

**Proof.** Since \(r_{f_1(\alpha_1)} = r_{\alpha_1} \neq r_{\alpha_2} = r_{f_1(\alpha_2)}\), it follows that both \(\{\alpha_1, \alpha_2\}\) and \(\{f_1(\alpha_1), f_1(\alpha_2)\}\) are linearly independent. Now set

\[X_1 = \{\alpha_1, \alpha_2\}, \quad W''_1 = \langle\{r_{\alpha_1}, r_{\alpha_2}\}\rangle, \quad \Phi''_1 = W''_1 X_1\]

and let \(\psi\) be the restriction of \(\phi\) on \(\Phi''_1\). Recall that by Lemma 2.3.6 we have \(f_1(\Phi_1^+) = \Phi^+\) and \(f_1^{-1}(\Phi^+) = \Phi_1^+\). Thus Proposition 2.1.1 yields that

\[
\begin{cases}
\langle \alpha_1, \phi(\alpha_2) \rangle \leq 0 \quad \text{and} \\
\langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \in \{\cos^2 \frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\} \cup [1, \infty)
\end{cases}
\]

if and only if \(\Phi''_1 = \Phi''_1^+ \cup \Phi''_1^-\), and this happens if and only if

\[
\{r_{f_1(\alpha_1)}, r_{f_1(\alpha_2)}\}\{f_1(\alpha_1), f_1(\alpha_2)\} \subseteq \text{PLC}(\{f_1(\alpha_1), f_1(\alpha_2)\}) \\
\cup \text{PLC}(-\{f_1(\alpha_1), f_1(\alpha_2)\})
\]

and by Lemma 2.3.18 above this happens if and only if

\[
(f_1(\alpha_1), f_1(\alpha_2)) \in (-\infty, -1] \cup \{-\cos \frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\}.
\]

\[\square\]
2.4. Dual Spaces of $V_1, V_2$ and the Tits cone

Throughout this section we assume that $S$ is finite.

For any real vector space $Y$, we define $Y^* = \text{Hom}(Y, \mathbb{R})$, and call it the algebraic dual space of $Y$. For each $i \in \{1, 2\}$, let $C_i$ be a convex cone in $V_i$ and we define

$$C_i^* = \{ f \in V_i^* \mid f(v) \geq 0 \text{ for all } v \in C_i \}.$$ 

Similarly for each convex cone $F_i$ in $V_i^*$ we define

$$F_i^* = \{ v \in V_i \mid g(v) \geq 0 \text{ for all } g \in F_i \}.$$ 

It is readily checked that such $C_i^*$ and $F_i^*$ are themselves convex cones.

**Lemma 2.4.1.** (i) For each $i \in \{1, 2\}$ the condition $0 \notin \text{PLC}(\Pi_i)$ implies the existence of an $f_i \in V_i^*$ such that $f_i(x) > 0$ for all $x \in \Pi_i$.

(ii) For each $i \in \{1, 2\}$, $(C_i^*)^* = \overline{C_i}$ for each convex cone $C_i$ in $V_i$ (where $\overline{C_i}$ is the topological closure of $C_i$ with respect to the standard topology on $V_i \cong \mathbb{R}^{d_i}$ with $d_i = \dim(\text{span}(V_i))$).

**Proof.** (i) For each $i \in \{1, 2\}$, set 

$$X_i = \{ \sum_{x \in \Pi_i} \lambda_x x \mid \lambda_x \geq 0 \text{ for all } x \text{ and } \sum_{x \in \Pi_i} \lambda_x = 1 \}.$$ 

Let $(\cdot, \cdot)_i$ be a positive definite inner product on $V_i$ and let $\| \cdot \|_i$ be the associated norm. Since $X_i$ is topologically closed it follows that there exists a $v_i \in X_i$ with $\|v_i\| = (v_i, v_i)_i$ being minimal. Now let $x_i$ be an arbitrary element of $X_i$ and consider the function 

$$g_i(t) = \| (1-t)v_i + tx_i \|^2_i \quad (t \in [0, 1]).$$ 

Since $X_i$ is convex, the choice of $v_i$ yields that $g_i$ attains a minimum at $t = 0$. Hence $g'_i(0) \geq 0$. But $g'_i(0) = 2(v_i, x_i - v_i)_i$; hence $(v_i, x_i)_i \geq (v_i, v_i)_i > 0$. It then follows that the linear functional $f_i$ defined by $f_i(x) = (x, v_i)_i$ takes strictly positive values on $\Pi_i$ completing the proof of (i).

(ii) Clearly for each convex cone $C_i \subset V_i, C_i \subseteq (C_i^*)^*$, furthermore, since $(C_i^*)^*$ is topologically closed it follows that $\overline{C_i} \subseteq (C_i^*)^*$. Thus it
only remains to prove that \((C_i^*)^* \subseteq \overline{C_i}\). Suppose for a contradiction that there exists \(v_i \in (C_i^*)^* \setminus \overline{C_i}\). Let \((,)_i\) be a positive definite inner product on \(V_i\) and let \(\|\|_i\) be the associated norm. Let \(u_i \in \overline{C_i}\) be chosen so that \(\|u_i - v_i\|_i\) is minimal. As in the proof of the (i) above, let \(x_i \in \overline{C_i}\) be arbitrary, and consider the function
\[
h_i(t) = \|(1 - t)u_i + tx_i - v_i\|_i^2 \quad (t \in [0, 1]).
\]
Since \(C_i\) is a convex cone it is clear that \(\overline{C_i}\) is also a convex cone. Then it follows that \(h_i\) attains a minimum at \(t = 0\). Thus
\[(2.4.1) \quad 0 \leq h'_i(0) = 2(x_i - u_i, u_i - v_i)_i.
\]
By specializing \(x_i\) to, say, \(2u_i\) and \(\frac{1}{2}u_i\) we see from (2.4.1) that
\[(2.4.2) \quad 0 = (u_i, u_i - v_i)_i.
\]
Consequently (2.4.1) yields that
\[(2.4.3) \quad 0 \leq (x_i, u_i - v_i)_i.
\]
Hence the linear functional \(f_i\) defined by \(f_i(x) = (x, u_i - v_i)_i\) is in \(C_i^*\). Now since \(v_i \in (C_i^*)^*\) it follows that \(0 \leq f_i(v_i) = (v_i, u_i - v_i)_i\) and observe that this and (2.4.2) together yield that \((u_i - v_i, u_i - v_i)_i \leq 0\).
Since \((,)_i\) is positive definite we have \(v_i = u_i\) contradicting that \(v_i \notin \overline{C_i}\).
\[\square\]

There is a natural action of \(W\) on \(V_i^* (i = 1, 2)\) as follows: if \(w \in W\) and \(f_i \in V_i^*\) then \(wf_i \in V_i^*\) is defined by
\[
(wf_i)(v_i) = f_i(w^{-1}v_i), \quad \text{for all } v_i \in V_i.
\]
For each \(i \in \{1, 2\}\), set \(P_i := \text{PLC}(\Pi_i) \cup \{0\}\). Then
\[
P_i^* = \{f \in V_i^* \mid f(x) \geq 0 \text{ for all } x \in \Pi_i\}.
\]
Generalizing the concept of a Tits cone as defined in [18] and in section 5.13 of [12], we define the Tits cone in this non-orthogonal setting to be
\[
U_i := \bigcup_{w \in W} wP_i^*.
\]
Observe that from these definitions we immediately have

\[(2.4.4)\]
\[U_i^* = \{ v \in V_i \mid (wf)v \geq 0 \text{ for all } f \in P_i^* \text{ and } w \in W \} \]
\[= \{ v \in V_i \mid f(w^{-1}v) \geq 0 \text{ for all } f \in P_i^* \text{ and } w \in W \} \]
\[= \bigcap_{w \in W} \{ v \in V_i \mid f(v) \geq 0 \text{ for all } f \in P_i^* \} \]
\[= \bigcap_{w \in W} \{ wv \in V_i \mid v \in (P_i^*)^* \} \]
\[= \bigcap_{w \in W} \{ wP_i \} \text{ (by Lemma 2.4.1(ii), since } P_i \text{ is a convex cone) \}
\[= \bigcap_{w \in W} wP_i \text{ (since } P_i \text{ is closed).} \]

Observe that then Lemma 2.4.1(i) yields that there are \( f_1 \in V_i^* \) and \( f_2 \in V_2^* \) such that \( f_1(x) \geq 1 \) for all \( x \in \Pi_1 \) and \( f_2(y) \geq 1 \) for all \( y \in \Pi_2 \). For the next result we fix one such pair of linear functionals \( \{f_1, f_2\} \).

**Proposition 2.4.2.** Let \( v_1 \in U_1^* \) and \( v_2 \in U_2^* \). Then \( \langle v_1, v_2 \rangle \leq 0 \).

**Proof.** Suppose for a contradiction that there exists \( v_1 \in U_1^* \) and \( v_2 \in U_2^* \) such that \( \langle v_1, v_2 \rangle > 0 \). Replace \( v_2 \) by a positive scalar multiple of itself if needed, we may assume that \( \langle v_1, v_2 \rangle = 1 \). Let
\[ \mathcal{B} = \{ x \in U_2^* \mid f_2(x) \leq f_2(v_2) \text{ and } \langle z, x \rangle \geq 1 \text{ for some } z \in U_1^* \text{ with } f_1(z) \leq f_1(v_1) \}. \]

Observe that \( v_2 \in \mathcal{B} \), so \( \mathcal{B} \neq \emptyset \).

Put \( \epsilon = \frac{2}{n_{f_1(v_1)}} \) where \( n = |S| \). We shall show that for any given \( x \in \mathcal{B} \) there exists \( y \in \mathcal{B} \) such that \( f_2(y) \leq f_2(x) - \epsilon \).

Given \( x \in \mathcal{B} \), let \( z = \sum_{a \in \Pi_1} \lambda_a a \in U_1^* \) (\( \lambda_a \geq 0 \) for all \( a \in \Pi_1 \)) be such that \( \langle z, x \rangle \geq 1 \) and \( f_1(z) \leq f_1(v_1) \). Observe that since
\[ \langle z, x \rangle = \sum_{a \in \Pi_1} \lambda_a \langle a, x \rangle \geq 1 \] it follows that there exists some \( a_0 \in \Pi_1 \) such that \( \lambda_{a_0} \langle a_0, x \rangle \geq \frac{1}{n} \), that is,

\[ \langle a_0, x \rangle \geq \frac{1}{\lambda_{a_0} n} \geq \frac{1}{nf_1(v_1)} = \epsilon \]

because \( \lambda_{a_0} \leq f_1(z) \leq f_1(v_1) \). Now let \( y = r_{\phi(a_0)}x \). It is clear from (2.4.4) that \( U_2^* \) is \( W \)-invariant. Now since \( x \in U_2^* \), it follows that \( y \in U_2^* \) too. Thus

\[
f_2(y) = f_2(x - 2\langle a_0, x \rangle \phi(a_0)) = f_2(x) - 2\langle a_0, x \rangle f_2(\phi(a_0)) \leq f_2(x) - \epsilon f_2(\phi(a_0)) \leq f_2(x) - \epsilon < f_2(v_2).
\]

We claim that \( y \in B \). To prove this we need to find some \( t \in U_1^* \) such that \( \langle t, y \rangle \geq 1 \) and \( f_1(t) \leq f_1(v_1) \).

First consider the case that \( \langle z, \phi(a_0) \rangle \geq 0 \). Put \( t = r_{a_0}z \). Then \( t \in U_1^* \) (since \( z \) is and \( U_1^* \) is \( W \)-invariant) and

\[
\langle t, y \rangle = \langle r_{a_0}z, r_{\phi(a_0)}x \rangle = \langle z, x \rangle \geq 1
\]

furthermore,

\[
f_1(t) = f_1(z) - 2\langle z, \phi(a_0) \rangle f_1(a_0) \leq f_1(z) \leq f_1(v_1)
\]

as required. Hence \( y \in B \) when \( \langle z, \phi(a_0) \rangle \geq 0 \).

Next if \( \langle z, \phi(a_0) \rangle < 0 \) then \( t = z \) will do; indeed

\[
\langle z, y \rangle = \langle z, x - 2\langle a_0, x \rangle \phi(a_0) \rangle = \langle z, x \rangle - 2 \langle a_0, x \rangle \langle z, \phi(a_0) \rangle \geq \langle z, x \rangle \geq 1.
\]
Furthermore, by our construction \( z \in U_1^\ast \) and \( f_1(z) \leq f_1(v_1) \). Hence when \( \langle z, \phi(a_0) \rangle < 0 \), \( y \in \mathcal{B} \) as well and this establishes the claim.

Starting with \( x = v_2 \), a finite number of iterations of the above process will produce a \( y \in \mathcal{B} \) with \( f_2(y) \) being negative, contradicting the fact that \( U_2^\ast \subseteq P_2 \) and \( f_2(P_2) \subseteq (0, \infty) \). \( \square \)

The following is a well-known result:

**Lemma 2.4.3.** ([13] 4.5.3) *Let \( H \) be a finite subgroup of \( W \). Then there exists \( w \in W \) and \( J \subseteq S \) such that \( W_J \) is a finite parabolic subgroup and \( wHw^{-1} \subseteq W_J \). \( \square \)

**Lemma 2.4.4.** The set
\[
\{ \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_2) \rangle \mid \alpha_1, \alpha_2 \in \Phi_1 \text{ and } \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_2) \rangle < 1 \}
\]
is finite.

**Proof.** Let \( \alpha_1, \alpha_2 \in \Phi_1 \) such that \( \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_2) \rangle < 1 \). By Corollary 2.3.17 we know that \( H := \langle \{ r_{\alpha_1}, r_{\alpha_2} \} \rangle \) is a finite dihedral subgroup of \( W \). Thus Lemma 2.4.3 yields that there are \( w \in W \) and \( J \subseteq S \) such that \( W_J \) is finite and \( wHw^{-1} \subseteq W_J \). In particular, there are \( x, y \in \Phi_1(W_J) \) such that \( \alpha_1 = wx \) and \( \alpha_2 = wy \). Now
\[
\langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_2) \rangle = \langle wx, w\phi(y) \rangle \langle wy, w\phi(x) \rangle = \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle
\]
\[
\in \{ \langle a, \phi(b) \rangle \langle b, \phi(a) \rangle \mid a, b \in \Phi_1 \text{ and } r_a, r_b \in \bigcup_{I \subseteq S, W_I \text{finite}} W_I \}.
\]
Since \( S \) is a finite set it follows that the last set on the right is finite proving the desired result. \( \square \)

Immediately from the above we have:

**Corollary 2.4.5.** There is a positive number \( \theta \) such that
\[
\theta < \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_2) \rangle \text{ for all } \alpha_1, \alpha_2 \in \Phi_1 \text{ with } 0 \neq \langle \alpha_1, \phi(\alpha_2) \rangle.
\]
\( \square \)
Chapter 3

The Dominance Hierarchy of Root Systems of Coxeter Groups

3.1. Introduction

If \( x \) and \( y \) are roots in the root system with respect to the standard (Tits) geometric realization of a Coxeter group \( W \), we say that \( x \) dominates \( y \) (written \( x \text{ dom}_W y \)) if \( wy \) is a negative root whenever \( wx \) is a negative root. We call a positive root \( x \) elementary if it does not dominate any positive root other than \( x \) itself. The set of all elementary roots is denoted by \( \mathcal{E} \). It has been proven by B. Brink and R. B. Howlett [Math. Ann. 296 (1993), 179–190] that \( \mathcal{E} \) is finite if (and only) if \( W \) is a finite rank Coxeter group. Amongst other things, this finiteness property enabled Brink and Howlett to establish the automaticity of all finite rank Coxeter groups. Later Brink has also given a complete description of the set \( \mathcal{E} \) for arbitrary finite rank Coxeter groups in [J. Algebra 206 (1998), 371–412]. But until the present, a systematic study of the dominance behaviour among non-elementary positive roots still remains to be completed. In this chapter we answer a collection of questions concerning dominance between such non-elementary positive roots. In particular, we show that for any finite rank Coxeter group and for any non-negative integer \( n \), the set of roots dominating precisely \( n \) other positive roots is finite. We then give both upper and lower bounds to the sizes of all such sets as well as an inductive algorithm to compute all such sets. In this chapter, for any given positive root \( x \), we obtain explicitly the set of positive roots dominated by \( x \), if there are any.
In Section 3.4, we study some cones closely related to the Tits cone and their connection with the dominance concept.

