Root systems and reflection representations of Coxeter groups

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Preface

The aim of this thesis is to investigate two topics relating to geometric realizations of Coxeter groups. The first is a class of non-orthogonal geometric realizations of Coxeter groups and the second is the study of dominance behaviour in the root systems associated with Coxeter groups extending those examined in [**6**] and [**5**].

For arbitrary Coxeter system (W, R), it is well known that W can be embedded into the orthogonal group of certain bilinear form $(\ , \ )$ on a real vector space V. The root system of W in V is a certain W-stable subset of V corresponding to the set of reflections in W. If the bilinear form $(\ , \ )$ is non-degenerate then of course V is W-isomorphic to its dual V^*, but since the form is not always non-degenerate it is sometimes useful to study both the representation of W on V and the contragredient representation on V^*. This motivates the slightly more general approach taken in this thesis, in which we consider a pair of real vector spaces V_1 and V_2 linked by a W-equivariant bilinear pairing satisfying a few extra conditions (which are guaranteed to hold in the case that $V_1 = V$ and $V_2 = V^*$). We show that in this situation W embeds (faithfully) in the general linear groups of each of V_1 and V_2, both images being generated by reflections. The classical theory of geometric realizations can be recovered as a special case. We define and study generalized root systems arising in such non-orthogonal geometric realizations of Coxeter systems, and compare them with the root systems arising from the standard geometric realizations. It turns out that it is natural to consider root systems in both V_1 and V_2; these are in W-equivariant bijective correspondence with each other and with the root system in the classical setting. Familiar properties of simple roots and positive roots generalize to the non-orthogonal case, although it is no longer necessarily true that the only scalar multiples of a root are itself and its negative.

The investigation of non-orthogonal geometric realizations will occupy Chapter 1 and Chapter 2.

In Chapter 3 we study a partial order in the root systems of Coxeter groups called dominance introduced by B. Brink and R. Howlett. We first examine the dominance of roots in the classical geometric realizations and then generalize these results to the non-orthogonal setting. In [**6**] and [**5**] dominance is only defined on the positive roots, but the definition extends naturally to all roots, and it turns out that the geometric characterization of dominance between roots remains unchanged. In [**6**], it is found that for all finite rank Coxeter groups, the set of positive roots dominating no positive roots other than themselves is finite, and in [**5**] such sets are explicitly computed. In this thesis we prove that for all natural numbers n, the set of roots dominating precisely n positive roots (denoted by D_n) is finite for all finite rank Coxeter groups. The set of positive roots is obviously the disjoint union of all these D_n's, and we examine the interaction of this decomposition with the action of W. For all infinite Coxeter groups of finite rank and for each n, it turns out that $D_n \neq \emptyset$, and we also compute an upper bound of $|D_n|$ (the size of D_n). In the classical case we study certain cones related to the Tits cone, and show how they are related to dominance.

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Statement of originality

This thesis contains no new material which has been accepted for the award of any other degree or diploma. All work in this thesis, except where duly acknowledged otherwise, is believed to be original.

Contents

Chapter 1

Non-orthogonal Geometric Realizations of Coxeter Groups

1.1. Paired reflection representations

Let S be an arbitrary set and suppose that each unordered pair $\{s,t\}$ of elements in S is assigned an $m_{st} \in \mathbb{Z} \cup \{\infty\}$, subject to the conditions that $m_{ss} = 1$ for all s in S and $m_{st} \geq 2$ for all distinct s, t in S. Suppose that V_1 and V_2 are vector spaces over the real field $\mathbb{R}$, and suppose that there exists a bilinear map $\langle\ ,\ \rangle : V_1 \times V_2 \to \mathbb{R}$ and sets $\Pi_1 = \{\ \alpha_s \mid s \in S\ \} \subseteq V_1$ and $\Pi_2 = \{\ \beta_s \mid s \in S\ \} \subseteq V_2$ such that the following conditions hold:

(C1) Π_1 spans V_1 and Π_2 spans V_2;

(C2) $\langle \alpha_s, \beta_s \rangle = 1$, for all $s \in S$;

(C3) $\langle \alpha_s, \beta_t \rangle \leq 0$, for all distinct $s, t \in S$;

(C4) for all s, $t \in S$,

$$\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle = \begin{cases} \cos^2(\pi/m_{st}) & \text{if } m_{st} \neq \infty, \\ \gamma^2,\ \text{for some } \gamma \geq 1 & \text{if } m_{st} = \infty; \end{cases}$$

(C5) for all $s, t \in S$, $\langle \alpha_s, \beta_t \rangle = 0$ if and only if $\langle \alpha_t, \beta_s \rangle = 0$;

(C6) $\sum_{s \in S} \lambda_s \alpha_s = 0$ with $\lambda_s \geq 0$ for all s implies $\lambda_s = 0$ for all s, and $\sum_{s \in S} \lambda_s \beta_s = 0$ with $\lambda_s \geq 0$ for all s implies $\lambda_s = 0$ for all s.

Note that (C4) and (C5) combined imply that $\langle \alpha_s, \beta_t \rangle$ and $\langle \alpha_t, \beta_s \rangle$ are zero if and only if $m_{st} = 2$. We can also express (C6) more compactly as $0 \notin \mathrm{PLC}(\Pi_1)$ and $0 \notin \mathrm{PLC}(\Pi_2)$, where $\mathrm{PLC}(A)$ (the *positive linear combinations of* A) is defined to be

$$\{ \sum_{a \in A} \lambda_a a \mid \lambda_a \geq 0 \text{ for all } a \in A, \text{ and } \lambda_{a'} > 0 \text{ for some } a' \in A \}.$$

Definition 1.1.1. In the above situation, if conditions (C1) to (C6) are satisfied then we call $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ a *Coxeter datum.* The m_{st} (for $s, t \in S$) are called the *Coxeter parameters* of $\mathscr{C}$.

Throughout this chapter, $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ will be a fixed Coxeter datum with Coxeter parameters m_{st}.

Definition 1.1.2. For each $s \in S$ let $\rho_{V_1}(s)$ and $\rho_{V_2}(s)$ be the linear transformations on V_1 and V_2 defined by

$$\rho_{V_1}(s)(x) = x - 2\langle x, \beta_s\rangle\alpha_s$$

for all $x \in V_1$, and

$$\rho_{V_2}(s)(y) = y - 2\langle \alpha_s, y\rangle\beta_s$$

for all $y \in V_2$. For each $i \in \{1, 2\}$ let $R_i := \{\rho_{V_i}(s) \mid s \in S\}$, and let W_i be the subgroup of $\mathrm{GL}(V_i)$ generated by R_i.

Since $\langle\alpha_s, \beta_s\rangle = 1$ for all $s \in S$ (by (C2) above), we find that $\rho_{V_1}(s)(\alpha_s) = -\alpha_s$ and $\rho_{V_2}(s)(\beta_s) = -\beta_s$. It follows that for all $v \in V_1$,

$$\rho_{V_1}(s)\rho_{V_1}(s)(v) = \rho_{V_1}(s)(v - 2\langle v, \beta_s\rangle\alpha_s) = v - 2\langle v, \beta_s\rangle\alpha_s + 2\langle v, \beta_s\rangle\alpha_s = v$$

showing that $\rho_{V_1}(s)$ is an involution. Similarly, $\rho_{V_2}(s)$ is also an involution.

The following result follows readily from these definitions:

Proposition 1.1.3. *Let $x \in V_1$ and $y \in V_2$. Then for all $s \in S$,*

$$\langle\rho_{V_1}(s)(x), \rho_{V_2}(s)(y)\rangle = \langle x, y\rangle.$$

Proof. By Definition 1.1.2 and the bilinearity of $\langle\ ,\ \rangle$, we find that

$$\begin{aligned}\langle\rho_{V_1}(s)(x), \rho_{V_2}(s)(y)\rangle &= \langle x - 2\langle x, \beta_s\rangle\alpha_s, y - 2\langle\alpha_s, y\rangle\beta_s\rangle \\ &= \langle x, y\rangle - 4\langle x, \beta_s\rangle\langle\alpha_s, y\rangle + 4\langle x, \beta_s\rangle\langle\alpha_s, y\rangle\langle\alpha_s, \beta_s\rangle \\ &= \langle x, y\rangle. \end{aligned}$$ □

The principal result of this section is that (W_1, R_1) and (W_2, R_2) are isomorphic Coxeter systems, in the sense of Definition 3 of Chapitre 4 of [**10**]. Recall that (W, R) is a Coxeter system if and only if W is a group and R a generating set for W, and in terms of these generators W is defined by a set of relations of the form $(rr')^{m(r,r')} = 1$, where $m(r,r') = m(r',r) > 1$ whenever r, r' are distinct elements of R and $m(r,r')$ is defined, and $m(r,r) = 1$ for all $r \in R$. We shall show that if $R = \{ r_s \mid s \in S \}$ is in bijective correspondence with S and $m(r_s, r_t) = m_{st}$ whenever $m_{st} < \infty$, with $m(r_s, r_t)$ undefined otherwise, then (W_1, R_1) and (W_2, R_2) are both isomorphic to the Coxeter system (W, R). We first show (Proposition 1.1.9 below) that (W_1, R_1) and (W_2, R_2), satisfy the required relations, after which it suffices to prove the following theorem:

Theorem 1.1.4. *If W is any group generated by a set $R := \{r_s \mid s \in S\}$ and satisfying the relations $(r_s r_t)^{m_{st}} = 1$ for all $s, t \in S$ such that $m_{st} < \infty$, and if $f\colon W \to W_1$ is a group homomorphism satisfying $f(r_s) = \rho_{V_1}(s)$ for all $s \in S$, then f is necessarily injective.*

Since $W_1 = \langle R_1 \rangle$, any such homomorphism f must also be surjective. So, choosing (W, R) to be the Coxeter system corresponding to the parameters m_{st}, and observing that Proposition 1.1.9 guarantees the existence of a homomorphism $f\colon W \to W_1$ satisfying $f(r_s) = \rho_{V_1}(s)$ for all s, it follows that f is an isomorphism, as required.

Of course analogous statements apply if W_1 is replaced by W_2; hence the claim that (W_1, R_1) and (W_2, R_2) are isomorphic Coxeter systems will be established once we have proved Theorem 1.1.4 and the analogous result for W_2. This will occupy the rest of this section.

Lemma 1.1.5. *If s is any element of S, then $\alpha_s \notin \mathrm{PLC}(\Pi_1 \setminus \{\alpha_s\})$ and $\beta_s \notin \mathrm{PLC}(\Pi_2 \setminus \{\beta_s\})$. Furthermore, if $s, t \in S$ with $s \neq t$, then $\{\alpha_s, \alpha_t\}$ is linearly independent, and so is $\{\beta_s, \beta_t\}$.*

Proof. Suppose for a contradiction that $\alpha_s = \sum_{t \in S\setminus\{s\}} \lambda_t \alpha_t$, where $\lambda_t \geq 0$ for all $t \in S \setminus \{s\}$. By (C2) and (C3) of Definition 1.1.1,

$$1 = \langle \alpha_s, \beta_s \rangle = \sum_{t \in S\setminus\{s\}} \lambda_t \langle \alpha_t, \beta_s \rangle \leq 0$$

which is absurd. Therefore $\alpha_s \notin \mathrm{PLC}(\Pi_1 \setminus \{\alpha_s\})$. Now suppose that $s, t \in S$ with $s \neq t$. Obviously α_s and α_t are both nonzero (by Condition (C2)), and by what has just been proved, α_s is not a positive scalar multiple of α_t. So to prove that $\{\alpha_s, \alpha_t\}$ is linearly independent, we only need to show that α_s is not a negative scalar multiple of α_t. But if $\alpha_s = -\lambda\alpha_t$ for some $\lambda > 0$ then $0 = \alpha_s + \lambda\alpha_t$, contradicting (C6) of Definition 1.1.1. Hence $\{\alpha_s, \alpha_t\}$ is linearly independent, as required.

Essentially the same argument can also be used to prove linear independence of $\{\beta_s, \beta_t\}$. □

Note that the above yields that for each $i \in \{1, 2\}$ and distinct $s, t \in S$ we have $\rho_{V_i}(s) \neq \rho_{V_i}(t)$.

Lemma 1.1.6. *Suppose that $s, t \in S$ such that $m_{st} \notin \{1, 2, \infty\}$. Then for all $n \in N$,*

$$(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = \frac{\sin(2n+1)\theta}{\sin\theta}\alpha_s + \frac{-\cos\theta}{\langle\alpha_t, \beta_s\rangle}\frac{\sin(2n\theta)}{\sin\theta}\alpha_t$$

and

$$(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = \frac{\sin(2n+1)\theta}{\sin\theta}\beta_s + \frac{-\cos\theta}{\langle\alpha_s, \beta_t\rangle}\frac{\sin(2n\theta)}{\sin\theta}\beta_t$$

where $\theta = \pi/m_{st}$.

Proof. Recall from Definition 1.1.2 that $\rho_{V_1}(s)\alpha_t = \alpha_t - 2\langle\alpha_t, \beta_s\rangle\alpha_s$, and $\rho_{V_1}(s)\alpha_s = \alpha_s - 2\langle\alpha_s, \beta_s\rangle\alpha_s = -\alpha_s$. Similar formulas apply to $\rho_{V_1}(t)$. Thus the matrix of $\rho_{V_1}(s)\rho_{V_1}(t)$ in its action on the subspace with basis $\{\alpha_s, \alpha_t\}$ is

$$\begin{pmatrix} -1 & -2\langle\alpha_t, \beta_s\rangle \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\langle\alpha_s, \beta_t\rangle & -1 \end{pmatrix} = \begin{pmatrix} 4\cos^2\theta - 1 & 2\langle\alpha_t, \beta_s\rangle \\ -2\langle\alpha_s, \beta_t\rangle & -1 \end{pmatrix}$$

since $\langle\alpha_s, \beta_t\rangle\langle\alpha_t, \beta_s\rangle = \cos^2\theta$. To prove the desired result, we only need to compute the nth power of the above matrix. Since $m_{st} \neq 2$ it follows that $\langle \alpha_s, \beta_t \rangle \neq 0 \neq \langle \alpha_t, \beta_s \rangle$, and we observe that

$$\begin{aligned}
&\begin{pmatrix} 4\cos^2\theta - 1 & 2\langle \alpha_t, \beta_s \rangle \\ -2\langle \alpha_s, \beta_t \rangle & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{-\cos\theta}{\langle \alpha_s, \beta_t \rangle} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4\cos^2\theta - 1 & -2\cos\theta \\ 2\cos\theta & -1 \end{pmatrix} \begin{pmatrix} \frac{-\cos\theta}{\langle \alpha_t, \beta_s \rangle} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{\sin\theta} \begin{pmatrix} \frac{-\cos\theta}{\langle \alpha_s, \beta_t \rangle} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin 3\theta & -\sin 2\theta \\ \sin 2\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \frac{-\cos\theta}{\langle \alpha_t, \beta_s \rangle} & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Now an induction on n yields that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\begin{pmatrix} 4\cos^2\theta - 1 & 2\langle \alpha_t, \beta_s \rangle \\ -2\langle \alpha_s, \beta_t \rangle & 1 \end{pmatrix}^n \\
&= \frac{1}{\sin\theta} \begin{pmatrix} \frac{-\cos\theta}{\langle \alpha_s, \beta_t \rangle} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(2n+1)\theta & -\sin 2n\theta \\ \sin 2n\theta & -\sin(2n-1)\theta \end{pmatrix} \begin{pmatrix} \frac{-\cos\theta}{\langle \alpha_t, \beta_s \rangle} & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Expanding the right hand side and examining the coefficients in the first column yields the formula

$$(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = \frac{\sin(2n+1)\theta}{\sin\theta}\alpha_s + \frac{-\cos\theta}{\langle \alpha_t, \beta_s \rangle} \frac{\sin 2n\theta}{\sin\theta}\alpha_t.$$

The other formula follows by a similar calculation. □

Remark 1.1.7. From Lemmas 1.1.6 and 1.1.5 we can see that if $i \in \{1, 2\}$ and $m_{st} \notin \{2, \infty\}$ then $\operatorname{ord}(\rho_{V_i}(s)\rho_{V_i}(t)) \geq m_{st}$. Indeed, in the subspace with basis $\{\alpha_s, \alpha_t\}$ the elements

$$\alpha_s, (\rho_{V_1}(s)\rho_{V_1}(t))(\alpha_s), (\rho_{V_1}(s)\rho_{V_1}(t))^2(\alpha_s), \ldots, (\rho_{V_1}(s)\rho_{V_1}(t))^{m_{st}-1}(\alpha_s)$$

are all distinct, and in the subspace with basis $\{\beta_s, \beta_t\}$ the elements

$$\beta_s, (\rho_{V_2}(s)\rho_{V_2}(t))(\beta_s), (\rho_{V_2}(s)\rho_{V_2}(t))^2(\beta_s), \ldots, (\rho_{V_2}(s)\rho_{V_2}(t))^{m_{st}-1}(\beta_s)$$

are all distinct.

Lemma 1.1.8. *Suppose that $s, t \in S$ such that $m_{st} = \infty$. If we set $\theta = \ln(\gamma + \sqrt{\gamma^2 - 1}) = \cosh^{-1}(\gamma)$, where $\gamma = \sqrt{\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle}$, then*

(i) *for all* $n \in \mathbb{N}$

$$(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = \frac{\sinh(2n+1)\theta}{\sinh\theta}\alpha_s + \frac{-\gamma}{\langle \alpha_t, \beta_s\rangle}\frac{\sinh(2n\theta)}{\sinh\theta}\alpha_t$$

and

$$(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = \frac{\sinh(2n+1)\theta}{\sinh\theta}\beta_s + \frac{-\gamma}{\langle \alpha_s, \beta_t\rangle}\frac{\sinh(2n\theta)}{\sinh\theta}\beta_t,$$

(ii) *for each* $i \in \{1, 2\}$, $\rho_{V_i}(s)\rho_{V_i}(t)$ *has infinite order in* $\mathrm{GL}(V_i)$.

Proof. (i) The matrix of $\rho_{V_1}(s)\rho_{V_1}(t)$ in its action on the subspace with basis $\{\alpha_s, \alpha_t\}$ is

$$\begin{pmatrix} -1 & -2\langle \alpha_t, \beta_s\rangle \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\langle \alpha_s, \beta_t\rangle & -1 \end{pmatrix} = \begin{pmatrix} 4\gamma^2 - 1 & 2\langle \alpha_t, \beta_s\rangle \\ -2\langle \alpha_s, \beta_t\rangle & -1 \end{pmatrix}$$

since $\langle \alpha_s, \beta_t\rangle\langle \alpha_t, \beta_s\rangle = \gamma^2$ (by (C4) of Definition 1.1.1). As in the proof of Lemma 1.1.6, to prove the required result we only need to compute the nth power of the above matrix. Since $m_{st} = \infty$, it follows that $\langle \alpha_s, \beta_t\rangle \neq 0 \neq \langle \alpha_t, \beta_s\rangle$, and we have

$$\begin{aligned} &\begin{pmatrix} 4\gamma^2 - 1 & 2\langle \alpha_t, \beta_s\rangle \\ -2\langle \alpha_s, \beta_t\rangle & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\gamma}{\langle \alpha_s, \beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4\gamma^2 - 1 & -2\gamma \\ 2\gamma & -1 \end{pmatrix}\begin{pmatrix} \frac{-\gamma}{\langle \alpha_t, \beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{\sinh\theta}\begin{pmatrix} \frac{-\gamma}{\langle \alpha_s, \beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sinh 3\theta & -\sinh 2\theta \\ \sinh 2\theta & -\sinh\theta \end{pmatrix}\begin{pmatrix} \frac{-\gamma}{\langle \alpha_t, \beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Now an induction yields that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\begin{pmatrix} 4\gamma^2 - 1 & 2\langle \alpha_t, \beta_s\rangle \\ -2\langle \alpha_s, \beta_t\rangle & -1 \end{pmatrix}^n \\ &= \begin{pmatrix} \frac{-\gamma}{\langle \alpha_s, \beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\sinh(2n+1)\theta}{\sinh\theta} & \frac{-\sinh(2n\theta)}{\sinh\theta} \\ \frac{\sinh(2n\theta)}{\sinh\theta} & \frac{-\sinh(2n-1)\theta}{\sinh\theta} \end{pmatrix}\begin{pmatrix} \frac{-\gamma}{\langle \alpha_t, \beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, for each $n \in \mathbb{N}$,

$$(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = \frac{\sinh(2n+1)\theta}{\sinh\theta}\alpha_s + \frac{-\gamma}{\langle \alpha_t, \beta_s\rangle}\frac{\sinh(2n\theta)}{\sinh\theta}\alpha_t.$$

A similar argument shows that

$$(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = \frac{\sinh(2n+1)\theta}{\sinh\theta}\beta_s + \frac{-\gamma}{\langle\alpha_s,\beta_t\rangle}\frac{\sinh(2n\theta)}{\sinh\theta}\beta_t,$$

completing the proof of (i).