We stress that all notations used in this chapter are exactly the same as those used in previous chapters.

Let $N : W \to \Phi^+$ be the function defined by

$$N(w) = \{ x \in \Phi^+ \mid wx \in \Phi^- \}.$$

It is a well-known fact that

$$N(w) = \{ x \in \Phi^+ \mid l(wr_x) < l(w) \} \quad \text{and} \quad l(w) = |N(w)|$$

(for example, Proposition 5.7 of [12] or Proposition 4.4.6 of [13]). If $\Phi'$ is a subset of $\Phi$ such that whenever $x, y \in \Phi'$, $r_xy \in \Phi'$, then we call $\Phi'$ a root subsystem of $\Phi$. Now if $W'$ is a reflection subgroup of $W$, then we set

$$\Phi(W') = \{ a \in \Phi \mid r_a \in W' \}.$$

Observe that $\Phi(W')$ is a root subsystem of $\Phi$. We call $\Phi(W')$ the root subsystem corresponding to $W'$.

Recall that Lemma 2.2.4 yields $f_1(\Delta_1(W')) = f_2(\Delta_2(W'))$ for all reflection subgroups $W'$ of $W$. This observation enables us to define the canonical roots of $W'$ in $\Phi$.

**Definition 3.1.1.** For any reflection subgroup $W'$ of $W$, let

$$\Delta(W') = f_1(\Delta_1(W')) = f_2(\Delta_2(W')).$$

**Remark 3.1.2.** Recall that $T := \bigcup_{w \in W} w Rw^{-1}$ denotes the set of reflections in $W$. Since for all $\alpha \in \Phi_1$ we have $r_\alpha = r_{f_1(\alpha)}$, it follows that

$$S(W') = \{ r_x \in T \mid x \in \Delta(W') \}.$$

With this justification, we call $\Delta(W')$ the canonical roots for $\Phi(W')$.

**Proposition 3.1.3.** (i) Suppose that $W'$ is a reflection subgroup of $W$. Suppose that $a$ and $b \in \Delta(W')$ with $a \neq b$. Then

$$(a, b) \in \{ -\cos(\pi/n) \mid n \in \mathbb{N}, n \geq 2 \} \cup (-\infty, -1].$$
Let $X \subseteq \Phi^+$ be a set such that whenever $a, b \in X$ $a \neq b$ the condition

$$(a, b) \in \{-\cos(\pi/n) \mid n \in \mathbb{N}, n \geq 2\} \cup (-\infty, -1]$$

is satisfied. If we set $W' = \langle r_x \mid x \in X \rangle$, then $X = \Delta(W')$.

**Proof.** (i) Follows from Proposition 2.2.12 and Proposition 2.3.19.

(ii) Follows from Proposition 2.2.15 and Proposition 2.3.19. □

### 3.2. Rank 2 Root Subsystems

Suppose that $\Phi(\langle r_a, r_b \rangle)$ is an infinite rank 2 root system with canonical roots $a$ and $b$. In this section we classify the root subsystems of $\Phi(\langle r_a, r_b \rangle)$.

Since $\Phi(\langle r_a, r_b \rangle)$ is infinite, it follows from Lemma 2.3.3 (ii) that $\langle r_a, r_b \rangle$ (the dihedral subgroup generated by $r_a$ and $r_b$) must be infinite. Thus it follows from Proposition 4.5.4 of [13] that $(a, b) \leq -1$. Let $\theta = \cosh^{-1}(-(a, b))$, and for each integer $i$, set $c_i \equiv \frac{\sinh(i\theta)}{\sinh \theta}$. Then

$$r_a a = -a, \quad r_b b = -b,$$

$$(r_a r_b)a = \frac{\sinh(3\theta)}{\sinh \theta} a + \frac{\sinh(2\theta)}{\sinh \theta} b = c_3 a + c_2 b$$

and

$$(r_b r_a)b = \frac{\sinh(2\theta)}{\sinh \theta} a + \frac{\sinh(3\theta)}{\sinh \theta} b = c_2 a + c_3 b.$$ 

An induction shows that for all integer $i$,

$$\begin{align*}
(r_a r_b)^i a &= c_{2i+1} a + c_{2i} b; \\
r_b (r_a r_b)^i a &= c_{2i+1} a + c_{2i+2} b; \\
(r_b r_a)^i b &= c_{2i} a + c_{2i+1} b; \\
r_a (r_b r_a)^i b &= c_{2i+2} a + c_{2i+1} b.
\end{align*}$$

These calculations lead to the following well known result:

**Lemma 3.2.1.**

$$\Phi(\langle r_a, r_b \rangle) = \{ c_i a + c_{i\pm 1} b \mid i \in \mathbb{Z} \}.$$
Observe from Lemma 3.2.1 and (3.2.1) above that \( w(c_n a + c_n \pm_1 b) = a \) for some \( w \in \langle r_a, r_b \rangle \) if and only if \( n \) is odd whereas \( w'(c_m b + c_m \pm_1 a) = b \) for some \( w' \in \langle r_a, r_b \rangle \) if and only if \( m \) is odd. Furthermore, if \( i, j \) are integers such that \( w(c_i a + c_i \pm_1 b) = c_j a + c_j \pm_1 b \) for some \( w \in \langle r_a, r_b \rangle \), then \( i \equiv j \pmod{2} \).

**Proposition 3.2.2.** Suppose that \( \Phi' \) is a root subsystem of \( \Phi(\langle r_a, r_b \rangle) \). Then \( \Phi' \) is at most a rank 2 root subsystem.

**Proof.** Suppose for a contradiction that there are at least three canonical generators \( x, y \) and \( z \) for the subsystem \( \Phi' \). By Lemma 3.2.1, there are three integers \( m, n \) and \( p \) such that

\[
\begin{align*}
x &= c_m a + c_m \pm_1 b; \\
y &= c_n a + c_n \pm_1 b; \\
z &= c_p a + c_p \pm_1 b.
\end{align*}
\]

If either \( x = c_m a + c_m +_1 b \) and \( y = c_n a + c_n +_1 b \) or \( x = c_m a + c_m -_1 b \) and \( y = c_n a + c_n -_1 b \), then \( (x, y) = \cosh((m - n)\theta) \geq 1 \), contradicting Proposition 3.1.3 (i). Without loss of generality, we may assume that \( x = c_m a + c_m +_1 b \), and \( y = c_n a + c_n -_1 b \). Now if \( z = c_p a + c_p +_1 b \), then \( (x, z) = \cosh((m - p)\theta) \geq 1 \), contradicting Proposition 3.1.3 (i); on the other hand if \( z = c_p a + c_p -_1 b \), then \( (z, y) = \cosh((n - p)\theta) \geq 1 \), again contradicting Proposition 3.1.3 (i). Therefore \( \Phi' \) has at most two canonical generators, that is, \( \Phi' \) is at most rank 2.

Suppose that \( x = c_m a + c_m +_1 b \) and \( y = c_n a + c_n +_1 b \ (m, n \in \mathbb{Z}) \), are roots in \( \Phi(\langle r_a, r_b \rangle) \). To classify the rank 2 root subsystems of the form \( \Phi(\langle r_x, r_y \rangle) \), it suffices to compute explicitly the canonical roots of \( \Phi(\langle r_x, r_y \rangle) \).

**Lemma 3.2.3.** Suppose that \( x = c_m a + c_m +_1 b \) and \( y = c_n a + c_n -_1 b \) are positive roots in \( \Phi(\langle r_a, r_b \rangle) \) (that is, \( m \) is a non-negative integer
and $n$ is a positive integer). Then $x$ and $y$ are the canonical roots for $\Phi(\langle r_x, r_y \rangle)$. In particular, $\Phi(\langle r_x, r_y \rangle)$ is infinite in size.

**Proof.** By Proposition 3.1.3 (i), to show that $x$ and $y$ are the canonical roots for $\Phi(\langle r_x, r_y \rangle)$, it is enough to show that $(x, y) \leq -1$. Indeed, $(x, y) = -\cosh((n + m)\theta) \leq -1$. Thus Proposition 4.5.4 of [13] yields that the dihedral subgroup $\langle r_x, r_y \rangle$ is infinite, and consequently Lemma 2.3.3 (ii) yields that $\Phi(\langle r_x, r_y \rangle)$ is infinite in size.

□

Next we compute the canonical roots of $\Phi(\langle r_x, r_y \rangle)$, where $x$ is of the form $c_m a + c_{m+1} b$ and $y$ is of the form $c_n a + c_{n+1} b$, $m \neq n$.

**Proposition 3.2.4.** Suppose that $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n+1} b$ ($m, n \in \mathbb{Z}$, $m \neq n$), are roots in $\Phi(\langle r_x, r_y \rangle)$.

(i) $\Phi(\langle r_x, r_y \rangle) = \{ \pm (c_k(m-n) - m a + c_k(m-n) - m - 1) b) \mid k \in \mathbb{Z} \}$. In particular this root subsystem is infinite in size.

(ii) The canonical roots of $\Phi(\langle r_x, r_y \rangle)$ are of the form $c_i a + c_{i+1} b$ and $c_j a + c_{j+1} b$, where

\[
i = \min \{ k(m-n) - m \mid k \in \mathbb{Z} \text{ and } k(m-n) - m > 0 \}
\]

and

\[
j = \min \{ k(m-n) + m \mid k \in \mathbb{Z} \text{ and } k(m-n) + m \geq 0 \}.
\]

**Proof.** (i) Clearly $\Phi(\langle r_x, r_y \rangle)$ consists of all the roots of the form

\[
(r_x r_y)^l x, \quad r_y (r_x r_y)^l x, \quad (r_y r_x)^l y \quad \text{and} \quad r_x (r_y r_x)^l y,
\]

where $l$ ranges over $\mathbb{Z}$. Depending on the choice of $m$ and $n$, we have the following three cases to consider:

(1) Both $m$ and $n$ are even.

(2) Exactly one of $m$ and $n$ is even.

(3) Both $m$ and $n$ are odd.

First suppose that (1) is the case. Then equation (3.2.1) yields that $x = (r_b r_a)^{m/2} b$, and $y = (r_b r_a)^{n/2} b$. Consequently

\[
r_x = r_b (r_a r_b)^m = (r_b r_a)^m r_b \quad \text{and} \quad r_y = r_b (r_a r_b)^n = (r_b r_a)^n r_b.
\]
Thus \( r_x r_y = (r_b r_a)^{m-n} \) and \( r_y r_x = (r_b r_a)^{n-m} \). Hence (3.2.1) yields that for any integer \( l \),

\[
(r_x r_y)^l x = (r_b r_a)^{(m-n)}(r_b r_a)^{m/2} b
= c_{2l(m-n)+m} a + c_{2l(m-n)+m+1} b
\]

and

\[
r_y (r_x r_y)^l x = r_b (r_a r_b)^n (r_b r_a)^{(m-n)}(r_b r_a)^{m/2} b
= r_b (r_b r_a)^{(m-n)-n+m/2} b
= r_a (r_b r_a)^{(m-n)-n+m/2-1} b
= c_{2l(m-n)+m-2} a + c_{2l(m-n)+m-1} b.
\]

By symmetry, we deduce that \( (r_y r_x)^l y = c_{2l(m-n)+n} a + c_{2l(m-n)+n+1} b \) and \( r_x (r_y r_x)^l y = c_{2l(m-n)+n-2} a + c_{2l(m-n)+n-1} b \).

Next suppose that (2) is the case. We may assume, without loss of generality, that \( m \) is even. Then \( x = (r_b r_a)^{m/2} b \) and \( y = r_b (r_a r_b)^{(n-1)/2} a \).

Similar rank 2 calculations as in (1) above yield that for any integer \( l \),

\[
(r_x r_y)^l x = c_{2l(m-n)+m} a + c_{2l(m-n)+m+1} b,
\]

\[
r_y (r_x r_y)^l x = c_{2l(m-n)+m-2} a + c_{2l(m-n)+m-1} b,
\]

\[
(r_y r_x)^l y = c_{2l(m-n)+n} a + c_{2l(m-n)+n+1} b,
\]

\[
r_x (r_y r_x)^l y = c_{2l(m-n)+n-2} a + c_{2l(m-n)+n-1} b.
\]

Now suppose that (3) is the case. Then \( x = r_b (r_a r_b)^{(m-1)/2} a \) and \( y = r_b (r_a r_b)^{(n-1)/2} a \). Again, a rank 2 calculation yields that for every integer \( l \),

\[
(r_x r_y)^l x = c_{2l(m-n)+m} a + c_{2l(m-n)+m+1} b,
\]

\[
r_y (r_x r_y)^l x = c_{2l(m-n)+m-2} a + c_{2l(m-n)+m-1} b,
\]

\[
(r_y r_x)^l y = c_{2l(m-n)+n} a + c_{2l(m-n)+n+1} b,
\]

\[
r_x (r_y r_x)^l y = c_{2l(m-n)+n-2} a + c_{2l(m-n)+n-1} b.
\]
Thus we see that $\Phi(\langle r_x, r_y \rangle)$ consists of all roots of the form

\begin{align}
(3.2.2) & \quad c_{2l(m-n)+m}a + c_{2l(m-n)+m+1}b, \\
(3.2.3) & \quad c_{2l(n-m)+n}a + c_{2l(n-m)+n+1}b
\end{align}

and

\begin{align}
(3.2.4) & \quad c_{2l(m-n)+m-2n}a + c_{2l(m-n)+m-2n-1}b, \\
(3.2.5) & \quad c_{2l(n-m)+n-2m}a + c_{2l(n-m)+n-2m-1}b,
\end{align}

where $l$ ranges over $\mathbb{Z}$.

Now let us consider all roots of the form (3.2.4) and (3.2.5). Observe that $2l(m-n) + m - 2n = 2l + 1(m-n) - m$ is of the form of an even multiple of $(m-n)$ minus $m$, whereas on the other hand $2l(n-m) + n - 2m = -(2l + 1)(m-n) - m$ is of the form of an odd multiple of $(m-n)$ minus $m$. Thus we conclude that the set of roots of the form (3.2.4) and (3.2.5) is exactly $\{ c_k(m-n)-m a + c_k(m-n)-m-1 b \mid k \in \mathbb{Z} \}$. Similarly we could also see that the set of roots of the form (3.2.2) and (3.2.3) is exactly $\{ c_k(m-n)+m a + c_k(m-n)+m+1 b \mid k \in \mathbb{Z} \}$. Finally we observe that for each integer $l$, $c_{-l} = \frac{\sinh(-l\theta)}{\sinh\theta} = -c_l$. Thus

$$c_{k(m-n)-m}a + c_k(m-n)-m-1 b = -(c_{-k(m-n)+m}a + c_{-k(m-n)+m+1}b).$$

Therefore

$$\Phi(\langle r_x, r_y \rangle) = \{ \pm(c_{k(m-n)-m}a + c_k(m-n)-m-1 b) \mid k \in \mathbb{Z} \}.$$

Observe that, in particular, the root subsystem $\Phi(\langle r_x, r_y \rangle)$ is infinite in size. This completes the proof of (i).

(ii) Let $\alpha$ and $\beta$ be the canonical roots for $\Phi(\langle r_x, r_y \rangle)$. Then $\alpha = c_i a + c_{i-1} b$ and $\beta = c_j a + c_{j+1} b$ for some positive integer $i$ and some nonnegative integer $j$. Indeed, by Lemma 3.2.1, the only other possibilities are either $\alpha = c_i a + c_{i+1} b$ and $\beta = c_j a + c_{j+1} b$ or $\alpha = c_i a + c_{i-1} b$ and $\beta = c_i a + c_{j-1} b$. But then $(\alpha, \beta) = \cosh((i-j)\theta) \geq 1$, contradicting
3.2. Rank 2 Root Subsystems

Proposition 3.1.3 (i). Now by (i) above

\[ \Phi(\langle r_x, r_y \rangle) = \{ c_k(m-n)-m a + c_k(m-n)-m b, c_k(m-n)+m a + c_k(m-n)+m b \mid k \in \mathbb{Z} \}. \]

Thus \( i \) must be of the form

\[ k(m - n) - m, \; k \in \mathbb{Z}, \]

and \( j \) must be of the form

\[ k'(m - n) + m, \; k' \in \mathbb{Z}. \]

Since \( \alpha \) and \( \beta \) are the canonical roots for \( \Phi(\langle r_x, r_y \rangle) \), it follows that \( i \) and \( j \) must be as small as possible subject to the requirement that both \( \alpha \) and \( \beta \) are positive roots. Therefore we conclude that \( i \) must be the least positive integer of the form \( k(m - n) - m \), where \( k \) is an integer; and \( j \) must be the least nonnegative integer of the form \( k'(m - n) + m \), where \( k' \) is an integer.

\[ \Box \]

Finally we look at the root subsystem generated by \( x = c_{m+1} a + c_m b \) and \( y = c_{n+1} a + c_n b \).

Proposition 3.2.5. Suppose that \( x = c_{m+1} a + c_m b \) and \( y = c_{n+1} a + c_n b \) are roots in \( \Phi(\langle r_x, r_y \rangle) \).

(i) \( \Phi(\langle r_x, r_y \rangle) = \{ \pm(c_k(m-n)-m-1 a + c_k(m-n)-m b) \mid k \in \mathbb{Z} \}. \) In particular, this root subsystem is infinite in size.

(ii) The canonical roots for \( \Phi(\langle r_x, r_y \rangle) \) are \( c_i a + c_{i-1} b \) and \( c_j a + c_{j+1} b \), where

\[ i = \min\{ k(m - n) + m \mid k \in \mathbb{Z} \text{ and } k(m - n) + m > 0 \} \]

and

\[ j = \min\{ k(m - n) - m \mid k \in \mathbb{Z} \text{ and } k(m - n) - m \geq 0 \}. \]

Proof. Follows from Proposition 3.2.4 with the roles of \( a \) and \( b \) interchanged. \( \Box \)
Combining Lemma 3.2.2, Lemma 3.2.3, Proposition 3.2.4 and Proposition 3.2.5, we immediately deduce that:

**Corollary 3.2.6.** Suppose that $\Phi'$ is a root subsystem of $\Phi(\langle r_a, r_b \rangle)$. Then either $|\Phi'| = 2$ or $\Phi'$ is an infinite rank 2 subsystem. □

3.3. The Dominance Hierarchy

**Definition 3.3.1.** (i) For $x$ and $y \in \Phi$, we say that $x$ dominates $y$, written $x \text{ dom}_W y$ if
\[
\{ w \in W \mid w \cdot x \in \Phi^- \} \subseteq \{ w \in W \mid w \cdot y \in \Phi^- \}.
\]

(ii) For each $x \in \Phi^+$, set
\[
D(x) = \{ y \in \Phi^+ \mid y \neq x \text{ and } x \text{ dom}_W y \},
\]
and for each $n \in \mathbb{N}$, define
\[
D_n = \{ x \in \Phi^+ \mid |D(x)| = n \}
\]
(the set of positive roots $x$ that dominate exactly $n$ other positive roots).

In [6] and [5] dominance is only defined on $\Phi^+$, and it is found in [6] that it is a partial order on $\Phi^+$. Here we have generalized the notion of dominance to the whole of $\Phi$. It can be readily seen that this generalized dominance is a partial order on $\Phi$. It also turns out that the geometric characterization of dominance remains the same as we extend the definition to cover all of $\Phi$. It is clear from the above definition that
\[
\Phi^+ = \bigcup_{n \in \mathbb{N}} D_n.
\]
In this chapter we study the above decomposition. We already knew a good deal about $D_0$ from [6] and [5]: if $W$ is finite then $D_0 = \Phi^+$, whereas if $W$ is an infinite Coxeter group of finite rank, then $|D_0| < \infty$. Observe that in the latter case $\bigcup_{n \in \mathbb{N}, n \geq 1} D_n$ will be an infinite set. One major result of this chapter (Theorem 3.3.9 and Corollary 3.3.10 below)
is that if $S$ is finite then $D_n$ is finite for all natural numbers $n$. We also give an upper bound for $|D_n|$. But first we need a few elementary results.

**Lemma 3.3.2.** (i) If $x$ and $y \in \Phi^+$, then $x \, \text{dom}_W \, y$ if and only if $(x, y) \geq 1$ and $\text{dp}(x) \geq \text{dp}(y)$.
(ii) Dominance is $W$-invariant: if $x \, \text{dom}_W \, y$ then for any $w \in W$, $wx \, \text{dom}_W \, wy$.
(iii) Suppose that $x, y \in \Phi$, and $x \, \text{dom}_W \, y$. Then $-y \, \text{dom}_W \, -x$.
(iv) Suppose that $x \in \Phi^+$ and $y \in \Phi^-$. Then $x \, \text{dom}_W \, y$ if and only if $(x, y) \geq 1$.
(v) Let $x, y \in \Phi$. Then there is dominance between $x$ and $y$ if and only if $(x, y) \geq 1$.

**Proof.** (i) See [6, Lemma 2.3].
(ii) Clear from the definition of dominance.
(iii) Suppose for a contradiction that there exists $w \in W$ such that $w(-y) \in \Phi^-$ and $w(-x) \in \Phi^+$. Then $w(y) \in \Phi^+$ yet $w(x) \in \Phi^-$, contradicting the assumption that $x \, \text{dom}_W \, y$.
(iv) Suppose that $x \, \text{dom}_W \, y$. Since dominance is $W$-invariant, it follows that $r_y x \, \text{dom}_W \, r_y y$. Because $r_y y = -y \in \Phi^+$, so $r_y x \in \Phi^+$. Thus part (i) yields that $(r_y x, r_y y) \geq 1$. Since $(, )$ is $W$-invariant, it follows that $(x, y) \geq 1$.

Conversely, suppose that $x \in \Phi^+$, $y \in \Phi^-$ with $(x, y) \geq 1$. Then clearly $r_y x = x - 2(x, y)y \in \Phi^+$. Thus $r_y x$ and $r_y y = -y$ are both positive. Then it follows from part (i) that there is dominance between $r_y x$ and $r_y y$. Since dominance is $W$-invariant, it follows that there is dominance between $x$ and $y$. Finally, since $x \in \Phi^+$ and $y \in \Phi^-$, it is clear that $x \, \text{dom}_W \, y$.
(v) Suppose that $x, y \in \Phi^-$. Then part (i) yields that there is dominance between $-x$ and $-y$ if and only if $(-x, -y) = (x, y) \geq 1$. This combined with part (i) and part (iv) above yields the desired result. 
\[\square\]
Lemma 3.3.3. Suppose that $x, y \in \Phi$ with $x \neq y$, $x \text{ dom}_W y$ and $y \in D_0$. Then $r_y x \in \Phi^+$.

Proof. Suppose for a contradiction that $r_y x \in \Phi^-$. Lemma 3.3.2 (ii) then yields that $r_y x \text{ dom}_W r_y y = -y$. Now Lemma 3.3.2 (iii) yields that $y \text{ dom}_W -r_y x \in \Phi^+$. Since $y \in D_0$, this forces $-r_y x = y$, contradicting $x \neq y$. □

Proposition 3.3.4. Suppose that $x, y \in \Phi$ are distinct with $x \text{ dom}_W y$.

(i) Let $a, b$ be the canonical roots for the root subsystem $\Phi(\langle r_x, r_y \rangle)$. Then there is a $w \in \langle r_x, r_y \rangle$ such that either $wx = a$ and $wy = -b$ or $wx = b$ and $wy = -a$.

In particular, $(a, b) = -(x, y)$.

(ii) $x \text{ dom}_W y$ if and only if $x \text{ dom}_{\langle r_x, r_y \rangle} y$.

(iii) $(r_y x, x) \leq -1$ and $(r_y x, y) \leq -1$, and in particular, $r_y x$ cannot dominate either $x$ or $y$.

Proof. (i) Since $|(x, y)| \geq 1$ it follows from Proposition 2.3.16 that $\langle a, b \rangle = \langle r_x, r_y \rangle$ is infinite, and hence $(a, b) \leq -1$ by Proposition 3.1.3. Thus $(a, -b) \geq 1$, and $a \text{ dom}_W -b$, and similarly $b \text{ dom}_W -a$. Using the $W$-invariance of dominance it follows readily that there are two dominance chains in the root subsystem $\Phi(\langle r_a, r_b \rangle)$, as follows:

(3.3.1) \[ \cdots \text{ dom}_W r_a r_b r_a(b) \text{ dom}_W r_a r_b(a) \text{ dom}_W a \text{ dom}_W (-b) \text{ dom}_W r_b(-a) \text{ dom}_W r_b r_a(-b) \text{ dom}_W \cdots \]

and

(3.3.2) \[ \cdots \text{ dom}_W r_b r_a r_b(a) \text{ dom}_W r_b r_a(b) \text{ dom}_W r_b(a) \text{ dom}_W b \text{ dom}_W (-a) \text{ dom}_W r_a(-b) \text{ dom}_W r_a r_b(-a) \text{ dom}_W \cdots \].

Observe that each element of $\Phi(\langle r_a, r_b \rangle)$ lies in exactly one of the above chains, and the negative of any element of one of these chains lies in the other. Thus $x', y' \in \Phi(\langle r_a, r_b \rangle)$ are in the same chain if and only if $(x', y') \geq 1$ and in different chains if and only if $(x', y') \leq -1$. 

From (3.3.1) we see that the roots dominated by $a$ are all negative, and from (3.3.2) we see that the roots dominated by $b$ are all negative. Since we may choose $w \in \langle r_a, r_b \rangle$ such that $wx = a$ or $wx = b$, and since $wx \dom_W wy$, it follows that either

\begin{align}
wx = a & \quad \text{and} \quad wy \in \Phi(\langle r_a, r_b \rangle) \cap \Phi^- \\
wx = b & \quad \text{and} \quad wy \in \Phi(\langle r_a, r_b \rangle) \cap \Phi^-.
\end{align}

Suppose that $wx = a$. Then $(a, -wy) = (wx, -wy) = -(x, y) \leq -1$ and $-wy \in \Phi(\langle r_x, r_y \rangle) \cap \Phi^+$. Hence it follows from Proposition 3.1.3 (ii) that \{a, -wy\} is the set of canonical roots for $\Phi(\langle r_x, r_y \rangle)$, forcing $-wy = b$. Similarly, in the case $wx = b$ we may conclude that $wy = -a$.

Finally, observe that in either case, $(a, b) = -(wx, wy) = -(x, y)$.

(ii) First suppose that $x \dom_W y$. Then

\begin{align}
\{ w \in W \mid wx \in \Phi^- \} & \subseteq \{ w \in W \mid wy \in \Phi^- \},
\end{align}

and taking the intersection with $\langle r_x, r_y \rangle$ gives

\begin{align}
\{ w \in \langle r_x, r_y \rangle \mid wx \in \Phi^- \} & \subseteq \{ w \in \langle r_x, r_y \rangle \mid wy \in \Phi^- \},
\end{align}

which shows that $x \dom_{\langle r_x, r_y \rangle} y$.

Conversely, suppose that $x \dom_{\langle r_x, r_y \rangle} y$. By Lemma 3.3.2 (v) applied with $W$ replaced by $\langle r_x, r_y \rangle$ we see that $(x, y)' \geq 1$ where $(x, y)'$ is the restriction of $(x, y)$ to the subspace spanned by $x$ and $y$. Thus $(x, y) \geq 1$, applying Lemma 3.3.2 (v) again yields that either $x \dom_W y$ or $y \dom_W x$. But the latter alternative would imply that $y \dom_{\langle r_x, r_y \rangle} x$, by the first part of this proof, contrary to the fact that $x \dom_{\langle r_x, r_y \rangle} y$ and $x \neq y$.

(iii) Since $x \dom_W y$, Lemma 3.3.2 (v) yields that $(x, y) \geq 1$. Then $(r_y x, y) = (x, -y) \leq -1$ and hence there is no dominance between $r_y x$ and $y$. Also $(r_y x, x) = (x, x) - 2(x, y)^2 \leq -1$ and consequently there is no dominance between $x$ and $r_y x$, proving (iii).

□
Remark 3.3.5. Observe from the proof of Proposition 3.3.4 (ii), that if \( x, y \in \Phi^+ \) are distinct, and \( x \dom_W y \), then the depth of \( x \) relative to \( (W', S(W')) \) is greater than the depth of \( y \) relative to \( (W', S(W')) \).

Lemma 3.3.6. Suppose that \( x, y \in \Phi \) are distinct with \( x \dom_W y \). Let \( a \) and \( b \) be the canonical roots for \( \Phi(\langle r_x, r_y \rangle) \). Then either

\[
    x = c_m a + c_{m+1} b \quad \text{and} \quad y = c_{m-1} a + c_m b
\]

or

\[
    x = c_m a + c_{m-1} b \quad \text{and} \quad y = c_{m-1} a + c_{m-2} b
\]

for some integer \( m \).

Proof. Since \( x, y \in \Phi(\langle r_a, r_b \rangle) \), Lemma 3.2.1 yields that

\[
    x = c_m a + c_{m\pm1} b \quad \text{and} \quad y = c_n a + c_{n\pm1} b
\]

for some integers \( m \) and \( n \). If either \( x = c_m a + c_{m+1} b \) and \( y = c_n a + c_{n-1} b \) or \( x = c_m a + c_{m-1} b \) and \( y = c_n a + c_{n+1} b \), then

\[
    (x, y) = -\cosh((n + m)\theta) \leq -1
\]

contradicting \( x \dom_W y \). Therefore there are only two possibilities

\[
    \begin{align*}
    (3.3.5) & \quad \begin{cases} x = c_m a + c_{m+1} b \\ y = c_n a + c_{n+1} b \end{cases} \\
    (3.3.6) & \quad \begin{cases} x = c_{m+1} a + c_m b \\ y = c_{n+1} a + c_n b. \end{cases}
    \end{align*}
\]

First suppose that (3.3.5) is the case. Since \( a \) and \( b \) are the canonical roots for \( \Phi(\langle r_a, r_b \rangle) = \Phi(\langle r_x, r_y \rangle) \), it follows from Proposition 3.2.4 (ii) that there are integers \( k_1 \) and \( k_2 \) such that

\[
    1 = k_1 (m - n) - m \quad \text{and} \quad 0 = k_2 (m - n) + m.
\]

But then \( k_1 + k_2 = \frac{1+m}{m-n} + \frac{m}{m-n} = \frac{1}{m-n} \in \mathbb{Z} \). Clearly this is only possible when \( m - n = \pm 1 \). On the other hand, since \( x \dom_W y \), it is readily
seen that \( m > n \), giving us

\[
x = c_m a + c_{m+1} b \quad \text{and} \quad y = c_{m-1} a + c_m b.
\]

If (3.3.6) is the case then similar reasoning as above yields that

\[
x = c_m a + c_{m-1} b \quad \text{and} \quad y = c_{m-1} a + c_{m-2} b.
\]

\[\square\]

**Remark 3.3.7.** Let \( x, y, a \) and \( b \) be as in Proposition 3.3.4 (i) and Lemma 3.3.6 above. Then in fact \( x \) and \( y \) are consecutive terms in precisely one of the dominance chains (3.3.1) or (3.3.2).