(ii) In the subspace with basis $\{\alpha_s, \alpha_t\}$, it is clear from (i) above that the elements $\alpha_s, (\rho_{V_1}(s)\rho_{V_2}(t))(\alpha_s), (\rho_{V_1}(s)\rho_{V_2}(t))^2(\alpha_s), \cdots$, are all distinct, and therefore $\rho_{V_1}(s)\rho_{V_1}(t)$ has infinite order in $\mathrm{GL}(V_1)$. In the same way, $\rho_{V_2}(s)\rho_{V_2}(t)$ has infinite order in $\mathrm{GL}(V_2)$. □

Proposition 1.1.9. *Suppose that s, t are distinct elements of S such that $m_{st} \neq \infty$. Then*

(i) $\rho_{V_1}(s)\rho_{V_1}(t)$ *has order* m_{st} *in* $\mathrm{GL}(V_1)$;
(ii) $\rho_{V_2}(s)\rho_{V_2}(t)$ *has order* m_{st} *in* $\mathrm{GL}(V_2)$.

Proof. Recall that $m_{st} = 1$ if and only if $s = t$. So in this case the statement is simply that $\rho_{V_i}(s)^2 = 1$ for each i and all $s \in S$. We have already noted that this is true.

Let $\alpha \in V_1$ be arbitrary and let $s, t \in S$ be distinct. We see that

$$\begin{aligned}(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha) &= \rho_{V_1}(s)(\alpha - 2\langle\alpha,\beta_t\rangle\alpha_t)\\ &= \alpha - 2\langle\alpha,\beta_s\rangle\alpha_s - 2\langle\alpha,\beta_t\rangle(\alpha_t - 2\langle\alpha_t,\beta_s\rangle\alpha_s)\\ &= \alpha + (4\langle\alpha,\beta_t\rangle\langle\alpha_t,\beta_s\rangle - 2\langle\alpha,\beta_s\rangle)\alpha_s - 2\langle\alpha,\beta_t\rangle\alpha_t.\end{aligned}$$

In the case that $m_{st} = 2$ we see that

$$(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha) = (\rho_{V_1}(t)\rho_{V_1}(s))(\alpha) = \alpha - 2\langle\alpha,\beta_s\rangle)\alpha_s - 2\langle\alpha,\beta_t\rangle\alpha_t,$$

so that $\rho_{V_1}(s)$ and $\rho_{V_1}(t)$ commute. Hence $\mathrm{ord}(\rho_{V_1}(s)\rho_{V_1}(t)) = 2$.

It remains to consider the case when $m_{st} > 2$. In the cases $\alpha = \alpha_s$ and $\alpha = \alpha_t$ the formula above gives

$$(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha_s) = (4\cos^2(\frac{\pi}{m_{st}}) - 1)\alpha_s - 2\langle\alpha_s,\beta_t\rangle\alpha_t$$

and

$$(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha_t) = 2\langle\alpha_t,\beta_s\rangle\alpha_s - \alpha_t,$$

and therefore the action of $\rho_{V_1}(s)\rho_{V_1}(t)$ on $\mathbb{R}\{\alpha_s, \alpha_t, \alpha\}$ may be represented by the following matrix M:

$$M = \begin{pmatrix} 4\cos^2(\pi/m_{st}) - 1 & 2\langle \alpha_t, \beta_s \rangle & 4\langle \alpha, \beta_t \rangle \langle \alpha_t, \beta_s \rangle - 2\langle \alpha, \beta_s \rangle \\ -2\langle \alpha_s, \beta_t \rangle & -1 & -2\langle \alpha, \beta_t \rangle \\ 0 & 0 & 1 \end{pmatrix}.$$

It is readily checked that M has distinct eigenvalues $e^{i\frac{2\pi}{m_{st}}}$, $e^{-i\frac{2\pi}{m_{st}}}$ and 1, and hence has order m_{st}. Consequently $(\rho_{V_1}(s)\rho_{V_1}(t))^{m_{st}}(\alpha) = \alpha$. Since α is arbitrary, it follows that $(\rho_{V_1}(s)\rho_{V_1}(t))^{m_{st}} = 1$ in $\mathrm{GL}(V_1)$. Thus we have $\mathrm{ord}(\rho_{V_1}(s)\rho_{V_1}(t)) \leq m_{st}$, and in view of Remark 1.1.7 above, it follows that $\mathrm{ord}(\rho_{V_1}(s)\rho_{V_1}(t))$ is precisely m_{st}.

The proof that $\mathrm{ord}(\rho_{V_2}(s)\rho_{V_2}(t)) = m_{st}$ is entirely similar. □

Combining Lemma 1.1.8(ii) and Proposition 1.1.9, we have

Corollary 1.1.10. *For each $i \in \{1, 2\}$, and for all $s, t \in S$,*

$$\mathrm{ord}(\rho_{V_i}(s)\rho_{V_i}(t)) = m_{st}.$$

□

Utilizing the formulas in Lemma 1.1.6 and Lemma 1.1.8 we may deduce the following:

Lemma 1.1.11. *Suppose that $s, t \in S$, and let n be an integer such that $0 \leq n < m_{st}$. Write*

$$\underbrace{\cdots \rho_{V_1}(t)\rho_{V_1}(s)\rho_{V_1}(t)}_{n\ \textit{factors}}\alpha_s = \lambda_n \alpha_s + \mu_n \alpha_t$$

and

$$\underbrace{\cdots \rho_{V_1}(s)\rho_{V_1}(t)\rho_{V_1}(s)}_{n\ \textit{factors}}\alpha_t = \lambda'_n \alpha_s + \mu'_n \alpha_t.$$

Then $\lambda_n \geq 0$, $\mu_n \geq 0$, $\lambda'_n \geq 0$ and $\mu'_n \geq 0$.

Proof. If $m_{st} = 2$ then $\langle \alpha_s, \beta_t \rangle = 0 = \langle \alpha_t, \beta_s \rangle$, giving $\rho_{V_1}(t)\alpha_s = \alpha_s$ and $\rho_{V_1}(s)\alpha_t = \alpha_t$. So $\lambda_1 = \mu'_1 = 1$ and $\mu_1 = \lambda'_1 = 0$. Since obviously also $\lambda_0 = \mu'_0 = 1$ and $\mu_0 = \lambda'_0 = 0$, the statement in the Lemma holds when $m_{st} = 2$. Thus we may assume that $m_{st} \geq 3$. Note in particular that this gives $\langle \alpha_s, \beta_t \rangle \neq 0 \neq \langle \alpha_t, \beta_s \rangle$.

Now suppose that n is even, so that

$$\underbrace{\cdots \rho_{V_1}(t)\rho_{V_1}(s)\rho_{V_1}(t)}_{n \text{ factors}} \alpha_s = (\rho_{V_1}(s)\rho_{V_1}(t))^{n/2}\alpha_s.$$

If $m_{st} < \infty$ then Lemma 1.1.6 gives

$$\lambda_n = \frac{\sin(n+1)\theta}{\sin\theta} \quad \text{and} \quad \mu_n = \frac{-\cos\theta}{\langle \alpha_t, \beta_s \rangle} \frac{\sin n\theta}{\sin\theta},$$

where $\theta = \pi/m_{st}$, while if $m_{st} = \infty$ then Lemma 1.1.8 gives

$$\lambda_n = \frac{\sinh(n+1)\theta}{\sinh\theta} \quad \text{and} \quad \mu_n = \frac{-\gamma}{\langle \alpha_t, \beta_s \rangle} \frac{\sinh n\theta}{\sinh\theta},$$

where $\gamma = \sqrt{\langle \alpha_t, \beta_s \rangle \langle \alpha_s, \beta_t \rangle} \geq 1$ and $\theta = \ln(\gamma + \sqrt{\gamma^2 - 1})$. Observe that if $0 \leq n < m_{st} < \infty$ then θ, $n\theta$ and $(n+1)\theta$ all lie in the interval $[0, \pi]$, and since also $\langle \alpha_t, \beta_s \rangle < 0$ it follows that $\lambda_n \geq 0$ and $\mu_n \geq 0$ whenever n is even and m_{st} is finite. The same conclusion follows when n is even and $m_{st} = \infty$, since in this case $\theta > 0$ (and $\langle \alpha_t, \beta_s \rangle < 0$ is still satisfied).