**Theorem 3.3.8.** \( D_1 \subseteq \{ r_a b \mid a, b \in D_0 \} \). Furthermore, if \(|S| < \infty \) then \(|D_1| \leq |D_0|^2 - |D_0|\).

**Proof.** Suppose that \( x \in D_1 \) and \( \{ y \} = D(x) \). Clearly \( y \in D_0 \). By Lemma 3.3.3, we know that \( r_y x \in \Phi^+ \). Thus to prove Theorem 3.3.8, we only need to show that \( r_y x \in D_0 \).

Suppose for a contradiction that \( r_y x \in \Phi^+ \setminus D_0 \). Then there exists \( z \in \Phi^+ \setminus \{ r_y x \} \) with \( r_y x \ dom_W z \). Since dominance is \( W \)-invariant, it follows that \( x \ dom_W r_y z \). Because \( z \in \Phi^+ \), so clearly \( r_y z \neq y \). Hence \( r_y z \in \Phi^- \), because \( D(x) = \{ y \} \). Thus \( (z, y) > 0 \). Then

\[
1 \leq (r_y x, z) = (x - 2(x, y)y, z)
\]

\[
= (x, z) - 2(x, y)(y, z).
\]

Since we know that \( (x, y) \geq 1 \ (x \ dom_W y) \) and \( (z, y) > 0 \), it follows that \( 1 \leq (x, z) \). Therefore Lemma 3.3.2 (v) yields that either \( x \ dom_W z \) or \( z \ dom_W x \). Suppose that \( z \ dom_W x \). Then \( r_y x \ dom_W z \ dom_W x \), contradicting Proposition 3.3.4 (iii). On the other hand, if \( x \ dom z \), then our construction forces \( z = y \). But then \( r_y x \ dom y \), again contradicting Proposition 3.3.4 (iii).

Thus \( r_y x \in D_0 \), as required. Since \( x \in D_1 \) was arbitrary, it follows that

\[
D_1 \subseteq \{ r_a b \mid a, b \in D_0 \}.
\]
Finally, since $D_1$ does not contain elements of the form $r_au$, $a \in D_0$, it follows that

$$D_1 \subseteq \{ r_ab \mid a, b \in D_0 \} \setminus D_0.$$  

In the case that $|S| < \infty$, we have $|D_0| < \infty$ (by [6, Theorem 2.8]), so it follows from the above that $|D_1| \leq |D_0|^2 - |D_0|$.

The above treatment of $D_1$ can be generalised to $D_n$ for arbitrary $n \in \mathbb{N}$. Indeed we have:

**Theorem 3.3.9.** For $n \in \mathbb{N}$,

$$D_n \subseteq \{ r_ab \mid a \in D_0, b \in \bigcup_{m \leq n-1} D_m \}.$$  

**Proof.** The case $n = 1$ has been covered by Theorem 3.3.8, so we may assume that $n > 1$.

Let $x \in D_n$, and suppose that $D(x) = \{ y_1, y_2, \ldots, y_n \}$, with $y_n$ being minimal with respect to dominance. Observe that $y_n \in D_0$. Then Lemma 3.3.3 yields that $r_{yn}x \in \Phi^+$. Hence either $r_{yn}x \in D_0$ or $r_{yn}x \in \Phi^+ \setminus D_0$.

If $r_{yn}x \in \Phi^+ \setminus D_0$, let $z \in D(r_{yn}x)$. We will show that there are at most $(n-1)$ possible values for $z$. Observe that this establishes

$$r_{yn}x \in \bigoplus_{m \leq n-1} D_m.$$  

Since $r_{yn}x \text{ dom } z$, Lemma 3.3.2 (ii) yields that $x \text{ dom}_{W} r_{yn}z$. Hence either $r_{yn}z = y_i$, for $1 \leq i \leq n - 1$ or $r_{yn}z \in \Phi^-$. If $r_{yn}z \in \Phi^-$, then $(y_n, z) > 0$, yielding

$$1 \leq (r_{yn}x, z) = (x - 2(x, y_n)y_n, z) = (x, z) - 2(x, y_n)(y_n, z).$$  

Now $(y_n, z) > 0$ and $(x, y_n) \geq 1$ (since $x \text{ dom}_{W} y_n$), therefore we must have $(x, z) \geq 1$. Similar to the proof of Theorem 3.3.8, we can conclude that $x \text{ dom}_{W} z$. Hence $z \in \{y_1, \ldots, y_n\}$. By Proposition 3.3.4(iii)
we know that $z \neq y_n$. Thus if $r_{y_n}z \in \Phi^-$, then $z \in \{y_1, \ldots, y_{n-1}\}$.

Summing up, if $z \in D(r_{y_n}x)$, then

$$z \in \{r_{y_n}(y_i) \mid r_{y_n}(y_i) \in \Phi^+, i \in \{1, \ldots, n-1\}\} \cup \{y_i \mid r_{y_n}(y_i) \in \Phi^-, i \in \{1, \ldots, n-1\}\}$$

and this is clearly a disjoint union of size $n - 1$. Thus $r_{y_n}(x) \in D_m$, for some $m \leq n - 1$, and hence

$$D_n \subseteq \{r_ab \mid a \in D_0, b \in \biguplus_{m \leq n-1} D_m\}.$$

It turns out that we can obtain reasonably nice upper bonds for $|D_n|$, indeed we can deduce immediately:

**Corollary 3.3.10.** Suppose that $|S| < \infty$. Then for $n \in \mathbb{N}$, $|D_n| < \infty$.

Indeed

$$|D_n| \leq |D_0|^{n+1} - |D_0|^n.$$

**Proof.** Induction on $n$. The case $n = 1$ has been shown in Theorem 3.3.8 and thus we may assume that $n > 1$. Since $D_i \cap D_j = \emptyset$ if $i \neq j$, it follows from Theorem 3.3.9 that

$$D_n \subseteq \{r_ab \mid a \in \mathfrak{S}, b \in \biguplus_{m \leq n-1} D_m \} \setminus \biguplus_{m < n} D_m.$$

Hence in the case $|S| < \infty$,

(3.3.7)

$$|D_n| \leq |D_0||D_0| + |D_1| + \cdots + |D_{n-1}| - (|D_0| + |D_1| + \cdots + |D_{n-1}|)$$

$$= (|D_0| - 1)(|D_0| + |D_1| + \cdots + |D_{n-1}|).$$

Now the inductive hypothesis yields that

$$|D_0| + |D_1| + |D_2| + \cdots + |D_{n-1}|$$

$$\leq |D_0| + |D_0|^2 - |D_0| + |D_0|^3 - |D_0|^2 + \cdots$$

$$+ |D_0|^{n-1} - |D_0|^{n-2} + |D_0|^n - |D_0|^{n-1}$$

$$= |D_0|^n.$$
Thus (3.3.7) yields that $|D_n| \leq |D_0|^{n+1} - |D_0|^n$, as required.

\[\square\]

Remark 3.3.11. (i) It is seen in the proof of Theorem 3.3.9 that if $x, y \in \Phi^+$ with $x \text{ dom } y$ and $x \in D_n$, then $r_y x \in D_m$, for some $m < n$. It turns out that we can say a bit more on this and we shall do so in Proposition 3.3.16 and Proposition 3.3.24 below.

(ii) Having shown that for all $n \in \mathbb{N}$, $|D_n| < \infty$ if $|S| < \infty$, it is not immediately clear, at this stage, that for all $n \in \mathbb{N}$, $D_n \neq \emptyset$. Lemma 3.3.12 to Corollary 3.3.23 below will, amongst other things, establish that $D_n \neq \emptyset$ for all $n \in \mathbb{N}$ if $W$ is a finite-rank infinite Coxeter group.

Lemma 3.3.12. For $n \in \mathbb{N}$,

$$\{ w a \mid a \in D_0, w \in W \text{ with } l(w) < n \} \cap D_n = \emptyset.$$

\textbf{Proof.} Suppose not. Then there exists some $n \in \mathbb{N}$ and $x = wa \in D_n$, with $a \in D_0$, $w \in W$ and $l(w) < n$. Let $D(x) = \{ y_1, \ldots, y_n \}$. Since dominance is $W$-invariant, it follows that

$$w^{-1} x = a \text{ dom } w^{-1} y_1, \ldots, w^{-1} y_n.$$  

Note that $a \notin \{ w^{-1} y_1, \ldots, w^{-1} y_n \}$, for otherwise $x \in \{ y_1, y_2, \ldots, y_n \}$ which is absurd. Then $w^{-1} y_1, \ldots, w^{-1} y_n \in \Phi^-$ (since $a \in D_0$). Hence $y_1, \ldots, y_n \in N(w^{-1})$, but this gives a contradiction to the fact that $|N(w^{-1})| = l(w^{-1}) = l(w) < n$. \[\square\]

Lemma 3.3.13.

$$RD_0 \subseteq -D_0 \uplus D_0 \uplus D_1.$$  

\textbf{Proof.} Let $r \in R$ and $x \in D_0$ be arbitrary. If $rx \in \Phi^+$, then Lemma 3.3.12 above yields that $rx \in D_0 \uplus D_1$. On the other hand, if $rx \in \Phi^-$, then $x \in \Pi$ and $r = r_x$. Thus $rx = -x \in -\Pi$. Since $r$ and $x$ were chosen arbitrarily, it follows that $RD_0 \subseteq -\Pi \uplus D_0 \uplus D_1$. \[\square\]

Generalising Lemma 3.3.13, we have:
Lemma 3.3.14. For all $n \geq 1$

$$RD_n \subseteq D_{n-1} \uplus D_n \uplus D_{n+1}.$$ 

Proof. Suppose that $n \geq 1$, and let $x \in D_n$, and $z \in \Pi$ be arbitrary. Since $x \neq z$ it follows that $r_zx \in \Phi^+$. 

Suppose for a contradiction that $r_zx \in D_m$ for some $m \geq n + 2$. Let $D(r_zx) = \{y_1, \ldots, y_m\}$. Then $x \dom_W r_zy_1, \ldots, r_zy_m$, for dominance is $W$-invariant. Since $x \in D_n$, and $m \geq n + 2$, it follows that there are $1 \leq i < j \leq m$ such that $r_zy_i \in \Phi^-$ and $r_zy_j \in \Phi^-$. But this is impossible since $r_z$ could only make one positive root negative. Therefore we may conclude that $r_zx \notin D_m$ where $m \geq m + 2$. A similar argument also shows that $r_zx \notin D_m'$ where $m' \leq n - 2$, and we are done. □

Lemma 3.3.15. Suppose that $x, y$ are in $\Phi^+$ with $y \preceq x$. Let $w \in W$ be such that $x = wy$ and $dp(x) = dp(y) + l(w)$. Then $y \in D_m$ implies that $x \in D_n$ for some $n \geq m$. Furthermore, $wD(y) \subseteq D(x)$. 

Proof. It is enough to show that the desired result holds in the case that $w = r_a$ for some $a \in \Pi$. The more general proof then follows from an induction on $l(w)$. 

Since $x = r_ay$ and $y \prec x$ it follows from Lemma 2.3.3 (v) that $(a, y) < 0$. Let $D(y) = \{z_1, z_2, \ldots, z_m\}$. Then Lemma 3.3.2 (v) yields that $a \notin D(y)$. Since $a \in \Pi$, this in turn implies that $r_aD(y) \subseteq \Phi^+$. Since dominance is $W$-invariant, it follows that $x \dom_W r_azi$ for all $i \in \{1, 2, \ldots, m\}$. Hence $\{r_azi, r_a^2z_2, \ldots, r_a^mz_m\} \subseteq D(x)$, and thus $x \in D_n$ for some $n \geq m$. □

The next proposition, somewhat an analogy to Lemma 2.3.3 (v) above, has many applications, among which, we can deduce for arbitrary positive root $x$, the integer $n$ for which $x \in D_n$. Furthermore, it enables us to compute $D(x)$ explicitly as well as to obtain an algorithm to compute all the $D_n$'s systematically.

Proposition 3.3.16. Suppose that $x \in D_n$, $n \geq 1$, and $a \in \Pi$. Then

(i) $r_ax \in D_{n-1}$ if and only if $(x, a) \geq 1$;
(ii) \( r_a x \in D_{n+1} \) if and only if \( (x, a) \leq -1 \);
(iii) \( r_a x \in D_n \) if and only if \( (x, a) \in (-1, 1) \).

**Proof.** (i) Suppose that \( x \in D_n, a \in \Pi \) such that \( r_a x \in D_{n-1} \). Let \( D(x) = \{ z_1, z_2, \ldots, z_n \} \). Then \( r_a x \text{ dom}_W r_a z_i \) for all \( i \in \{1, 2, \ldots, n\} \) since dominance is \( W \)-invariant. Thus at least one of \( r_a z_1, \ldots, r_a z_n \) must be negative. Without loss of generality, we may assume that \( r_a z_1 \in \Phi^- \). Since \( a \in \Pi \), it follows that \( a = z_1 \). Then \( (z, a) \geq 1 \) by Lemma 3.3.2 (v) since \( x \text{ dom}_W a \).

Conversely, suppose that \( x \in D_n \) and \( a \in \Pi \) such that \( (x, a) \geq 1 \). Then by Lemma 3.3.2 (v) \( x \text{ dom}_W a \); furthermore, Lemma 2.3.3 (v) yields that \( r_a x \prec x \). Hence Lemma 3.3.15 yields that

\[
(3.3.8) \quad r_a D(r_a x) \subseteq D(x).
\]

Now suppose for a contradiction that \( r_a x \notin D_{n-1} \). Then Lemma 3.3.14 yields that \( r_a x \in D_n \cup D_{n+1} \). From (3.3.8) it is clear that \( r_a x \notin D_{n+1} \). But if \( r_a x \in D_n \), then (3.3.8) yields that \( r_a D(r_a x) = D(x) \). Observe that \( a \in D(x) \) and \( a \notin r_a D(r_a x) \), producing a contradiction as desired. This completes the proof of (i).

(ii) Replace \( x \) by \( r_a x \) in (i) above then we may obtain the desired result.

(iii) Follows from (i), (ii) and Lemma 3.3.14.

\[\square\]

**Definition 3.3.17.** For each \( x \in \Phi^+ \), define

\[
S(x) = \{ w \in W \mid l(w) = dp(x) - 1 \text{ and } w^{-1} x \in \Pi \},
\]

\[
T(x) = \{ w \in W \mid l(w) = dp(x) \text{ and } w^{-1} x \in \Phi^- \}.
\]

In other words, for \( x \in \Phi^+ \), \( S(x) \) (respectively, \( T(x) \)) consists of all \( w \in W \) of minimal length such that \( w^{-1} x \in \Pi \) (respectively, \( w^{-1} x \in \Phi^- \)).

**Proposition 3.3.18.** Suppose that \( x \in \Phi^+ \). Then \( x \in D_n \) where \( n = | \{ b \in N(w^{-1}) \mid x \text{ dom}_W b \} | \) for all \( w \in S(x) \). In particular,

\[
D(x) = \{ b \in N(w^{-1}) \mid (x, b) \geq 1 \}
\]
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for all \( w \in S(x) \).

**Proof.** Let \( x \in \Phi^+ \) and write \( x = wa \) where \( w \in S(x) \) and \( a \in \Pi \). Let \( w = r_{a_1} \cdots r_{a_l} \) with \( l = l(w) \). For all \( i \in \{2, \ldots, l\} \),

\[
\begin{align*}
  w^{-1}(r_{a_1}r_{a_2} \cdots r_{a_{i-2}})a_i-1 &= r_{a_i} \cdots r_{a_1}r_{a_1} \cdots r_{a_{i-2}}a_i-1 \\
  &= r_{a_i} \cdots r_{a_1}r_{a_i-1}a_i-1 \\
  &= -r_{a_i} \cdots r_{a_1}a_i-1.
\end{align*}
\]

Since \( l(r_{i-1} \cdots r_i) = l(r_1 \cdots r_{i-1})-1 \), it follows from Lemma 2.3.3 (iv) that \( r_{a_i} \cdots r_{a_i}a_i-1 \in \Phi^+ \). Thus

\[
(3.3.9) \quad w^{-1}(r_{a_1}r_{a_2} \cdots r_{a_{i-2}})a_i-1 = -r_{a_i} \cdots r_{a_i}a_i-1 \in \Phi^-.
\]

Now by Proposition 3.3.16, we can immediately deduce that \( x \in D_n \) where

\[
\begin{align*}
  n &= | \{ i : (a_{i-1}, r_{a_{i+1}} \cdots r_{a_i}a) \leq -1 \} | \\
  &= | \{ i : (r_{a_1} \cdots r_{a_{i-1}}(a_{i-1}), r_{a_1} \cdots r_{a_i}) \leq -1 \} | \\
  &= | \{ i : (r_{a_1} \cdots r_{a_{i-1}}(a_{i-1}), x) \leq -1 \} | \\
  &= | \{ i : (-r_{a_1} \cdots r_{a_{i-2}}(a_{i-1}), x) \leq -1 \} | \\
  &= | \{ b \in N(w^{-1}) : (-b, x) \leq -1 \} | \\
  &= | \{ b \in N(w^{-1}) : (-b, x) \geq 1 \} |.
\end{align*}
\]

Lemma 3.3.2 (v) yields that either \( x \dom_{W} b \) or \( b \dom_{W} x \). Since all such \( b \) are in \( N(w^{-1}) \) where \( w \in S(x) \), it follows that \( w^{-1}x \in \Pi \) and \( w^{-1}b \in \Phi^- \). Thus \( b \) cannot dominate \( x \). So we may conclude that

\[
(3.3.10) \quad n = | \{ b \in N(w^{-1}) \mid x \dom_{W} b \} |
\]

for all \( w \in S(x) \). But (3.3.10) says precisely that \( D(x) \subseteq N(w^{-1}) \) and

\[
\begin{align*}
  D(x) &= \{ b \in N(w^{-1}) \mid x \dom_{W} b \} \\
  &= \{ b \in N(w^{-1}) \mid (x, b) \geq 1 \}.
\end{align*}
\]

\( \square \)
Immediately from Proposition 3.3.18, we have:

**Corollary 3.3.19.** Let \( x \in \Phi^+ \). Then \( D(x) \subseteq \bigcap_{w \in S(x)} N(w^{-1}) \). And \( D(x) = \{ b \in N(w^{-1}) \mid (x,b) \geq 1 \} \) is independent of the particular choice of \( w \in S(x) \).

It turns out that we can also say something about the roots in \( \bigcap_{w \in S(x)} N(w^{-1}) \setminus D(x) \). Indeed in the next two lemmas we deduce that if \( b \in \bigcap_{w \in S(x)} N(w^{-1}) \), then \( (x,b) > 0 \).

**Lemma 3.3.20.** Suppose that \( x \in \Phi^+ \), \( w \in T(x) \) and \( b \in N(w^{-1}) \). Then \( (b,x) > 0 \).

**Proof.** If \( dp(x) = 1 \), then \( x \in \Pi \). Hence \( T(x) = \{ r_x \} \) and \( x = b \), and so \( (b,x) = 1 \) as required.

Thus we may assume that \( dp(x) > 1 \) and proceed by an induction on \( dp(x) \). Let \( a \in \Pi \cap N(w^{-1}) \). Then

\[
l(r_aw) = l(w^{-1}r_a) = l(w^{-1}) - 1 = l(w) - 1.
\]

Now since \((r_aw)^{-1}(r_ax) = w^{-1}x \in \Phi^-\), it follows that

\[
dp(r_ax) \leq l(r_aw) < l(w) = dp(x).
\]

Thus Lemma 2.3.3 (v) yields that \( (a,x) > 0 \). If \( b = a \) then we are done. So we may assume that \( b \neq a \) and let \( w' = r_aw \). Observe that then \( w' \in T(r_ax) \). Since \( b \in N(w^{-1}) \), it follows that \( w'^{-1}r_ab = w^{-1}r_ar_a = w^{-1}b \in \Phi^- \). Thus \( r_ab \in N(w'^{-1}) \). And so the inductive hypothesis yields that \( (r_ab,r_ax) > 0 \). Since \((, )\) is \( W \)-invariant, it follows that \( (b,x) > 0 \) as required. \( \square \)

**Lemma 3.3.21.** Suppose that \( x \in \Phi^+ \), \( w \in S(x) \) and \( b \in N(w^{-1}) \). Then \( (b,x) > 0 \).

**Proof.** Follows from Lemma 3.3.20 and the fact that for each \( w \in S(x) \) there is a \( w' \in T(x) \) such that \( N(w^{-1}) \subset N(w'^{-1}) \). \( \square \)
Lemma 3.3.22. For \( n \in \mathbb{N} \), if \( D_n = \emptyset \), then \( D_m = \emptyset \) for all \( m \in \mathbb{N} \) such that \( m > n \).

Proof. Suppose for a contradiction that there exists \( n \in \mathbb{N} \) such that \( D_n = \emptyset \) and yet \( D_{n+1} \neq \emptyset \). Let \( x \in D_{n+1} \). Then Lemma 3.3.14 yields that \( r_ax \notin D_n \) for all \( a \in \Pi \). Hence if \( a \in \Pi \) such that \( r_ax < x \) then \( r_ax \in D_{n+1} \) still. Write \( x = wb \), where \( b \in \Pi \), and \( w \in S(x) \). Let \( w = r_{a_1} \cdots r_{a_l} \) (\( a_i \in \Pi \), for \( i = 1, \ldots, l \)) be a reduced expression for \( w \). Then for all \( i \in \{1, \ldots, l \} \), we have that \( r_{a_i} \cdots r_{a_2}r_{a_1}x \in D_{n+1} \). In particular, \( b = r_{a_1} \cdots r_{a_1}x \in D_{n+1} \), contradicting the fact that \( b \in \Pi \subset D_0 \).

Corollary 3.3.23. Suppose that \( W \) is a finite rank infinite Coxeter group. Then for all nonnegative integers \( n \), \( D_n \neq \emptyset \).

Proof. It is clear from the definition of the \( D_n \)'s that \( \Phi^+ = \biguplus_{n \geq 0} D_n \).
Since \( W \) is an infinite Coxeter group, it follows that \( |\Phi^+| = \infty \). On the other hand, since \( W \) is of finite rank, Theorem 3.3.9 yields that \( |D_n| < \infty \). Thus Lemma 3.3.22 implies that \( D_n \neq \emptyset \) for all nonnegative integers \( n \).

The following is a generalization of Proposition 3.3.16:

Proposition 3.3.24. Let \( x \in D_n \) with \( n > 0 \), and let \( a \in \Phi^+ \). Then

(i) \( |D(r_ax)| < n \) if \( (x, a) \geq 1 \);

(ii) \( |D(r_ax)| > n \) if \( (x, a) \leq -1 \).

Proof. (i) If \( \text{dp}(a) = 1 \) then this is just Propostion 3.3.16. Suppose now that \( \text{dp}(a) > 1 \), and proceed by induction.

Write \( a = r_b c \), where \( b \in \Pi \), \( c \in \Phi^+ \) such that

\[
\text{dp}(a) = \text{dp}(c) + 1
\]

Now since \( (x, a) = (x, r_b c) = (r_b x, c) \geq 1 \), it follows from the inductive hypothesis that

\[
|D(r_c(r_b x))| < |D(r_b x)|.
\]
Now we have three possibilities:

1) \((b, x) \geq 1\);
2) \((b, x) \leq -1\);
3) \((b, x) \in (-1, 1)\).

If 1) is the case, then Proposition 3.3.16 yields that \(r_b x \in D_{n-1}\).
And thus we have in this case

\[
|D(r_a x)| = |D(r_b (r_c r_b x))| \\
\leq |D(r_c (r_b x))| + 1 \quad \text{(follows from Proposition 3.3.16)} \\
\leq |D(r_b x)| \quad \text{(follows from (3.3.12))} \\
= n - 1
\]
as required.

If 2) is the case, then Proposition 3.3.16 yields that \(r_b x \in D_{n+1}\),
and \((b, r_c(r_b x)) = (b, r_b x - 2(r_b x, c)c) = (b, r_b x) - 2(x, a)(b, c)\). By
Lemma 2.3.3 (v), equation (3.3.11) above yields that \((b, c) < 0\). Hence

\[
(3.3.13) \quad (b, r_c(r_b x)) > (b, r_b x) \geq 1.
\]

Then Proposition 3.3.16 yields that

\[
|D(r_a x)| = |D(r_b (r_c r_b x))| \\
= |D(r_c r_b x)| - 1 \quad \text{(by (3.3.13) above)} \\
\leq |D(r_b x)| - 2 \quad \text{(by (3.3.12))} \\
\leq n - 1 \quad \text{(since } r_b x \in D_{n+1} \text{ in case 2)}
\]
as required.
If 3) is the case, then we are done unless $|D(r_c(r_bx))| = n - 1$, and $(b, r_c r_b x) \leq -1$. But this is impossible, since
\[
(b, r_c r_b x) \\
= (b, r_b x) - 2(r_b x, c)(b, c) \\
= (b, r_b x) - 2(a, x)(b, c) \\
> (b, r_b x) \quad (\text{since } (a, x) \geq 1, \text{ and } (b, c) < 0) \\
> -1.
\]
Thus $|D(r_a x)| = |D(r_b r_c r_b x)| < n$ in this case too. This completes the proof of (i).

(ii) Replace $x$ by $r_a x$, then apply (i) above.

Next we give an algorithm to systematically compute all the $D_n$'s for an arbitrary finite-rank Coxeter group $W$.

**Lemma 3.3.25.** Suppose that $x \in D_n$ with $n \geq 1$. Then there exists $y \in D_{n-1}$ such that $y < x$.

**Proof.** Suppose that the contrary is true. Let $x \in D_n$ be such that there is no root in $D_{n-1}$ preceding $x$. Write $x = w a$, where $a \in \Pi$, and $w \in S(x)$. Let $w = r_{a_1} \cdots r_{a_l}$ $(a_i \in \Pi, i = 1, \ldots, l)$ be a reduced expression for $w$. Then $a = r_{a_1} \cdots r_{a_l} x$. Observe that in such case, for all $i \in \{1, \ldots, l - 1\}$, $r_{a_i} \cdots r_{a_1} x < r_{a_{i-1}} \cdots r_{a_1} x$. Thus Lemma 2.3.3 (v) yields that $(r_{a_{i-1}} \cdots r_{a_1} x, a_i) > 0$ for $i \in \{2, \cdots, l\}$. So by Proposition 3.3.16, our assumption that $x$ is not preceded by any root in $D_{n-1}$ is equivalent to the condition that $(r_{a_{i-1}} \cdots r_{a_1} x, a_i) \in (0, 1)$ for all $i \in \{2, \cdots, l\}$. Proposition 3.3.16 (iii) then yields that
\[
r_{a_i} r_{a_{i-1}} \cdots r_{a_1} x \in D_n
\]
for all $i \in \{1, \cdots, l\}$. In particular, $a = r_{a_l} \cdots r_{a_1} x \in D_n$ where $n \geq 1$, contradicting that $a \in \Pi \subset D_0$. □
Proposition 3.3.26. Suppose that $W$ is a finite-rank Coxeter group. For $n \geq 1$, there is an algorithm to compute $D_n$ provided that $D_{n-1}$ is known.

Proof. We outline such an algorithm:

1) Set $D = \emptyset$.
2) Enumerate all the elements of $D_{n-1}$ in some order, that is, write $D_{n-1} = \{x_1, \cdots, x_m\}$, where $m = |D_{n-1}|$.
3) Starting with $x_1$, apply all simple reflections $r_a$ ($a \in \Pi$), to $x_1$, one at a time. If $(a, x_1) \leq -1$, then add $r_a x_1$ to $D$.
4) Repeat 3) to $x_2, \cdots, x_m$.
5) Enumerate all the elements of the modified set $D$ in some order, that is, write $D = \{x'_1, x'_2, \cdots, x'_{|D|}\}$.
6) Starting with $x'_1$, apply all simple reflections $r_a$ ($a \in \Pi$) to $x'_1$, one at a time. If $(a, x'_1) \in (-1, 0)$ and $r_a x'_1 \notin D$, then add $r_a x'_1$ to $D$.
7) Repeat 6) to $x'_2, \cdots, x'_{|D|}$.
8) Repeat steps 5) to 7) above.
9) Repeat 8) until no new elements can be added to $D$.
10) Set $D_n = D$.

Next we show that the above algorithm will be able to produce all elements of $D_n$ within a finite number of iterations.

Let $x \in D_n$ ($n \geq 1$) be arbitrary. Lemma 3.3.25 yields that there exists a $y \in D_{n-1}$ with $y \prec x$. Write $x = wy$ for some $w \in W$ with $l(w) = \text{dp}(x) - \text{dp}(y)$. Let $w = r_{a_1} r_{a_2} \cdots r_{a_l}$ ($a_1, \ldots, a_l \in \Pi$) be a reduced expression for $w$. Then

$$y \prec r_{a_1} y \prec r_{a_{l-1}} r_{a_l} y \prec \cdots \prec r_{a_1} r_{a_2} \cdots r_{a_l} y = x.$$}

Since $x \in D_n$ and $y \in D_{n-1}$, it follows from Lemma 3.3.15 that

$$r_{a_1} y, r_{a_{l-1}} r_{a_l} y, \cdots, r_{a_2} r_{a_3} \cdots r_{a_l} y \in D_{n-1} \cup D_n$$
Furthermore, there exists \( i \in \{1, 2, \ldots, l\} \) such that

\[
y \in D_{n-1}
\]

\[
ra_i y \in D_{n-1}
\]

\[
\vdots
\]

\[
ra_{i+1} ra_{i+2} \cdots ra_i y \in D_{n-1}
\]

and

\[
ra_i (ra_{i+1} ra_{i+2} \cdots ra_i y) \in D_n
\]

\[
ra_{i-1} ra_i (ra_{i+1} ra_{i+2} \cdots ra_i y) \in D_n
\]

\[
\vdots
\]

\[
ra_1 ra_2 \cdots ra_i y = x \in D_n.
\]

Since \( ra_{i+1} ra_{i+2} \cdots ra_i y \in D_{n-1} \), it follows that \( ra_i ra_{i+1} ra_{i+2} \cdots ra_i y \) is an element of \( D_n \) obtainable by going through steps 3) and 4) above. This in turn implies that \( ra_{i-1} ra_i \cdots ra_i y \) is an element obtainable by going through steps 5) to 7). It then follows that \( ra_{i-2} ra_{i-1} ra_i \cdots ra_i y \) and so on are all obtainable by (repeated) application of step 8). In particular \( x = ra_1 \cdots ra_i y \) can be obtained after \( (i-2) \) iterations of step 8). Thus \( x \) can be obtained by going through steps 1) to 8), with step 8) repeated finitely many times. Since \( x \in D_n \) was arbitrary, it follows that every element of \( D_n \) can be obtained from the above algorithm in this manner with step 8) repeated finitely many times.