Next suppose that n is odd. Then

$$\begin{aligned} \underbrace{\cdots \rho_{V_1}(t)\rho_{V_1}(s)\rho_{V_1}(t)}_{n \text{ factors}} \alpha_s &= \rho_{V_1(t)}(\rho_{V_1}(s)\rho_{V_1}(t))^{(n-1)/2}\alpha_s \\ &= \rho_{V_1}(s)(\rho_{V_1}(s)\rho_{V_1}(t))^{(n+1)/2}\alpha_s. \end{aligned}$$

Observe that applying $\rho_{V_1}(t)$ will not change the coefficient of α_s and applying $\rho_{V_1}(s)$ will not change the coefficient of α_t. Hence the coefficient of α_s in $\rho_{V_1}(t)(\rho_{V_1}(s)\rho_{V_1}(t))^{(n-1)/2}\alpha_s$ is λ_{n-1} and the coefficient of α_t in $\rho_{V_1}(s)(\rho_{V_1}(s)\rho_{V_1}(t))^{(n+1)/2}\alpha_s$ is μ_{n+1}; that is, $\lambda_n = \lambda_{n-1}$ and $\mu_n = \mu_{n+1}$. Applying Lemma 1.1.6 and Lemma 1.1.8 yields

$$\lambda_n = \lambda_{n-1} = \begin{cases} \dfrac{\sin n\theta}{\sin\theta} & \text{if } m_{st} < \infty \\[2ex] \dfrac{\sinh n\theta}{\sinh\theta} & \text{if } m_{st} = \infty \end{cases}$$

and

$$\mu_n = \mu_{n+1} = \begin{cases} \dfrac{-\cos\theta}{\langle \alpha_t, \beta_s \rangle} \dfrac{\sin(n+1)\theta}{\sin\theta} & \text{if } m_{st} < \infty \\ \dfrac{-\gamma}{\langle \alpha_t, \beta_s \rangle} \dfrac{\sinh(n+1)\theta}{\sinh\theta} & \text{if } m_{st} = \infty, \end{cases}$$

where θ and γ are as previously, and it follows that $\lambda_n \geq 0$ and $\mu_n \geq 0$ when n is odd and $n < m_{st}$. Finally by symmetry, $\lambda'_n \geq 0$ and $\mu'_n \geq 0$ whenever $n < m_{st}$. □

Remark 1.1.12. The same argument applies equally well if we replace, respectively, $\rho_{V_1}(s)$, $\rho_{V_1}(t)$, α_s and α_t by $\rho_{V_2}(s)$, $\rho_{V_2}(t)$, β_s and β_t.

Definition 1.1.13. Let (W, R) be a Coxeter system. The *length* function of W with respect to R is the function $l\colon W \to N$ defined by

$$l(w) = \min\{\, n \in \mathbb{N} \mid w = r_1 r_2 \cdots r_n, \text{ where } r_1, r_2, \ldots, r_n \in R \,\},$$

for all $w \in W$. We say that $w = r_1 r_2 \cdots r_n$ with $r_1, r_2, \ldots, r_n \in R$ is a *reduced expression* for w if $l(w) = n$.

Let W and f be as in the statement of Theorem 1.1.4. The homomorphism $f\colon W \to W_1$ permits us to define an action of W on V_1 by $wx := (f(w))x$ for all $w \in W$ and $x \in V_1$.

Proposition 1.1.14. *Let $w \in W$ and $s \in S$. If $l(wr_s) \geq l(w)$, then $w\alpha_s \in \mathrm{PLC}(\Pi_1)$.*

Proof. Choose $w \in W$ of minimal length such that the assertion fails for some $\alpha_s \in \Pi_1$, and choose such an α_s. Certainly $w \neq 1$, since $1\alpha_s = \alpha_s$ is trivially a positive linear combination of Π_1. Thus $l(w) > 1$, and we may choose $t \in S$ such that $w_1 = wr_t$ has length $l(w) - 1$. If $l(w_1 r_s) \geq l(w_1)$, then $l(w_1 r) \geq l(w_1)$ for both $r = r_s$ and $r = r_t$. Alternatively, if $l(w_1 r_s) < l(w_1)$, we define $w_2 = w_1 r_s$, and note that $l(w_2 r) \geq l(w_2)$ will hold for both $r = r_s$ and $r = r_t$ if $l(w_2 r_t) \geq l(w_2)$. If this latter condition is not satisfied then we define $w_3 = w_2 r_t$. Continuing in this way we find, for some positive integer

k, a sequence of elements $w_0 = w, w_1, w_2, \cdots, w_k$ with $l(w_i) = l(w) - i$ for all $i = 0, 1, 2, \cdots, k$, and, when $i < k$,

$$w_{i+1} = \begin{cases} w_i r_s & \text{if } i \text{ is odd} \\ w_i r_t & \text{if } i \text{ is even} \end{cases}$$

Now since $0 \leq l(w_k) = l(w) - k$, we conclude that $l(w)$ is an upper bound for the possible values of k. Choosing k to be as large as possible, we deduce that $l(w_k r) \geq l(w_k)$ for both $r = r_s$ and $r = r_t$, for otherwise the process described above would allow a w_{k+1} to be found, contrary to the definition of k. By the minimality of our original counterexample it follows that $w\alpha_s$ and $w\alpha_t$ are both in $\mathrm{PLC}(\Pi_1)$.

We have $w = w_k v$, where v is an alternating product of r_s's and r_t's, ending in r_t, and with k factors altogether. Obviously this means that $l(v) \leq k$. But $w = w_k v$ gives $l(w) \leq l(w_k) + l(v)$, so it follows that $l(v) \geq l(w) - l(w_k) = k$, and hence $l(v) = k$. Furthermore, in view of the hypothesis that $l(wr_s) \geq l(w)$, and since $w_k v r_s = w r_s$, we have

$$l(w_k) + l(vr_s) \geq l(wr_s) \geq l(w) = l(w_k) + k = l(w_k) + l(v),$$

and hence $l(vr_s) \geq l(v)$. In particular, v cannot have a reduced expression in which the final factor is r_s, for if so vr_s would have a strictly shorter expression.

Since r_s and r_t satisfy the defining relations of the dihedral group of order $2m_{st}$, it follows that every element of the subgroup generated by r_s and r_t has an expression of length less than $m_{st}+1$ as an alternating product of r_s and r_t. Thus $l(v) \leq m_{st}$. Moreover, if m_{st} is finite then the two alternating products of length m_{st} define the same element; so $l(v)$ cannot equal m_{st}, as v has no reduced expression ending with r_s. Thus Lemma 1.1.11 above yields $v\alpha_s = \lambda_1\alpha_s + \mu_1\alpha_t$ for some nonnegative coefficients λ_1 and μ_1. Hence

$$w\alpha_s = w_k v\alpha_s = w_k(\lambda_1\alpha_s + \mu_1\alpha_t) = \lambda_1 w_k\alpha_s + \mu_1 w_k\alpha_t \in \mathrm{PLC}(\Pi_1)$$

since $w_k\alpha_s, w_k\alpha_t \in \mathrm{PLC}(\Pi_1)$. This contradicts our original choice of w and α_s as a counterexample to the statement of the proposition; so if $w \in W$, $s \in S$, with $l(wr_s) \geq l(w)$, then $w\alpha_s \in \mathrm{PLC}(\Pi_1)$. □

Now we are ready to complete the proof of Theorem 1.1.4:

Proof of Theorem 1.1.4. Suppose, for a contradiction, that the kernel of f is nontrivial, and choose w in the kernel of f with $w \neq 1$. Then $l(w) > 0$, and we may write $w = w'r_s$ for some $s \in S$ and $w' \in W$ with $l(w') = l(w) - 1$. Since $l(w'r_s) > l(w')$, Proposition 1.1.14 yields $w'\alpha_s \in \mathrm{PLC}(\Pi_1)$. But

$$\alpha_s = w\alpha_s = (w'r_s)\alpha_s = w'(r_s\alpha_s) = w'(-\alpha_s) = -w'\alpha_s$$

and hence $0 = \alpha_s + w'\alpha_s \in \mathrm{PLC}(\Pi_1)$, contradicting our original assumption that $0 \notin \mathrm{PLC}(\Pi_1)$ (condition (C6)). Therefore the kernel of f is trivial, as required. □

By Proposition 1.1.9 we may define a homomorphism $g\colon W \to W_2$ satisfying $g(r_s) = \rho_{V_2}(s)$ for all $s \in S$ and obtain an action of W on V_2. Applying exactly the same arguments that lead to Theorem 1.1.4 gives the analogous result for W_2 and V_2:

Theorem 1.1.15. *Suppose that W is any group generated by a set $R := \{r_s \mid s \in S\}$ satisfying the relations $(r_sr_t)^{m_{st}} = 1$ for all $s, t \in S$ such that $m_{st} < \infty$; and suppose that $g\colon W \to W_2$ is a group homomorphism satisfying $g(r_s) = \rho_{V_2}(s)$ for all $s \in S$. Then g is necessarily injective.* □

Combining Theorem 1.1.4 and Theorem 1.1.15, we have now shown that

$$W \cong W_1 \cong W_2,$$

where W is the abstract Coxeter group determined by the Coxeter parameters of our Coxeter datum $\mathscr{C}$. We refer to W_1 and W_2 as the *realizations of* W in V_1 and V_2 respectively.

1.2. Root Systems and Canonical Coefficients

Having established in Section 1.1 that W_1 and W_2 are Coxeter groups, we now wish to develop a theory of root systems for W_1 and W_2.

As above, let W be the abstract Coxeter group determined by the Coxeter datum $\mathscr{C}$, and let $R = \{ r_s \mid s \in S \}$ be its distinguished generating set. Also as above we use the isomorphisms $W \to W_1$ and $W \to W_2$ to define an action of W on V_1 and an action of W on V_2 (so that $r_s x = x - 2\langle x, \beta_s \rangle \alpha_s$ and $r_s y = y - 2\langle \alpha_s, y \rangle \beta_s$ for all $x \in V_1$ and $y \in V_2$, and all $s \in S$). Observe that these actions are faithful, in the sense that if $wx = x$ for all x (in V_1 or V_2) then $w = 1$.

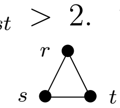
An easy induction on $l(w)$ yields the following extension to Proposition 1.1.3:

Lemma 1.2.1. *Let $x \in V_1$, $y \in V_2$. Then $\langle x, y \rangle = \langle wx, wy \rangle$, for all $w \in W$.* □

The following definition is a natural generalization of the concept of the root system of a Coxeter group, as defined in [**10**] and [**12**], for example.

Definition 1.2.2. (i) Define $\Phi_1 := W(\Pi_1) = \{ w\alpha_s \mid w \in W, s \in S \}$, and $\Phi_2 := W(\Pi_2) = \{ w\beta_s \mid w \in W, s \in S \}$. For each $i \in \{1, 2\}$, we call Φ_i the *root system* for V_i, and its elements the *roots* of W in V_i. We call Π_i the set of *simple roots* in Φ_i, and also refer to Π_i as the *root basis* for Φ_i.

(ii) For each $i \in \{1, 2\}$, set $\Phi_i^+ := \Phi_i \cap \mathrm{PLC}(\Pi)$, and $\Phi_i^- = -\Phi_i^+$. We call Φ_i^+ the set of *positive roots* in Φ_i, and Φ_i^- the set of *negative roots* in Φ_i.

For each $i \in \{1, 2\}$, we adopt the traditional diagrammatic description of simple roots Π_i: draw a graph that has one vertex for each $s \in S$, and join the vertices corresponding to $s, t \in S$ by an edge labelled by m_{st} if $m_{st} > 2$. The label m_{st} is often omitted if $m_{st} = 3$. Thus the diagram 

corresponds to $\Pi_1 = \{ \alpha_r, \alpha_s, \alpha_t \}$

and $\Pi_2 = \{\, \beta_r, \beta_s, \beta_t \,\}$. Unlike in the case of the standard geometric realization of Coxeter groups, this diagram does not uniquely determine the individual values of $\langle \alpha_r, \beta_s \rangle$, $\langle \alpha_s, \beta_r \rangle$, $\langle \alpha_s, \beta_t \rangle$, $\langle \alpha_t, \beta_s \rangle$, $\langle \alpha_r, \beta_t \rangle$ and $\langle \alpha_t, \beta_r \rangle$, instead we can only tell that

$$\langle \alpha_r, \beta_s \rangle \langle \alpha_s, \beta_r \rangle = \langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle = \langle \alpha_t, \beta_r \rangle \langle \alpha_r, \beta_t \rangle = \tfrac{1}{4}.$$

Suppose that we know

$$\begin{aligned} \langle \alpha_r, \beta_s \rangle &= -1/4, & \langle \alpha_s, \beta_r \rangle &= -1, \\ \langle \alpha_s, \beta_t \rangle &= -1/6, & \langle \alpha_t, \beta_s \rangle &= -3/2, \\ \langle \alpha_t, \beta_r \rangle &= -1/10, & \langle \alpha_r, \beta_t \rangle &= -5/2. \end{aligned}$$

Then

$$\begin{aligned} &r_s\alpha_r = \alpha_r - 2\langle \alpha_r, \beta_s \rangle \alpha_s = \alpha_r + \tfrac{1}{2}\alpha_s; \\ &(r_r r_s)\alpha_r = r_r(\alpha_r + \tfrac{1}{2}\alpha_s) = \tfrac{1}{2}\alpha_s; \\ &(r_t r_r r_s)\alpha_r = r_t(\tfrac{1}{2}\alpha_s) = \tfrac{1}{2}\alpha_s + \tfrac{1}{6}\alpha_t; \\ &(r_s r_t r_r r_s)\alpha_r = r_s(\tfrac{1}{2}\alpha_s + \tfrac{1}{6}\alpha_t) = \tfrac{1}{6}\alpha_t; \\ &(r_r r_s r_t r_r r_s)\alpha_r = r_r(\tfrac{1}{6}\alpha_t) = \tfrac{1}{6}\alpha_t + \tfrac{1}{30}\alpha_r; \\ &(r_t r_r r_s r_t r_r r_s)\alpha_r = r_t(\tfrac{1}{6}\alpha_t + \tfrac{1}{30}\alpha_r) = \tfrac{1}{30}\alpha_r. \end{aligned}$$

Remark 1.2.3. We see from the above example that it is possible for a non-trivial positive scalar multiple of a root to also be a root, lying in the same W-orbit as the root itself. Clearly if $w\alpha = \lambda\alpha$, where $1 \neq \lambda \in \mathbb{R}$, then $w^n\alpha = \lambda^n\alpha$, showing that there are infinitely many scalar multiples of α in Φ_1. And of course, all roots in the W-orbit of α will possess this same property. Despite the fact that in this regard Φ_1 and Φ_2 are different from root systems defined in orthogonal geometric realizations, it nevertheless turns out that all major properties of root systems can be generalized to the non-orthogonal setting. We begin with the observation that any root in Φ_i can be expressed as a linear combination of simple roots from Π_i with coefficients being all non-positive or all non-negative:

Lemma 1.2.4. *For each* $i \in \{1,2\}$, $\Phi_i = \Phi_i^+ \uplus \Phi_i^-$, *where* $\uplus$ *denotes disjoint union.*

Proof. The condition (C6), which says that $0 \notin \mathrm{PLC}(\Pi_i)$ (for each $i \in \{1,2\}$), ensures that $\Phi_i^+ \cap \Phi_i^- = \emptyset$.

Let $x \in \Phi_1$. By Definition 1.2.2(i) there exists $w \in W$ and $\alpha_s \in \Pi_1$ with $x = w\alpha_s$. Let $w' = wr_s$, noting that the lengths of w and w' differ by at most 1. Now if $l(w') \geq l(w)$ then Proposition 1.1.14 yields that $x = w\alpha_s \in \Phi_i^+$. On the other hand if $l(w) \geq l(w')$, then Proposition 1.1.14 yields that $w'\alpha_s \in \Phi_i^+$, and this in turn gives $x = wr_s(\alpha_s) = w(-\alpha_s) = -w'\alpha_s \in \Phi_1^-$. This yields that $\Phi_i = \Phi_1^+ \uplus \Phi_1^-$. The same reasoning also shows that $\Phi_2 = \Phi_2^+ \uplus \Phi_2^-$. □

The preceding proof shows, incidentally, that $l(wr_s) = l(w)$ is not a possibility, since a root $z \in \Phi_i$ $(i = 1,2)$ cannot simultaneously be positive and negative. This observation natually leads to the following key combinatorial fact of the action of W on Φ_i, $i = 1.2$:

Corollary 1.2.5. *If* $w \in W$ *and* $s \in S$, *then*

$$l(wr_s) = \begin{cases} l(w) + 1 \text{ if } w\alpha_s \in \Phi_1^+, \text{ and } w\beta_s \in \Phi_2^+, \\ l(w) - 1 \text{ if } w\alpha_s \in \Phi_1^-, \text{ and } w\beta_s \in \Phi_2^-. \end{cases}$$

□

Remark 1.2.6. From the above, we observe that $w\alpha_s \in \Phi_1^+$ if and only if $w\beta_s \in \Phi_1^+$, and $w\alpha_s \in \Phi_1^-$ if and only if $w\beta_s \in \Phi_1^-$, for all $w \in W$ and $s \in S$.

Remark 1.2.7. Since Π_1 and Π_2 need not be linearly independent, although we know from Lemma 1.2.4 that each root in Φ_i is expressible as a linear combination of simple roots from Π_i with coefficients all being of the same sign, that expression need not be unique. Thus the concept of *the coefficient of a simple root in a given root* is potentially ambiguous. The rest of this section is devoted to finding a canonical way of expressing any given root as a linear combination of simple roots, by specifying the *canonical coefficients* of the simple roots in a given root.

Let V_1' be a vector space over $\mathbb{R}$ with basis $\Pi_1' := \{\, \alpha_s' \mid s \in S \,\}$ in bijective correspondence with S, and let V_2' be a vector space over $\mathbb{R}$ with basis $\Pi_2' := \{\, \beta_s' \mid s \in S \,\}$, also in bijective correspondence with S. Define linear maps $\pi_1 \colon V_1' \to V_1$ and $\pi_2 \colon V_2' \to V_2$ by requiring that

$$\pi_1\Big(\sum_{s\in S} \lambda_s \alpha_s'\Big) = \sum_{s\in S} \lambda_s \alpha_s \quad \text{and} \quad \pi_2\Big(\sum_{s\in S} \mu_s \beta_s'\Big) = \sum_{s\in S} \mu_s \beta_s,$$

for all $\lambda_s, \mu_s \in \mathbb{R}$, and define a bilinear map $\langle\ ,\ \rangle \colon V_1' \times V_2' \to \mathbb{R}$ by requiring that

$$\langle \alpha_s', \beta_t' \rangle = \langle \alpha_s, \beta_t \rangle$$

for all s, $t \in S$. Observe that

$$\langle x', y' \rangle = \langle \pi_1(x'), \pi_2(y') \rangle$$

for all $x' \in V_1'$ and $y' \in V_2'$.

With these definitions $\mathscr{C}' = (S, V_1', V_2', \Pi_1', \Pi_2', \langle\ ,\ \rangle)$ is clearly a Coxeter datum having the same parameters as our original Coxeter datum $\mathscr{C}$, and thus corresponding to the same abstract Coxeter system.

For each $s \in S$, define linear transformations $\rho_{V_1'}(s) \colon V_1' \to V_1'$ and $\rho_{V_2'}(s) \colon V_2' \to V_2'$ by

$$\rho_{V_1'}(s)(x') = x' - 2\langle x', \beta_s' \rangle \alpha_s',$$

for all $x' \in V_1'$, and

$$\rho_{V_2'}(s)(y') = y' - 2\langle \alpha_s', y' \rangle \beta_s',$$

for all $y' \in V_2'$, noting that $\pi_i \rho_{V_i'}(s) = \rho_{V_i}(s)\pi_i$ for all $s \in S$ and $i \in \{1, 2\}$. For each $i \in \{1, 2\}$, let $R_i' = \{\, \rho_{V_i'}(s) \mid s \in S \,\}$, and let W_i' be the subgroup of $\mathrm{GL}(V_i')$ generated by R_i'.