Finally \( W \) is of finite rank, so \( |D_n| < \infty \). Therefore step 9) will only be repeated a finite number of times and hence the algorithm will terminate completing the proof. \( \square \)

**Corollary 3.3.27.** If \( |S| < \infty \), then we may compute \( D_n \), for all \( n \in \mathbb{N} \).

**Proof.** [5] gives a complete description of \( D_0 \) when \( |S| < \infty \). Now combine [5] and Proposition 3.3.26, the result follows immediately. \( \square \)
Now a question worth asking is whether, given an arbitrary (finite rank) Coxeter group \( W \), the successive \( D_n \)'s expand or contract in size? At this stage, unfortunately, a full answer has not yet been found. However we do know that for a family of special subsets \( D_{n,1} \)'s of the \( D_n \)'s to be defined below, the successive \( D_{n,1} \)'s tend to increase in size.

For \( I \subseteq \Pi \), the Coxeter group generated by \( \{ r_a \mid a \in I \} \) is denoted by \( W_I \), and the root system of \( W_I \) on the subspace of \( V \) spanned by \( I \) is denoted by \( \Phi_I \). As usual, write \( \Phi_I^+ \) for \( \Phi_I \cap \text{PLC}(I) \) and write \( \Phi_I^- \) for \( -\Phi_I^+ \). The Coxeter graph \( \Gamma(I) \) of \( W_I \) has vertex set \( I \), and two vertices \( a_r \) and \( a_s \) are adjoined by an edge of weight \( m_{rs} \), where \( r, s \in S \) and \( m_{rs} \neq 2 \).

Each root \( x \) can be written uniquely as \( \sum_{a \in \Pi} \lambda_a a \), and we say \( \lambda_a \) is the coefficient (witten coeff \( x \)) of \( a \) in \( x \). The support (written supp \( x \)) of \( x \) is the set of all \( a \in \Pi \) with coeff \( a \) of \( x \) \( \neq 0 \), and \( \Gamma(x) \) defined by \( \Gamma(\text{supp}(x)) \) is the corresponding graph. It is readily seen that \( \Gamma(\text{supp}(x)) \) is finite and connected.

**Definition 3.3.28.** For \( n \in \mathbb{N} \), set

\[
D_{n,1} = D_n \cap \{ \beta \in \Phi_I^+ \mid \text{coeff}_x(\beta) = 1 , \text{ for some } x \in \Pi \}.
\]

In Proposition 3.3.30 below we shall prove that, amongst other things, \( |D_{n,1}| \geq |D_{m,1}| \), if \( n > m \).

**Lemma 3.3.29.** (4.4 Lemma of [5]) Let \( \beta \) be a positive root and \( x \in \Pi \) with coeff \( x(\beta) \) = 1. Then \( x \preceq \beta \); that is, there exists a \( w \in W_{\text{supp}(x)\setminus\{x\}} \) such that \( \beta = wx \) and \( dp(\beta) = l(w) + 1 \).

**Proposition 3.3.30.** Let \( I \subseteq \Pi \), \( x \in I \) and \( I_1, I_2 \subseteq I \) be such that \( \Gamma(I_1 \setminus \{x\}) \) and \( \Gamma(I_2 \setminus \{x\}) \) are unions of connected components of \( \Gamma(I \setminus \{x\}) \) with \( I = I_1 \cup I_2 \), and \( I_1 \cap I_2 = \{x\} \). Then
3.3. The Dominance Hierarchy

(i) \[ \phi : \{ (\beta_1, \beta_2) \in \Phi^+_I \times \Phi^+_I \mid \text{coeff}_x(\beta_1) = \text{coeff}_x(\beta_2) = 1 \} \rightarrow \{ \alpha \in \Phi^+_I \mid \text{coeff}_x(\alpha) = 1 \} \]

\[ (\beta_1, \beta_2) \mapsto \beta_1 + \beta_2 - x \]

defines a bijection. Moreover, \( dp(\phi(\beta_1, \beta_2)) = dp(\beta_1) + dp(\beta_2) - 1 \) and \( \beta_1, \beta_2 \preceq \phi(\beta_1, \beta_2) \) for \( \beta_1 \in \Phi^+_I \), with \( \text{coeff}_x(\beta_i) = 1 \) \((i = 1, 2)\).

(ii) \( \phi \) restricts to a bijection

\[ \{ (\beta_1, \beta_2) \in (\Phi^+_I \cap D_i) \times (\Phi^+_I \cap D_j) \mid \text{coeff}_x(\beta_1) = \text{coeff}_x(\beta_2) = 1 \} \leftrightarrow \{ \beta \in \Phi^+_I \cap D_{i+j} \mid \text{coeff}_x(\beta) = 1 \}. \]

Proof. (i) Lemma 4.2 of [5].

(ii) Let \( \beta \in \Phi^+_I \) with \( \text{coeff}(\beta) = 1 \). Then Lemma 3.3.29 above yields that \( \beta = wx \), for some \( w \in W_{I \setminus \{x\}} \). We may write \( w = w_1w_2 \) where \( w_1 \in W_{I_1 \setminus \{x\}} \) and \( w_2 \in W_{I_2 \setminus \{x\}} \) with \( l(w) = l(w_1) + l(w_2) \).

Observe that under this construction, \( W_{I_1 \setminus \{x\}} \) commutes with \( W_{I_2 \setminus \{x\}} \) and \( W_{I_1 \setminus \{x\}} \) fixes \( \Phi^+_I \setminus \{x\} \) pointwise and vice versa. Set \( \beta_1 = w_1x \) and \( \beta_2 = w_2x \).

We first show that \( \beta_1 \in D_i, \beta_2 \in D_j \) implies that \( \beta \in D_{i+j} \).

Suppose for a contradiction that \( \beta \in D_n \) for some \( n \in \mathbb{N} \) and \( n \neq i + j \). Let \( \{ z_1, \ldots, z_i \} = D(\beta_1) \) and \( \{ z_{i+1}, \ldots, z_{i+j} \} = D(\beta_2) \).

Since \( \beta_1 = w_1x \text{ dom}_W z_1, \ldots, z_i \), it follows from the \( W \)-invariance of dominance that \( x \text{ dom}_W w_1^{-1}z_1, \ldots, w_1^{-1}z_i \).

Since \( x \in \Pi \) it follows that \( w_1^{-1}z_1, \ldots, w_1^{-1}z_i \in \Phi^- \). Consequently \( z_1, \ldots, z_i \in N(w_1^{-1}) \subseteq \Phi^+_{I_1 \setminus \{x\}} \). Similar reasoning also yields that \( z_{i+1}, \ldots, z_{i+j} \in N(w_2^{-1}) \subseteq \Phi^+_{I_2 \setminus \{x\}} \). Observe that in particular, we have
$D(\beta_1) \cap D(\beta_2) = \emptyset$. By Lemma 3.3.15,

$$w_2 D(\beta_1) = w_2 D(w_1 x) \subseteq D(w_2 w_1 x) \quad \text{(since } w_1 x < w_2 w_1 x)$$

$$= D(wx)$$

$$= D(\beta).$$

Similarly $w_1 D(\beta_2) \subseteq D(\beta)$. Since $w_2$ fixes $z_1, \ldots, z_i$ pointwise and $w_1$ fixes $z_{i+1}, \ldots, z_{j+i}$ pointwise, it follows that

(3.3.14) \quad \{ z_1, \ldots, z_i, z_{i+1}, \ldots, z_{i+j} \} \subseteq D(\beta).

Observe that (3.3.14) implies that, in particular, $\beta \in D_n$ for some $n > i + j$. Thus there exists $z \in \Phi^+$ such that $z \in D(\beta)$ and

(3.3.15) \quad z \neq z_1, \ldots, z_i, z_{i+1}, \ldots, z_{i+j}.$$

$\beta = wx$ dom$_W z$ implies that $x$ dom$_W w^{-1} z$. Because $x \in \Pi$, this forces that $w^{-1} z \in \Phi^-$. Thus $z \in N(w^{-1}) \subseteq \Phi^+_I \{x\}$. Now the connectedness of $\Gamma(z)$ implies that either $z \in \Phi^+_I \{x\}$ or $z \in \Phi^+_I \{z\}$. Suppose that $z \in \Phi^+_I \{x\}$. Then $wx$ dom$_W z$ implies that

$$w_1 x = w_2^{-1}(wx) \text{ dom}_W w_2^{-1} z = z.$$ 

Then $z \in \{ z_1, \ldots, z_i \}$, contradicting (3.3.15). Next suppose that $z \in \Phi^+_I \{z\}$. Then $\beta = wx$ dom$_W z$ implies that

$$w_2 x = w_1^{-1}(wx) \text{ dom}_W w_1^{-1} z = z.$$ 

Then $z \in \{ z_{i+1}, \ldots, z_{i+j} \}$, again contradicting (3.3.15). Therefore if $\beta_1 \in D_i$ and $\beta_2 \in D_j$ then $\beta \in D_{i+j}$, as required.

The converse can be shown by a similar argument. \hfill \Box

### 3.4. The Imaginary Cone and Standard Dominance

The section is devoted to a preliminary study of the so called imaginary cone (introduced by Dyer in [3] and [4]) of a Coxeter group and a stronger form of dominance. In particular, we will show that if $x$ and
y are roots with $x \operatorname{dom} W y$, then $x - y$ is in this imaginary cone. We stress that throughout this section $S$ is assumed to be finite.

**Definition 3.4.1.** Let $V^*$ be the dual space of $V$ (recall that $W$ acts on $V^*$ via the following: for all $w \in W$, $x \in V$, and $f \in V^*$, $(wf)(x) := f(w^{-1}x)$). For any convex cone $C$ in $V$, define the dual of $C$ to be

$$C^* = \{ \phi \in V^* | \phi(v) \geq 0, \text{ for all } v \in C \},$$

and similarly for a convex cone $F$ in $V^*$, we define its dual to be

$$F^* = \{ v \in V | f(v) \geq 0, \text{ for all } f \in F \}.$$

Let $P = \operatorname{PLC}(\Pi) \cup \{0\}$, and we set $U = \bigcup_{w \in W} wP^*$.

**Remark 3.4.2.** Given the finite dimensinality of $V$ (and hence $V^*$), it is well known that if $C$ (respectively, $F$) is a convex cone in $V$ (respectively, $V^*$), then then $(C^*)^*$ (respectively, $(F^*)^*$) is the topological closure of $C$ (respectively, $F$) in $V$ with respect to the standard topology on $V \cong \mathbb{R}^{|S|}$ (respectively, $V^*$ with respect to the standard topology). Furthermore $C^*$ (respectively, $F^*$) is always a convex cone in $V$ (respectively, $V^*$), even if $C$ (respectively, $F$) is neither convex nor a cone in $V$ (respectively, $V^*$).

**Lemma 3.4.3.** Let $w \in W$ be arbitrary and suppose that $f \in wP^*$. Then $f(a) \geq 0$ for all but finitely many positive roots $a$.

**Proof.**

$$wP^* = \{ w\phi \in V^* | \phi(a) \geq 0 \text{ for all } a \in \Phi^+ \}$$

$$= \{ f \in V^* | (w^{-1}f)(a) \geq 0 \text{ for all } a \in \Phi^+ \}$$

$$= \{ f \in V^* | f(wa) \geq 0 \text{ for all } a \in \Phi^+ \}$$

$$= \{ f \in V^* | f(b) \geq 0 \text{ for all } b \in w\Phi^+ \}$$

$$= \{ f \in V^* | \text{ if } c \in \Phi^+ \text{ then } f(c) < 0 \text{ only if } c \in N(w^{-1}) \}$$

Since $N(w^{-1})$ is a finite set, the desired result follows. $\square$
**Lemma 3.4.4.** $U$ consists of all $\phi \in V^*$ such that $\phi(a) \geq 0$ for all but finitely many positive roots $a$.

**Proof.** Let

$$X := \{ \psi \in V^* | \psi(a) \geq 0 \text{ for all but finitely many } b \in \Phi^+ \}.$$ 

The previous lemma yields that $U \subseteq X$. Now let $\psi \in X$ be arbitrary. We shall show that $\psi \in U$ by an induction on the size of the set $\text{Neg}(\psi) := \{ b \in \Phi^+ | \psi(b) < 0 \}$. If $|\text{Neg}(\psi)| = 0$, then $\psi \in P^* \subset U$, so we may assume that $\text{Neg}(\psi) \neq \emptyset$. Then there must exist $b \in \Pi$ such that $\psi(b) < 0$. It is readily observed that the size of $\text{Neg}(r_b \psi)$ is one less than the size of $\text{Neg}(\psi)$. Indeed $\text{Neg}(r_b \psi) = r_b(\text{Neg}(\psi) \setminus \{b\})$. Hence the inductive hypothesis yields that $r_b \psi \in U$. Since $U$ is $W$-invariant, if follows that $\psi \in U$ since $U \subseteq U$. Since $\psi \in X$ was arbitrary, it follows that $X \subseteq U$. \qed

**Remark 3.4.5.** By the lemma above, we see that $U$ is a convex cone, and we call it the *Tits Cone*.

**Lemma 3.4.6.**

$$U^* = \bigcap_{w \in W} wP.$$ 

**Proof.**

$$U^* = \{ v \in V | f(v) \geq 0, \text{ for all } f \in U \}$$

$$= \{ v \in V | (w\phi)(v) \geq 0, \text{ for all } \phi \in P^*, \text{ and for all } w \in W \}$$

$$= \{ v \in V | \phi(w^{-1}v) \geq 0, \text{ for all } \phi \in P^*, \text{ and for all } w \in W \}$$

$$= \bigcap_{w \in W} \{ v \in V | \phi(w^{-1}v) \geq 0, \text{ for all } \phi \in P^* \}$$

$$= \bigcap_{w \in W} \{ wv \in V | \phi(v) \geq 0, \text{ for all } \phi \in P^* \}$$

$$= \bigcap_{w \in W} \{ wv \in V | v \in (P^*)^* \}$$

$$= \bigcap_{w \in W} wP \quad ((P^*)^* = P, \text{ since } P \text{ is topologically closed}).$$
Adopting the concept introduced in [3] and [4], we make the following definition.

**Definition 3.4.7.** We define the imaginary cone $Q$ of $W$ to be

$$Q = \{ v \in U^* \mid (v,a) \leq 0 \text{ for all but finitely many } a \in \Phi^+ \}.$$ 

Observe that $Q$ is indeed a convex cone.

**Lemma 3.4.8.** Suppose that $v \in V$ has the property that $(a,v) \leq 0$ for all $a \in \Pi$. Then $wv - v \in P$ for all $w \in W$. Moreover, if $v \in P$, then $v \in U^*$.

**Proof.** Use induction on $l(w)$. Noting that if $l(w) = 0$ then there is nothing to prove. If $l(w) \geq 1$, then we may write $w = w'r_a$ with $w' \in W$, $a \in \Pi$ and $l(w) = l(w') + 1$. Then

$$wv - v = w'r_a(v) - v = w'(v - 2(v,a)) - v = (w'v - v) + 2|(a,v)|w'a.$$ 

Observe that $w'a \in P$, and $w'v - v \in P$ by the inductive hypothesis. Since $P$ is a cone it follows that $wv - v \in P$ as required.

If $v \in P$, then $wv = (wv - v) + v \in P$ for all $w \in W$, and so $v \in \bigcap_{w \in W} w^{-1}P = U^*$.

Now we are ready to give an alternative characterization of $Q$:

**Proposition 3.4.9.**

$$Q = \{ wv \mid w \in W, v \in P \text{ such that } (v,a) \leq 0 \text{ for all } a \in \Phi^+ \}.$$ 

**Proof.** First set

$$X = \{ wv \mid w \in W, v \in PLC(\Pi) \text{ such that } (v,a) \leq 0 \text{ for all } a \in \Phi^+ \}.$$ 

Suppose that $u \in Q$. Lemma 3.4.6 yields that $u$ can indeed be expressed as $wv$ for some $w \in W$ and $v \in P$. In particular, $u \in P$. For
each \( b \in P \) set \( \text{Pos}(b) := \{ c \in \Phi^+ \mid (b, c) > 0 \} \). If \( \text{Pos}(u) = \emptyset \), then trivially \( u \in Q \) (taking \( w = 1 \)). Thus we may assume that \( \text{Pos}(u) \neq \emptyset \) and proceed by an induction on \( |\text{Pos}(v)| \). Choose \( a \in \Pi \) such that \((u, a) > 0\). Now it is readily checked that \( \text{Pos}(r_a u) = r_a(\text{Pos}(u) \setminus \{a\}) \).

Thus the inductive hypothesis yields that \( r_a u \in X \). Thus \( u \in X \) since \( X \) is clearly \( W \)-invariant. Since \( u \in Q \) was arbitrary, it follows that \( Q \subseteq X \).

Conversely, if \( x \in X \), then \( x = wv \) for some \( w \in W \) and \( v \in P \) such that \((v, a) \leq 0\) for all \( a \in \Pi \). By the previous lemma, \( x \in U^* \). Now let \( y \in \Phi^+ \). Then \((x, y) = (wv, y) = (v, w^{-1}y) \) and \((x, y) > 0\) only for those roots \( b \in N(w^{-1}) \). Clearly there are finitely (indeed, at most \( l(w) \)) many such roots. Thus \( x \in Q \). Since \( x \in X \) was arbitrary, it follows that \( X \subseteq Q \). \( \square \)

**Proposition 3.4.10.** Suppose that \( x, y \in \Phi \) are distinct with \( x \ \text{dom} \ W \ y \).

Then for all \( w \in W \), \( w(x - y) \in \text{PLC}(\Pi) \), that is, \( x - y \in U^* \).

**Proof.** Let \( W' \) be the (infinite) dihedral subgroup of \( W \) generated by \( r_x \) and \( r_y \). Let \( S(W') = \{s, t\} \) and \( \triangle(W') = \{\alpha_s, \alpha_t\} \). Proposition 3.3.4 (i) and Proposition 4.5.4 (i) of [13] combined yield that \( \alpha_s, \alpha_t \in \Phi^+ \) with \((\alpha_s, \alpha_t) = -(x, y) \leq -1 \). Since \( x \) and \( y \in \Phi(W') = W'\Delta(W') \), Lemma 3.2.1 yields that there are integers \( m \) and \( n \) such that

\[
x = \frac{\sinh(n \pm 1)\theta}{\sinh(\theta)} \alpha_s + \frac{\sinh(n\theta)}{\sinh(\theta)} \alpha_t \quad \text{and} \quad y = \frac{\sinh(m \pm 1)\theta}{\sinh(\theta)} \alpha_s + \frac{\sinh(m\theta)}{\sinh(\theta)} \alpha_t
\]

where \( \theta = \ln(- (\alpha_s, \alpha_t) + \sqrt{(\alpha_s, \alpha_t)^2 - 1}) = \cosh^{-1}(-(\alpha_s, \alpha_t)) \).

Keeping all notation as in Section 3.2, we write \( c_i \) for \( \frac{\sinh(i\theta)}{\sinh(\theta)} \), for all \( i \in \mathbb{N} \). Now let us consider the possible values of \((x, y)\):
(a) If \( x = c_{n+1}\alpha_s + c_n\alpha_t \) and \( y = c_{m+1}\alpha_s + c_m\alpha_t \), then
\[
(x, y) = \cosh((n - m)\theta) \geq 1.
\]
(b) If \( x = c_{n+1}\alpha_s + c_n\alpha_t \) and \( y = c_{m-1}\alpha_s + c_m\alpha_t \), then
\[
(x, y) = -\cosh((n + m)\theta) \leq -1.
\]
(c) If \( x = c_{n-1}\alpha_s + c_n\alpha_t \) and \( y = c_{m+1}\alpha_s + c_m\alpha_t \), then
\[
(x, y) = -\cosh((n + m)\theta) \leq -1.
\]
(d) If \( x = c_{n-1}\alpha_s + c_n\alpha_t \) and \( y = c_{m-1}\alpha_s + c_m\alpha_t \), then
\[
(x, y) = \cosh((n - m)\theta) \geq 1.
\]

Since \( x \dom_W y \), Lemma 3.3.2 (v) yields that \((x, y) \geq 1\). Therefore we can rule out cases (b) and (c) above and conclude that either
\[
x = \sinh(n + 1)\theta \alpha_s + \sinh(n\theta)\alpha_t, \quad y = \sinh(m + 1)\theta \alpha_s + \sinh(m\theta)\alpha_t;
\]
or
\[
x = \sinh(n - 1)\theta \alpha_s + \sinh(n\theta)\alpha_t, \quad y = \sinh(m - 1)\theta \alpha_t + \sinh(m\theta)\alpha_t.
\]

Next we shall show that \( n > m \). Suppose for a contradiction that \( m \geq n \). Then either \( x = y \) (when \( n = m \)), or there will be a \( w \in W' \) such that \( wx \in \Phi(W') \cap \Phi^- \) and yet \( wy \in \Phi(W') \cap \Phi^+ \) (when \( n < m \)), all contradicting the fact that \( x \dom_W y \). Since \( c_n > c_m \) if \( n > m \), it follows that \( x - y \in \PLC(\Pi) \). Finally because dominance is \( W \)-invariant, so for any \( w \in W \), repeat the above argument with \( x \) replaced by \( wx \) and \( y \) replaced by \( wy \), we may conclude that \( w(x - y) \in \PLC(\Pi) \). \( \square \)

**Theorem 3.4.11.** Suppose that \( x, y \in \Phi \) are distinct with \( x \dom_W y \). Then there exists \( w \in W \) such that \((w(x - y), z) \leq 0\), for all \( z \in \Phi^+ \), that is, \( x - y \in Q \).

**Proof.** By the previous proposition, we know that \( x - y \in U^* \). Thus if we can prove that there exists \( w \in W \) with \((w(x - y), z) \leq 0\) for all \( z \in \Phi^+ \), then \( x - y \in Q \) by Proposition 3.4.9 (since \( U^* \) is \( W \)-invariant).
Clearly it is enough to prove that \((w(x - y), z) \leq 0\) for all \(z \in \Pi\) and we give an algorithm to find such a \(w\) below.

Let \(W'\) be the (infinite) dihedral subgroup of \(W\) generated by \(r_x\) and \(r_y\), and let \(\triangle(W') = \{a_0, b_0\}\). By Proposition 3.3.4 (i), \(a_0, b_0 \in \Phi^+\), and \((a_0, b_0) = -(x, y) \leq -1\), and there is some \(u \in \langle r_x, r_y \rangle\) such that either

\[
(3.4.1) \quad u(x) = a_0 \quad \text{and} \quad u(y) = -b_0 \quad \text{or} \quad u(x) = b_0 \quad \text{and} \quad u(y) = -a_0.
\]

At any rate, \(u(x - y) = a_0 + b_0\). Obviously \((a_0 + b_0, a_0) \leq 0\) and \((a_0 + b_0, b_0) \leq 0\). However there may be \(c_1 \in \Pi\) such that \((a_0 + b_0, c_1) > 0\). If this is the case, set

\[
a_1 = r_{c_1}a_0
\]

and

\[
b_1 = r_{c_1}b_0.
\]

Observe that \((d, c_1) \leq 0\) for all \(d \in \Pi \setminus \{c_1\}\), so

\[
(3.4.2) \quad c_1 \in \text{supp}(a_0) \cup \text{supp}(b_0)
\]

Note that \(a_0 \neq c_1\) and \(b_0 \neq c_1\), since \((a_0 + b_0, c_1) > 0\) whereas \((a_0 + b_0, a_0) \leq 0\) and \((a_0 + b_0, b_0) \leq 0\). Thus \(a_1, b_1 \in \Phi^+\), and \((a_1, b_1) = (a_0, b_0) \leq -1\). Consequently Proposition 3.1.3 (ii) yields that \(a_1, b_1\) are the canonical roots for the root subsystem \(\Phi(\langle r_{a_1}, r_{b_1} \rangle)\).

Observe that applying \(r_{c_1}\) to \(a_0\) and \(b_0\) will only change the coefficient of \(c_1\), therefore (3.4.2) yields that

\[
\text{supp}(a_1) \cup \text{supp}(b_1) \subseteq \text{supp}(a_0) \cup \text{supp}(b_0).
\]

Furthermore,

\[
\sum_{a \in \Pi} \text{coeff}_a(a_1) + \sum_{a \in \Pi} \text{coeff}_a(b_1) < \sum_{a \in \Pi} \text{coeff}_a(a_0) + \sum_{a \in \Pi} \text{coeff}_a(b_0)
\]

since the coefficient for \(c_1\) has been strictly decreased as \((a_0 + b_0, c_1) > 0\).

Moreover, since \((a_0 + b_0, c_1) > 0\), it follows that at least one of \((a_0, c_1)\)
or \((b_0, c_1)\) must be strictly positive. Hence Lemma 2.3.3 (v) yields that
\[
dp(a_1) + dp(b_1) \leq dp(a_0) + dp(b_0).
\]
Repeat this process and we can obtain new pairs of positive roots
\[
\{a_2, b_2\}, \ldots, \{a_{m-1}, b_{m-1}\}, \{a_m, b_m\}
\]
with
\[
supp(a_m) \cup supp(b_m) \subseteq supp(a_{m-1}) \cup supp(b_{m-1}) \subseteq \cdots
\]
\[
\subseteq supp(a_0) \cup supp(b_0)
\]
so long as we can find a \(c_m \in \Pi\) such that \((a_{m-1} + b_{m-1}, c_m) > 0\).
And this process only terminates at \(\{a_n, b_n\}\) for some \(n\), if for all \(z \in \Pi\), \((a_n + b_n, z) \leq 0\). Thus if we could show that this process terminates at \(\{a_n, b_n\}\) for some finite \(n\), then we have in fact found a \(w = r_{c_n} \cdots r_{c_1} u \in W\) such that
\[
(w(x - y), z) = (r_{c_n} \cdots r_{c_1} (a_0 + b_0), z) \leq 0
\]
for all \(z \in \Pi\), which in turn will establish that \(x - y \in Q\).

Since the set of positive roots having depth less than a specific bound and support in a fixed finite subset of \(\Pi\) is finite, it then follows that the possible pairs of positive roots \(\{a_i, b_i\}\) obtained in this process must be finite too. Finally since
\[
\sum_{a \in \Pi} \coeff_a(a_j) + \sum_{a \in \Pi} \coeff_a(b_j) < \sum_{a \in \Pi} \coeff_a(a_i) + \sum_{a \in \Pi} \coeff_a(b_i)
\]
for all \(j > i\), therefore the sequence \(\{a_0, b_0\}, \{a_1, b_1\}, \ldots\) must terminate at \(\{a_n, b_n\}\).

Incidentally, observe that for all \(i \in \{1, \cdots, n\}\), where \(n\) is as above, \(r_{c_i} a_{i-1} \in \Phi^+\) and \(r_{c_i} b_{i-1} \in \Phi^+\). So we can easily deduce that \(r_{c_i} \cdots r_{c_1}(a_0) \in \Phi^+\), and \(r_{c_i} \cdots r_{c_1}(b_0) \in \Phi^+\), for all \(i \in \{1, \cdots, n\}\).

Hence, let \(w \in W\) be as above, (3.4.2) yields that either
\[
wx = r_{c_n} \cdots r_{c_1} u a_0 \in \Phi^+ \quad \text{and} \quad wy = r_{c_n} \cdots r_{c_1} u(-b_0) \in \Phi^-
\]
or
\[
wx = r_{c_n} \cdots r_{c_1} u b_0 \in \Phi^+ \quad \text{and} \quad wy = r_{c_n} \cdots r_{c_1} u(-a_0) \Phi^-.
\]
The observation made at the end of the proof of Theorem 3.4.11 yields that:

**Corollary 3.4.12.** Let $x, y \in \Phi$ be distinct with $x \dom_W y$. Then there exists $w \in W$ such that $wx \in \Phi^+$, $wy \in \Phi^-$, and $(w(x - y), z) \leq 0$ for all $z \in \Phi^+$.

**Corollary 3.4.13.** Suppose that $x, y \in \Phi$ are distinct with $x \dom_W y$. Then the following are equivalent:

(i) whenever $x \dom_W z \dom_W y$ for some $z \in \Phi$, then either $z = x$ or $z = y$;

(ii) there exists a $w \in W$ such that $w(x) \in D_0$ and $w(y) \in -D_0$.

**Proof.** (i) implies (ii): Let $w$ be as in the previous corollary. First we show that then $wx \in D_0$. Suppose for a contradiction that $wx \notin D_0$. Let $z \in D(wx)$. Then the last corollary yields $(wy, z) \geq 1$ too, for $(wy, z) \geq (wx, z) \geq 1$. Since the last Corollary yields that $wy \in \Phi^-$, it follows that $z \dom_W wy$. But this gives us $x \dom_W w^{-1}z \dom_W y$ with $x \neq w^{-1}z \neq y$, contradicting (i). Therefore $wx \in D_0$, as required.

(ii) implies (i): Clear.

**Definition 3.4.14.** Suppose that $x, y \in \Phi$, $x \dom_W y$ satisfy both (i) and (ii) of Corollary 3.4.13, then we say that the dominance between $x$ and $y$ is minimal.

**Proposition 3.4.15.** Suppose that $x, y \in \Phi$ are distinct, and $x \dom_W y$. Then the dominance between $x$ and $y$ with respect to the subgroup $\langle r_x, r_y \rangle$ is minimal.

**Proof.** Let $a$ and $b$ be the canonical roots for $\Phi(\langle r_x, r_y \rangle)$. Again as in the proof of Proposition 3.3.4 (i) we see that every root in $\Phi(\langle r_x, r_y \rangle)$
must be in exactly one of the following two dominance chains:

\[(3.4.3) \quad \cdots \dom_W r_a r_b r_a(b) \dom_W r_a r_b(a) \dom_W r_a(b) \dom_W a \\
\quad \dom_W (−b) \dom_W r_b(−a) \dom_W r_a(−b) \dom_W \cdots \]

and

\[(3.4.4) \quad \cdots \dom_W r_b r_a r_b(a) \dom_W r_b r_a(b) \dom_W r_b(a) \dom_W b \\
\quad \dom_W (−a) \dom_W r_a(−b) \dom_W r_a r_b(−a) \dom_W \cdots .\]

Upon inspecting (3.4.3) and (3.4.4), we can readily see that the only elementary roots in this root subsystem with respect to \(\dom_{\langle r_x, r_y \rangle}\) are \(a\) and \(b\). By Proposition 3.3.4 (i), we know that there is some \(w \in \langle r_x, r_y \rangle\) such that either

\[wx = a \text{ and } wy = −b\]

or

\[wx = b \text{ and } wy = −a,\]

therefore Corollary 3.4.13 yields that the dominance between \(x\) and \(y\) with respect to \(\langle r_x, r_y \rangle\) is minimal. \(\square\)

### 3.5. Dominance in \(\Phi_1\)

In this section we generalize the results obtained in Section 3.3 into the non-orthogonal geometric realization studied in Chapter 1 and Chapter 2. In particular, we prove an analogue of Theorem 3.3.9 and Corollary 3.3.10 adapted to the non-orthogonal setting, namely, for a suitable definition of roots, the set of roots dominating precisely \(n\) other positive roots is finite in size, for any positive integer \(n\). The conclusion drawn towards the end of this section is that the dominance concept, especially those results analogous to Theorem 3.3.9 and Corollary 3.3.10 are in fact dependent only on the underlying Coxeter group and not on the particular geometric realization used.

Keeping all notation as in Chapter 1, in this section we adapt the notion of dominance to the roots in \(\Phi_1\). We stress that all the results
obtained throughout this section equally apply to the roots in $\Phi_2$. We begin by define what is meant by saying that a root in $\Phi_1$ dominates another root in $\Phi_1$.