Defining (W, R) to be the abstract Coxeter system corresponding to the Coxeter datum $\mathscr{C}'$ (or $\mathscr{C}'$), Theorems 1.1.4 and 1.1.15 yield isomorphisms $f_1' \colon W \to W_1'$ and $f_2' \colon W \to W_2'$, and we use these to define actions of W on V_1' and V_2' via $wz' = (f_i'(w))(z')$ for all $w \in W$ and

$z' \in V_i'$, for each $i \in \{1,2\}$. Note that since by definition $f_i'(r_s) = \rho_{V_i'}(s)$ for all $s \in S$, we have

$$\pi_i f_i'(r_s) = f_i(r_s)\pi_i \qquad \text{for all } s \in S \text{ and } i \in \{1,2\}$$

where $f_1 \colon W \to W_1$ is the f of Theorem 1.1.4 and f_2 is the analogous isomorphism $W \to W_2$. Since W is generated by $\{\, r_s \mid s \in S \,\}$ it follows that $\pi_i f_i'(w) = f_i(w)\pi_i$ for all $w \in W$ and $i \in \{1,2\}$. That is,

$$\pi_i(wz') = w\pi_i(z') \qquad \text{for all } w \in W \text{ and } z' \in V_i' \tag{1.2.1}$$

for each $i \in \{1,2\}$. In other words, π_1 and π_2 are W-module homomorphisms.

Definition 1.2.8. Define $\Phi_1' := \{\, w\alpha_s' \mid w \in W \text{ and } s \in S\}$ (the root system for W in V_1') and $\Phi_2' := \{\, w\beta_s' \mid w \in W \text{ and } s \in S\}$ (the root system for W in V_2').

Since Π_1' and Π_2' are linearly independent, the expressions of roots in Φ_1' and Φ_2' in terms of Π_1' and Π_2' are unique. In particular, for each $i \in \{1,2\}$, the coefficient of a given simple root in Π_i' in any root of Φ_i' is uniquely determined. We will utilize this fact to specify a preferred way of expressing roots in Φ_1 and Φ_2 as linear combinations of elements from Π_1 and Π_2. The following proposition will be a key step:

Proposition 1.2.9. *For each $i \in \{1,2\}$, the restriction of π_i defines a W-equivariant bijection $\Phi_i' \to \Phi_i$.*

Notation: Define $\phi_1 \colon \Phi_1 \to \Phi_1'$ and $\phi_2 \colon \Phi_2 \to \Phi_2'$ to be the inverses of the bijections guaranteed by Proposition 1.2.9.

To prove Proposition 1.2.9, we need a few elementary results and some further notation.

Definition 1.2.10. For each $i \in \{1,2\}$, define an equivalence relation $\sim_i$ on Φ_i as follows: if $z_1, z_2 \in \Phi_i$ then $z_1 \sim_i z_2$ if and only if z_1 and z_2 are (nonzero) scalar multiples of each other. For each $z \in \Phi_i$, write $\widehat{z}$ for the equivalence class containing z, and set $\widehat{\Phi}_i = \{\, \widehat{z} \mid z \in \Phi_i \,\}$.

Observe that the action of W on Φ_i (for $i = 1, 2$) gives rise to a well-defined action of W on $\widehat{\Phi}_i$ satisfying $w\widehat{z} = \widehat{wz}$ for all $w \in W$ and all $z \in \Phi_i$.

Definition 1.2.11. For each $w \in W$, define

$$N_1(w) := \{\, \widehat{\alpha} \mid \alpha \in \Phi_1^+ \text{ and } w(\alpha) \in \Phi_1^- \,\}$$

and

$$N_2(w) := \{\, \widehat{\beta} \mid \beta \in \Phi_2^+ \text{ and } w(\beta) \in \Phi_2^- \,\}.$$

Observe that $N_1(w)$ and $N_2(w)$ can alternatively be characterised as follows:

$$N_1(w) := \{\, \widehat{\alpha} \mid \alpha \in \Phi_1^- \text{ and } w(\alpha) \in \Phi_1^+ \,\}$$

and

$$N_2(w) := \{\, \widehat{\beta} \mid \beta \in \Phi_2^- \text{ and } w(\beta) \in \Phi_2^+ \,\}.$$

Thus if $\widehat{z} \in \widehat{\Phi}_i$, then $\widehat{z} \in N_i(w)$ if and only if one of z and wz is positive and the other negative.

Lemma 1.2.12. (i) *If $s \in S$, then $N_1(r_s) = \{\widehat{\alpha_s}\}$ and $N_2(r_s) = \{\widehat{\beta_s}\}$.*
(ii) *Let $w \in W$. Then $N_1(w)$ and $N_2(w)$ both have cardinality $l(w)$.*
(iii) *Let $w_1, w_2 \in W$ and let $\dotplus$ denote set symmetric difference. Then*

$$N_i(w_1w_2) = w_2^{-1}N_i(w_1) \dotplus N_i(w_2) \qquad \text{for each } i \in \{1,2\}.$$

(iv) *Let $w_1, w_2 \in W$. Then for each $i \in \{1,2\}$,*

$$l(w_1w_2) = l(w_1) + l(w_2) \text{ if and only if } N_i(w_2) \subseteq N_i(w_1w_2).$$

Proof. (i) Let $s \in S$. Clearly $r_s(\alpha_s) = \alpha_s - 2\langle\alpha_s, \beta_s\rangle\alpha_s = -\alpha_s$, and so $\widehat{\alpha_s} \in N_1(r_s)$. Now let $\alpha \in \Phi_1^+$, and write $\alpha = \sum_{r\in S} \lambda_r\alpha_r$ with $\lambda_r \geq 0$ for all $r \in S$. Suppose that $r_s(\alpha) \in \Phi_1^-$; that is, $\alpha - 2\langle\alpha, \beta_s\rangle\alpha_s = -\sum_{r\in S} \mu_r\alpha_r$, for some μ_r with $\mu_r \geq 0$ for all $r \in S$. Then

$$0 = (\lambda_s + \mu_s - 2\langle\alpha, \beta_s\rangle)\alpha_s + \sum_{r\in S\setminus\{s\}} (\lambda_r + \mu_r)\alpha_r. \tag{1.2.2}$$

Now apply $\langle\ ,\beta_s\rangle$ to this expression. Since $\langle\alpha_s,\beta_s\rangle = 1$ and $\langle\alpha_r,\beta_s\rangle \le 0$ whenever r is in $S\setminus\{s\}$, we conclude that $\lambda_s+\mu_s-2\langle\alpha,\beta_s\rangle$ must be nonnegative. But this gives $0\in \mathrm{PLC}(\Pi_1)$, contradicting (C6) of Definition 1.1.1, unless all coefficients in (1.2.2) are zero. Thus $\lambda_r=\mu_r=0$ for all $r\in S\setminus\{s\}$, forcing α to be a positive scalar multiple of α_s. Hence the only positive roots in Φ_1 made negative by applying r_s are of the form $\lambda\alpha_s$, where λ is positive, that is, $N_1(r_s)=\{\widehat{\alpha_s}\}$.

Exactly the same reasoning gives that $N_2(r_s)=\{\widehat{\beta_s}\}$, and this completes the proof of (i).

(ii) If $w\in W$, define $n_1(w)=|N_1(w)|$ and $n_2(w)=|N_2(w)|$. We shall use induction on $l(w)$ to prove that $n_1(w)=l(w)$ for all $w\in W$. Exactly similar arguments can be used to prove that $n_2(w)=l(w)$ for all $w\in W$.

If $l(w)=0$ then $w=1$ and clearly $n_1(1)=l(1)=0$. Now assume that $l(w)>0$. Then there exist $s\in S$ and $w'\in W$ such that $w=w'r_s$ and $l(w')=l(w)-1$. If we can prove that

$$n_1(w')=n_1(w)-1 \tag{1.2.3}$$

then it will follow from the inductive hypothesis that

$$n_1(w)=n_1(w')+1=l(w')+1=l(w)$$

as required. Observe that (1.2.3) will follow if we can prove that

$$r_s(N_1(w)\setminus\{\widehat{\alpha_s}\})=N_1(w').$$

Let $\widehat{x}\in N_1(w')$, choosing the representative root x to be positive. Observe that $\widehat{x}\ne\widehat{\alpha_s}$, since $l(w'r_s)>l(w')$ and hence $w'\alpha_s\in\Phi_1^+$ by Corollary 1.2.5. So part (i) yields that $r_sx\in\Phi_1^+$. Moreover, $w(r_sx)=w'r_sr_sx=w'x\in\Phi_1^-$ (since $\widehat{x}\in N_1(w')$). Hence $r_s\widehat{x}\in N_1(w)$, and $\widehat{x}\in r_sN_1(w)$. But clearly $\widehat{x}\ne r_s\widehat{\alpha_s}$, and so it follows that

$$N_1(w')\subseteq r_s(N_1(w)\setminus\{\widehat{\alpha_s}\}).$$

Conversely, let $\widehat{x} \in N_1(w) \setminus \{\widehat{\alpha_s}\}$ with $x \in \Phi_1^+$. By part (i) $r_s x \in \Phi_1^+$; moreover, $w' r_s x = w r_s r_s x = wx \in \Phi_1^-$ (since $\widehat{x} \in N_1(w)$). Hence

$$r_s(N_1(w) \setminus \{\widehat{\alpha_s}\}) \subseteq N_1(w').$$

Therefore $r_s(N_1(w) \setminus \{\widehat{\alpha_s}\}) = N_1(w')$, as required. This completes the proof of (ii).

(iii) Suppose that $\widehat{x} \in N_1(w_1 w_2)$, where we may assume that $x \in \Phi_1^+$ without loss of generality. Observe that this gives $w_1 w_2 x \in \Phi_1^-$.