**Definition 3.5.1.** For $\alpha_1, \alpha_2 \in \Phi_1$, we say that $\alpha_1$ dominates $\alpha_2$ with respect to $W$ (written $\alpha_1 \text{ dom}_W \alpha_2$) if

$$\{ w \in W \mid w\alpha_1 \in \Phi_1^- \} \subseteq \{ w \in W \mid w\alpha_2 \in \Phi_1^- \}.$$

**Lemma 3.5.2.** Suppose that $\alpha_1, \alpha_2 \in \Phi_1$. Then $\alpha_1 \text{ dom}_W \alpha_2$ if and only if $f_1(\alpha_1) \text{ dom}_W f_1(\alpha_2)$.

**Proof.** By definition, $\alpha_1 \text{ dom}_W \alpha_2$ if and only if whenever $w \in W$ and $w\alpha_1 \in \Phi_1^-$ then $w\alpha_2 \in \Phi_1^-$. Since $f_1$ is $W$-equivariant, if follows from Lemma 2.3.6 that

$$w\alpha_1 \in \Phi_1^- \text{ if and only if } f_1(w\alpha_1) = wf_1(\alpha_1) \in \Phi^-$$

and

$$w\alpha_2 \in \Phi_1^- \text{ if and only if } f_1(w\alpha_2) = wf_1(\alpha_2) \in \Phi^-.$$

Thus $\alpha_1 \text{ dom}_W \alpha_2$ if and only if $f_1(\alpha_1) \text{ dom}_W f_1(\alpha_2)$, as required. \hfill $\square$

Lemma 3.5.2 combined with results obtained in Section 3.3 enables us to deduce the following facts concerning dominance in $\Phi_1$.

**Lemma 3.5.3.** Let $\alpha_1, \alpha_2 \in \Phi_1^+$ such that $\alpha_1$ is not a scalar multiple of $\alpha_2$ and $\alpha_1 \text{ dom}_W \alpha_2$. Then

(i) $\langle \alpha_1, \phi(\alpha_2) \rangle > 0$;

(ii) $(w\alpha_1) \text{ dom}_W (w\alpha_2)$ for all $w \in W$, that is, dominance is $W$-invariant;

(iii) $\text{dp}_1(\alpha_1) > \text{dp}_1(\alpha_2)$;

(iv) $-\alpha_2 \text{ dom}_W -\alpha_1$.

**Proof.** (i) Lemma 3.3.2 (i), Corollary 2.3.9 and Lemma 3.5.2 combined yield the desired result.

(ii) Follows from Lemma 3.3.2 (ii) and Lemma 3.5.2.
Lemma 3.5.4. Suppose that $W$ is a finite Coxeter group. Then the only dominance in $\Phi_1$ is of the form $\alpha \text{ dom } \lambda \alpha$ for some $\lambda > 0$.

Proof. If $W$ is finite, then there is a unique $w_0 \in W$ of maximal length. So $l(ws) < l(w)$ for all $s \in S$, and this implies that $w \alpha_s \in \Phi^-_1$ for all $s \in S$. Thus $w_0(\Phi^+_1) = \Phi^-_1$. Hence $N_1(w_0) = \hat{\Phi}_1$ and consequently $w \in W$, $l(w_0w) = l(w_0) - l(w)$. Furthermore, $w_0 = w_0^{-1}$. Now suppose that $\alpha \in \Phi^+_1$. Then $-w_0 \alpha \in \Phi^+_1$, and

$$dp_1(-w_0 \alpha) = \frac{1}{2}(l(r_{-w_0 \alpha}) + 1) \quad \text{(by Lemma 1.3.19 of Chapter 1)}$$
$$= \frac{1}{2}(l(w_0 r_\alpha w_0) + 1)$$
$$= \frac{1}{2}(l(w_0) - l(r_\alpha w_0) + 1)$$
$$= \frac{1}{2}(l(w_0) - l(w_0 r_\alpha) + 1)$$
$$= \frac{1}{2}(l(w_0) - l(w_0) + l(r_\alpha) + 1)$$
$$= \frac{1}{2}(l(r_\alpha) + 1)$$
$$= dp_1(\alpha) \quad \text{(again by Lemma 1.3.19 of Chapter 1)}.$$

Therefore the map sending each positive root $\alpha$ to $-w_0 \alpha$ is a depth preserving permutation.

Now suppose for a contradiction that $\alpha_1, \alpha_2 \in \Phi_1$ which are not scalar multiples of each other and $\alpha_1 \text{ dom}_W \alpha_2$. Without loss of generality we may assume that $\alpha_1, \alpha_2 \in \Phi^+_1$. Then Lemma 3.5.3 (v) yields that $-w_0 \alpha_2 \text{ dom}_W -w_0 \alpha_1$, and by Lemma 3.5.3 (iv) this in turn implies that

$$dp_1(\alpha_2) = dp_1(-w_0 \alpha_2) > dp_1(-w_0 \alpha_1) = dp_1(\alpha_1),$$

contradicting Lemma 3.5.3 (iv). □

Now we give a geometric characterization of dominance in $\Phi_1$:

Lemma 3.5.5. Let $\alpha_1$ and $\alpha_2$ be arbitrary roots in $\Phi^+_1$ which are not scalar multiples of each other. Then $\alpha_1 \text{ dom}_W \alpha_2$ if and only if the
The following three conditions are satisfied:

\[ \langle \alpha_2, \phi(\alpha_1) \rangle > 0, \quad \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \geq 1 \text{ and } dp_1(\alpha_1) > dp_1(\alpha_2). \]

**Proof.** Suppose that \( \alpha_1 \text{ dom}_W \alpha_2 \). By Lemma 3.5.3 (i) and (iv) we only need to prove that \( \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \geq 1 \).

If \( \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle < 1 \), then by Corollary 2.3.17, it follows that \( D \) the subgroup of \( W \) generated by \( r_{\alpha_1} \) and \( r_{\alpha_2} \) is finite. By Lemma 3.5.4 there is no non-trivial dominance in \( D \) as required.

Conversely, assume that \( \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \geq 1 \), \( \langle \alpha_1, \phi(\alpha_2) \rangle > 0 \) and \( dp_1(\alpha_1) > dp(\alpha_2) \). First consider the case that \( \alpha_2 = \alpha_r \) for some \( r \in S \). Clearly \( r_r \alpha_1 \in \Phi_1^+ \), for \( \alpha_1 \) is not a positive scalar multiple of \( \alpha_2 \) (because \( dp_1(\alpha_1) > dp_1(\alpha_2) \)), and

\[
\langle \alpha_1, \phi(\alpha_1) \rangle \langle r_r \alpha_1, \phi(\alpha_1) \rangle = \langle \alpha_1, \phi(\alpha_1) - 2\langle \alpha_r, \phi(\alpha_1) \rangle \beta_r \rangle \langle \alpha_1 - 2\langle \alpha_1, \beta_r \rangle \alpha_r, \phi(\alpha_1) \rangle
= ((\langle \alpha_1, \phi(\alpha_1) \rangle - 2\langle \alpha_1, \beta_r \rangle \langle \alpha_r, \phi(\alpha_1) \rangle)\rangle^2
\geq 1.
\]

Now direct calculations similar to those in Lemma 1.1.8 of Chapter 1 yield that there are infinitely many elements in \( \hat{\Phi}_1 \) of the form \( \lambda \alpha_1 + \mu r_r \alpha_1 \), where \( \lambda, \mu > 0 \). Suppose for a contradiction that \( \alpha_1 \) does not dominate \( \alpha_r \), and choose \( w \in W \) such that \( w \alpha_1 \in \Phi_1^- \) and \( w \alpha_r \in \Phi_1^+ \). Then

\[ w(r_r \alpha_1) = w(\alpha_1 - 2\langle \alpha_1, \beta_r \rangle \alpha_r) = w \alpha_1 + 2\langle \alpha_1, \beta_r \rangle (-w \alpha_r). \]

By Corollary 2.3.13, \( \langle \alpha_1, \beta_r \rangle > 0 \) (since \( \langle \alpha_r, \phi(\alpha_1) \rangle > 0 \)), so we see that \( w(r_r \alpha_1) \) is negative. So \( N_1(w) \) contains both \( \alpha_1 \) and \( r_r \alpha_1 \), and hence all the roots of the form \( \lambda \alpha_1 + \mu r_r \alpha_1 \) where \( \lambda, \mu > 0 \), contradicting the finiteness of \( N_1(w) \).
3.5. Dominance in \( \Phi_1 \)

Proceeding by induction on \( \text{dp}_1(\alpha_2) \), suppose that now \( \text{dp}_1(\alpha_2) > 1 \), and choose \( s \in S \) such that \( r_s \alpha_2 \prec_1 \alpha_2 \). Since \( \text{dp}_1(\alpha_1) > \text{dp}_1(\alpha_2) > 1 \), it follows that \( r_s \alpha_2 \in \Phi_1^+ \). Now

\[
\langle r_s \alpha_1, r_s \phi(\alpha_2) \rangle \langle r_s \alpha_2, r_s \phi(\alpha_1) \rangle = \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \geq 1
\]

and \( \langle r_s \alpha_2, r_s \phi(\alpha_1) \rangle = \langle \alpha_2, \phi(\alpha_1) \rangle > 0 \), and furthermore,

\[
\text{dp}(r_s \alpha_1) \geq \text{dp}_1(\alpha_1) - 1 > \text{dp}_1(\alpha_2) - 1 = \text{dp}_1(r_s \alpha_2).
\]

Hence the inductive hypothesis yields that \( r_s \alpha_1 \text{ dom}_W r_s \alpha_2 \), and by Lemma 3.5.3 (ii), this implies that \( \alpha_1 \text{ dom}_W \alpha_2 \).

Using similar arguments as those used in the proofs of Lemma 3.3.2 part (iv) and part (v), we may extend the last lemma to the following:

**Proposition 3.5.6.** Let \( \alpha_1, \alpha_2 \in \Phi_1 \). Then there is dominance between \( \alpha_1 \) and \( \alpha_2 \) if and only if

\[
\langle \alpha_1, \phi(\alpha_2) \rangle > 0 \quad \text{and} \quad \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \geq 1.
\]

Now we are able to prove the converse of Corollary 2.3.17:

**Lemma 3.5.7.** Suppose \( \alpha_1, \alpha_2 \in \Phi_1 \) with \( r_{\alpha_1} \neq r_{\alpha_2} \). If the subgroup of \( W \) generated by \( r_{\alpha_1} \) and \( r_{\alpha_2} \) is finite then \( \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle < 1 \).

**Proof.** Suppose for a contradiction that \( \langle r_{\alpha_1}, r_{\alpha_2} \rangle \) is finite and yet \( \langle \alpha_1, \phi(\alpha_2) \rangle \langle \alpha_2, \phi(\alpha_1) \rangle \geq 1 \). Since \( r_{\alpha_1} = r_{-\alpha_1} \), we may replace \( \alpha_1 \) by \(-\alpha_1\) if needed and assume that \( \langle \alpha_1, \phi(\alpha_2) \rangle \geq 0 \). Then Proposition 3.5.6 yields that there is dominance with respect to \( W \) between \( \alpha_1 \) and \( \alpha_2 \). Proposition 3.3.4 (ii) and Lemma 3.5.2 combined yield that there is dominance with respect to \( \langle r_{\alpha_1}, r_{\alpha_2} \rangle \) between \( \alpha_1 \) and \( \alpha_2 \), contradicting Lemma 3.5.4.

Recall the equivalence relation and the set of equivalence classes \( \widetilde{\Phi}_1 \) defined in the proof of Lemma 1.2.13.
Proposition 3.5.8. There is a \( W \)-equivariant bijection \( \tilde{f}_1 : \tilde{\Phi}_1 \to \Phi \) satisfying
\[
\tilde{f}_1(\tilde{\alpha}) = f(\alpha),
\]
for all \( \alpha \in \Phi_1 \).

Proof. Since \( f \) is \( W \)-equivariant, it follows that if such a map \( \tilde{f}_1 \) exists, then it must be \( W \)-equivariant. Furthermore, it can be readily checked that if such \( \tilde{f}_1 \) exists, then it must be bijective. Thus all that remains is to prove that such \( \tilde{f}_1 \) is well-defined, that is, we need to show that if \( \alpha \) and \( \lambda \alpha \) are in \( \Phi_1 \) with \( \lambda > 0 \) and \( \lambda \neq 1 \), then \( f_1(\alpha) = f_1(\lambda \alpha) \). Our proof will be based on an induction on the depth of \( \alpha \) (which is equal to the depth of \( \lambda \alpha \)).

If \( dp_1(\alpha) = 1 \), then \( \alpha = \mu \alpha_s \) for some \( s \in S \) and \( \mu > 0 \). Then Corollary 1.2.19 yields that \( \phi(\alpha) = \frac{1}{\mu} \beta_s \). Now Corollary 2.3.13 yields that \( \text{coeff}_{\alpha_t}(f_1(\alpha)) = 0 \) for all \( t \in S \setminus \{s\} \). Since \( \text{coeff}_{\alpha_s}(\phi(\alpha)) = 1 \), it follows from Proposition 2.3.14 that \( f_1(\alpha) = \gamma_s \). Similarly if \( \lambda \alpha \in \Phi_1^+ \), then \( f_1(\lambda \alpha) = \gamma_s \). Hence \( f_1(\alpha) = f_1(\lambda \alpha) \) when \( dp(\alpha) = 1 \).

Next we may assume that \( dp_1(\alpha) > 1 \), and choose \( t \in S \) such that \( r_t \alpha \prec_1 \alpha \). By Lemma 1.3.10 this means \( \langle \alpha, \beta_t \rangle > 0 \). Then \( \langle \lambda \alpha, \beta_t \rangle > 0 \) too, and so \( r_t(\lambda \alpha) \prec_1 (\lambda \alpha) \) as well. Now by the inductive hypothesis
\[
f_1(r_t(\alpha)) = f_1(r_t(\lambda \alpha)) = \gamma
\]
for some \( \gamma \in \Phi \). Then
\[
f_1(\alpha) = r_t f_1(r_t(\alpha)) = r_t \gamma = r_t(f_1(r_t(\lambda \alpha))) = f_1(\lambda \alpha)
\]
as required. \( \square \)

Definition 3.5.9. Let \( \tilde{\alpha}_1, \tilde{\alpha}_2 \in \tilde{\Phi}_1 \), we say that \( \tilde{\alpha}_1 \) dominates \( \tilde{\alpha}_2 \) with respect to \( W \) (written \( \tilde{\alpha}_1 \dom_W \tilde{\alpha}_2 \)), if \( \alpha_1 \dom_W \alpha_2 \). For each nonnegative integer \( n \), define
\[
\tilde{D}_{1,n} = \{ \tilde{\alpha} \in \tilde{\Phi}_1^+ | \tilde{\alpha} \text{ dominates exactly } n \text{ elements in } \tilde{\Phi}_1^+ \setminus \{\tilde{\alpha}\} \}.
\]

Proposition 3.5.10. For each nonnegative integer \( n \), \( \tilde{f}_1(\tilde{D}_{1,n}) = D_n \).
Proof. Follows readily from Proposition 3.5.8 and Lemma 3.5.2. □

Now we are ready to prove the following main result of this section:

Theorem 3.5.11. Suppose that $S$ is finite. Then $\hat{D}_{1,0}$ is finite in size. Furthermore, for each $n \in \mathbb{N}$, $|\hat{D}_{1,n}| < \infty$ and $|\hat{D}_{1,n}| \leq |D_0|^{n+1} - |D_0|^n$.

Proof. Follows from Theorem 3.3.8, Theorem 3.3.9, Proposition 3.5.10 and Corollary 3.3.10. □
References