If $w_2 x \in \Phi_1^+$ then $\widehat{x} \notin N_1(w_2)$, since x and $w_2 x$ are both positive; furthermore, $\widehat{w_2 x} \in N_1(w_1)$, since $w_2 x$ is positive and $w_1 w_2 x$ is negative. So $\widehat{x} \in w_2^{-1} N_1(w_1) \dot{+} N_1(w_2)$. On the other hand, if $w_2 x \in \Phi_1^-$ then $\widehat{x} \in N_1(w_2)$ and $\widehat{w_2 x} \notin N_1(w_1)$, again giving $\widehat{x} \in w_2^{-1} N_1(w_1) \dot{+} N_1(w_2)$. Since $\widehat{x}$ was chosen arbitrarily it follows that

$$N_1(w_1 w_2) \subseteq w_2^{-1} N_1(w_1) \dot{+} N_1(w_2).$$

Conversely, suppose that $\widehat{x} \in w_2^{-1} N_1(w_1) \dot{+} N_1(w_2)$, choosing $x \in \Phi_1^+$. Note that either $\widehat{x} \in N_1(w_2)$ and $\widehat{x} \notin w_2^{-1} N_1(w_1)$, giving $\widehat{x} \in N_1(w_2)$ and $w_2 \widehat{x} \notin N_1(w_1)$, or else $\widehat{x} \notin N_1(w_2)$ and $\widehat{x} \in w_2^{-1} N_1(w_1)$, in which case $\widehat{x} \notin N_1(w_2)$ and $w_2 \widehat{x} \in N_1(w_1)$.

If $\widehat{x} \in N_1(w_2)$ and $w_2 \widehat{x} \notin N_1(w_1)$, then $w_2 x$ must be opposite in sign to x, and $w_1(w_2 x)$ must be of the same sign as $w_2 x$. Thus $w_2 x \in \Phi_1^-$ and $w_1 w_2 x \in \Phi_1^-$. On the other hand, if $\widehat{x} \notin N_1(w_2)$ and $w_2 \widehat{x} \in N_1(w_1)$ then $w_2 x$ must be of the same sign as x and $w_1(w_2 x)$ must be opposite in sign to $w_2 x$. Thus $w_2 x \in \Phi_1^+$ and $w_1 w_2 x \in \Phi_1^-$. So $w_1 w_2 x \in \Phi_1^-$ in either case, and since $\widehat{x} \in w_2^{-1} N_1(w_1) \dot{+} N_1(w_2)$ was arbitrary it follows that

$$w_2^{-1} N_1(w_1) \dot{+} N_1(w_2) \subseteq N_1(w_1 w_2).$$

Since this reverses the inclusion proved above we conclude that equality holds, and since exactly similar arguments apply for N_2, this completes the proof of (iii).

(iv) By part (ii) above $l(w_1 w_2) = l(w_1) + l(w_2)$ if and only if $|N_1(w_1 w_2)| = |N_1(w_1)| + |N_1(w_2)| = |w_2^{-1} N_1(w_1)| + |N_1(w_2)|$. By part (iii) this happens if and only if $w_2^{-1}N_1(w_1) \cap N_1(w_2) = \emptyset$, and this in turn happens if and only if $N_1(w_2) \subseteq N_1(w_1w_2)$. As usual, the same reasoning applies for N_2. $\square$

If $w = r_{s_1}r_{s_2}\cdots r_{s_l}$ $(s_1, \cdots, s_l \in S)$ is such that $l(w) = l$, then it is easy to see that

$$N_1(w) = \{\widehat{\alpha_{s_l}},\, r_{s_l}\widehat{\alpha_{s_{l-1}}},\, r_{s_l}r_{s_{l-1}}\widehat{\alpha_{s_{l-2}}},\, \ldots,\, r_{s_l}r_{s_{l-1}}\ldots r_{s_2}\widehat{\alpha_{s_1}}\} \tag{1.2.4}$$

and

$$N_2(w) = \{\widehat{\beta_{s_l}},\, r_{s_l}\widehat{\beta_{s_{l-1}}},\, r_{s_l}r_{s_{l-1}}\widehat{\beta_{s_{l-2}}},\, \ldots,\, r_{s_l}r_{s_{l-1}}\ldots r_{s_2}\widehat{\beta_{s_1}}\}. \tag{1.2.5}$$

If $w_1\,, w_2 \in W$ such that $l(w_1w_2) = l(w_1) + l(w_2)$, then we call w_2 a *right hand segment* of w_1w_2.

Lemma 1.2.13. *W is finite if and only if $\widehat{\Phi_1}$ if finite, and if and only if $\widehat{\Phi_2}$ is finite.*

Proof. Since $|\Pi_1| = |S| = |\{\, r_s \mid s \in S\,\}| \leq |W|$, it follows that $|\Pi_1| < \infty$ whenever $|W| < \infty$. Then

$$\begin{aligned} |\Phi_1| &= |\{\, \widehat{wx} \mid w \in W \text{ and } x \in \Pi_1\,\}| \\ &\leq |\{\, wx \mid w \in W \text{ and } x \in \Pi_1\,\}| \\ &\leq |W||\Pi_1| \\ &< \infty. \end{aligned}$$

Conversely, assume that $|\widehat{\Phi_1}| < \infty$. Define an equivalence relation $\sim$ on Φ_1 as follows: for $x_1,\, x_2 \in \Phi_1$, $x_1 \sim x_2$ if there is a positive λ such that $x_1 = \lambda x_2$. For each $x \in \Phi_1$, write $\widetilde{x}$ for the equivalence class containing x and set $\widetilde{\Phi_1} := \{\, \widetilde{x} \mid x \in \Phi_1\,\}$. Since $-x$ is a root whenever x is a root, $|\widetilde{\Phi_1}| = 2|\widehat{\Phi_1}| < \infty$. The action of W on Φ_1 naturally induces a well-defined action of W on $\widetilde{\Phi_1}$ satisfying $w\widetilde{x} := \widetilde{wx}$. Now for each $w \in W$ define a map $\sigma_w \colon \widetilde{\Phi_1} \to \widetilde{\Phi_1}$ by $\sigma_w(\widetilde{x}) := \widetilde{wx}$ for all $\widetilde{x} \in \widetilde{\Phi_1}$. Then σ_w is a permutation of $\widetilde{\Phi_1}$, and furthermore, $w \mapsto \sigma_w$ is a homomorphism $\sigma : W \to \mathrm{Sym}(\widetilde{\Phi_1})$ (the symmetric group on $\widetilde{\Phi_1}$). Now if w is in the kernel of σ then $w\widetilde{x} = \widetilde{x}$ for all $x \in \Phi_1$, and in particular, $w\widetilde{x} = \widetilde{x}$ for all $x \in \Pi_1$. But by Corollary 1.2.5 this means that $l(wr_s) > l(w)$ for all $s \in S$, and therefore $w = 1$. Thus σ is injective, and

$$|W| \leq |\operatorname{Sym}(\widetilde{\Phi_1})| = |\widetilde{\Phi_1}|\,! < \infty$$

as required.

And as usual, exactly similar arguments yield that W is finite if and only if $\widehat{\Phi_2}$ is finite. □

Let $K \subseteq S$. If we define V_{1K} to be the subspace of V_1 spanned by $\Pi_{1K} = \{\, \alpha_s \mid s \in K \,\}$ and V_{2K} to be the subspace of V_2 spanned by $\Pi_{2K} = \{\, \beta_s \mid s \in K \,\}$, and let $\langle\ ,\ \rangle_K$ be the restriction of $\langle\ ,\ \rangle$ to $V_{1K} \times V_{2K}$, then clearly $(K, V_{1K}, V_{2K}, \Pi_{1K}, \Pi_{2K}, \langle\ ,\ \rangle_K)$ is a Coxeter datum with parameters $(\, m_{st} \mid s, t \in K \,)$. Write $W_K = \langle\, r'_s \mid s \in K \,\rangle$ for the corresponding abstract Coxeter group, and let $\eta\colon W_K \to W$ be the homomorphism defined by $r'_s \mapsto r_s$ for all $s \in K$. It follows immediately from the formulas for the actions of W on V_1 and W_K on V_{1K} that $r'_s v = r_s v$ for all $s \in K$ and $v \in V_{1K}$, and therefore $wv = \eta(w)v$ for all $w \in W_K$ and $v \in V_{1K}$. Since the action of W_{1K} on V_{1K} is faithful, it follows that η is injective. Thus W_K can be identified with the *standard parabolic subgroup* of W generated by the set $\{\, r_s \mid s \in K \,\}$.

Definition 1.2.14. Given $K \subseteq S$, we define Φ_{1K} and Φ_{2K} to be the root systems of W_K in V_{1K} and V_{2K} respectively, and Φ^+_{1K}, Φ^+_{2K} to be the corresponding sets of positive roots.

In other words,

$$\Phi_{1K} = \{\, w\alpha_r \mid w \in W_K \text{ and } r \in K \,\}$$

and

$$\Phi^+_{1K} = \Phi_{1K} \cap \operatorname{PLC}(\Pi_{1K}),$$

and similarly for Φ_{2K} and Φ^+_{2K}.

Remark 1.2.15. It is a consequence of Lemma 1.2.13 that W_K is finite if and only if $\widehat{\Phi_{1K}}$ and $\widehat{\Phi_{2K}}$ are finite.

It is well known (and in fact follows easily from Corollary 1.2.5 and Lemma 1.2.12) that if W_K is finite then there is a unique longest element $w_K \in W_K$, satisfying $N_1(w_K) = \{\, \widehat{\alpha} \mid \alpha \in \Phi_{1K} \,\}$. However, for our present purposes we require this only in the special case that K has cardinality 2.

Notation: Let r, $s \in S$ be such such $m_{rs} < \infty$. Define $w_{\{r,s\}} \in \langle r_r, r_s \rangle$ to be the element $r_r r_s r_r \cdots = r_s r_r r_s \cdots$, where there are m_{rs} factors on each side, the factors being alternately r_r and r_s.

Lemma 1.2.16. *Let $w \in W$ and $s \in S$ be such that $w\alpha_s = \nu\alpha_t$ for some positive scalar ν, and suppose that $r \in S$ such that $l(wr_r) < l(w)$. Then*

(i) $\langle r_r, r_s \rangle$ *is finite.*
(ii) $N_1(w_{\{r,s\}} r_s) = \widehat{\Phi_{1\{\,r,s\,\}}} \setminus \{\widehat{\alpha_s}\}$.
(iii) $l(wr_s w_{\{r,s\}}^{-1}) = l(w) - l(w_{\{r,s\}} r_s)$.

Proof. (i) Since $l(wr_r) < l(w)$, Corollary 1.2.5 yields that $w\alpha_r \in \Phi_1^-$. Furthermore, $w\widehat{\alpha_r} \neq \widehat{\alpha_t}$, for otherwise $\widehat{\alpha_r} = \widehat{\alpha_s}$, forcing $r = s$ by Lemma 1.1.5, contrary to the fact that $w\alpha_s$ is positive and $w\alpha_r$ is negative. Now observe that

$$\begin{aligned} wr_s(\alpha_r) &= w(\alpha_r - 2\langle \alpha_r, \beta_s \rangle \alpha_s) = w(\alpha_r) - 2\langle \alpha_r, \beta_s \rangle w(\alpha_s) \\ &= \underbrace{w(\alpha_r)}_{\in \Phi_1^- \setminus \mathbb{R}\{\alpha_t\}} - \underbrace{2\langle \alpha_r, \beta_s \rangle \alpha_t}_{\text{a scalar multiple of } \alpha_t}, \end{aligned}$$

and assume, for a contradiction, that $wr_s(\alpha_r) \in \mathrm{PLC}(\Pi_1)$. Rearranging the above equation gives

$$\underbrace{wr_s(\alpha_r) - w(\alpha_r)}_{\in \mathrm{PLC}(\Pi_1)} = \underbrace{-2\langle \alpha_r, \beta_s \rangle \alpha_t}_{\text{a positive scalar multiple of } \alpha_t},$$

and, since $-w(\alpha_r) \in (\Phi_1^+ \setminus \mathbb{R}\{\alpha_t\})$, the left hand side is not a scalar multiple of α_t. So, moving the α_t term from the left side to the right, this expresses $\lambda\alpha_t$, for some scalar λ, as a positive linear combination of $\Pi_1 \setminus \{\alpha_t\}$. But if $\lambda \leq 0$ this implies that $0 \in \mathrm{PLC}(\Pi_1)$, contradicting condition (C6) of a Coxeter datum, while if $\lambda > 0$ it implies that $\alpha_t \in \mathrm{PLC}(\Pi_1 \setminus \{\alpha_t\})$ contradicting Lemma 1.1.5. Therefore $wr_s(\alpha_r) \in \Phi_1^-$. Since both $wr_s(\alpha_r)$ and $wr_s(\alpha_s) = -\alpha_t$ are negative, it follows that $wr_s(\lambda\alpha_r + \mu\alpha_s)$ is a negative linear combination of Π_1 whenever $\lambda, \mu \geq 0$. Hence $wr_s(\lambda\alpha_r + \mu\alpha_s) \in \Phi_1^-$ whenever $\lambda\alpha_r + \mu\alpha_s \in \Phi^+_{1\{r,s\}}$. This says precisely that $\widehat{\Phi_{1\{r,s\}}} \subseteq N_1(wr_s)$. Since $N_1(wr_s)$ is a finite set of size $l(wr_s)$ by Lemma 1.2.12(ii), it follows from Remark 1.2.15 above that $\langle r_r, r_s\rangle$ must be finite. Therefore $w_{\{r,s\}}$ exists.

(ii) Since $w_{\{r,s\}}$ is the longest element in $W_{\{r,s\}}$ it follows that $l(w_{\{r,s\}}r_r) < l(w_{\{r,s\}})$ and $l(w_{\{r,s\}}r_s) < l(w_{\{r,s\}})$. So Corollary 1.2.5 yields that $w_{\{r,s\}}\alpha_r \in \Phi_1^-$ and $w_{\{r,s\}}\alpha_s \in \Phi_1^-$. This then implies that $\widehat{\Phi_{1\{r,s\}}} \subseteq N_1(w_{\{r,s\}})$. On the other hand $w_{\{r,s\}} = \underbrace{r_r r_s r_r \cdots}_{m_{rs} \text{ factors}} = \underbrace{r_s r_r r_s \cdots}_{m_{rs} \text{ factors}}$, so it follows from Lemma 1.2.12 (i) and the repeated application of Lemma 1.2.12 (iii) that $N_1(w_{\{r,s\}}) \subseteq \widehat{\Phi_{1\{r,s\}}}$ whence $N_1(w_{\{r,s\}}) = \widehat{\Phi_{1\{r,s\}}}$. Observe that $w_{\{r,s\}}r_s\alpha_s = -w_{\{r,s\}}\alpha_s \in \Phi_1^+$, and so

$$\widehat{\Phi_{1\{r,s\}}} \setminus \{\widehat{\alpha}\} \subseteq N_1(w_{\{r,s\}}r_s).$$

Thus to show the desired result, we only need to establish that

$$|N_1(w_{\{r,s\}}r_s)| = |\widehat{\Phi_{1\{r,s\}}}| - 1.$$

Indeed, Lemma 1.2.12 (ii) yields that $|N_1(w_{\{r,s\}}r_s)| = l(w_{\{r,s\}}r_s)$, and we have already checked that $l(w_{\{r,s\}}r_s) = l(w_{\{r,s\}}) - 1$, thus

$$|N_1(w_{\{r,s\}}r_s)| = l(w_{\{r,s\}}r_s) = l(w_{\{r,s\}}) - 1 = |\widehat{\Phi_{1\{r,s\}}}| - 1.$$

(iii) To show that $w_{\{r,s\}}r_s$ is a right hand segment of w, it is enough to show that $N_1(w_{\{r,s\}}r_s) \subseteq N_1(w)$ by Lemma 1.2.12(iv). So let α be an arbitrary positive root such that $\widehat{\alpha} \in N_1(w_{\{r,s\}}r_s)$, and observe by part (ii) that $\alpha = \lambda\alpha_r + \mu\alpha_s$ for some $\lambda > 0$ and $\mu \geq 0$. Thus

$$w\alpha = \lambda w\alpha_r + \mu w\alpha_s = \lambda w\alpha_r + \mu\nu\alpha_t.$$

Suppose for a contradiction that $w\alpha \in \Phi_1^+$. Then

$$\underbrace{w\alpha - \lambda w(\alpha_r)}_{\in \operatorname{PLC}(\Pi_1)} = \underbrace{\mu\nu\alpha_t}_{\text{a positive scalar multiple of } \alpha_t},$$

and, since $-w(\alpha_r) \in (\Phi_1^+ \setminus \mathbb{R}\{\alpha_t\})$, the left hand side is not a scalar multiple of α_t. So, moving the α_t term from the left side to the right, this expresses $\lambda'\alpha_t$, for some scalar λ', as a positive linear combination of $\Pi_1 \setminus \{\alpha_t\}$. But if $\lambda' \leq 0$ this implies that $0 \in \operatorname{PLC}(\Pi_1)$, contradicting condition (C6) of a Coxeter datum, while if $\lambda' > 0$ it implies that $\alpha_t \in \operatorname{PLC}(\Pi_1 \setminus \{\alpha_t\})$ contradicting Lemma 1.1.5. Therefore $w\alpha \in \Phi_1^-$, and therefore $\widehat{\alpha} \in N_1(w)$. Hence all elements of $N_1(w_{\{r,s\}}r_s)$ lie in $N_1(w)$, as required. □

Now we are ready to prove Proposition 1.2.9:

Proof of Proposition 1.2.9. Since $\Phi_1' = \{\, w\alpha_s' \mid w \in W \text{ and } s \in S \,\}$, to prove that the restriction of π_1 to Φ_1' is bijective it suffices to show that if $\pi_1(w\alpha_s') = \pi_1(v\alpha_t')$ for some $w, v \in W$ and $s, t \in S$, then $w\alpha_s' = v\alpha_t'$. Since π_1 is a W-homomorphism and $\pi_1(\alpha_s') = \alpha_s$ for all $s \in S$ we see that $\pi_1(w\alpha_s') = w\alpha_s$ and $\pi_1(v\alpha_t') = v\alpha_t$, and deduce that it suffices to prove the following statement: if $w\alpha_s = \alpha_t$ for some $w \in W$ and $s, t \in S$, then $w\alpha_s' = \alpha_t'$.

We assume that $w(\alpha_s) = \alpha_t$, and proceed by an induction on $l(w)$. The case $l(w) = 0$ reduces to the statement that if $\alpha_s = \alpha_t$ for some $s, t \in S$ then $\alpha_s' = \alpha_t'$, which is trivially true since Π_1 and Π_1' are both assumed to be in bijective correspondence with S. So we may assume that $l(w) > 0$, and choose $r \in S$ such that $l(wr_r) < l(w)$. Lemma 1.2.16 yields that $\langle r_r, r_s \rangle$ is a finite dihedral group (hence m_{rs} is finite), and

$$l(w(w_{\{r,s\}}r_s)^{-1}) = l(w) - l(w_{\{r,s\}}r_s).$$

We treat separately the cases m_{rs} even and m_{rs} odd.

If $m_{rs} = 2k$ is even, then $w_{\{r,s\}} = (r_r r_s)^k = (r_s r_r)^k$, and then the formulas in Lemma 1.1.6 yield

(1.2.6)

$$\begin{aligned}(w_{\{r,s\}} r_s)\alpha_s &= -w_{\{r,s\}}\alpha_s = -(r_s r_r)^k \alpha_s \\ &= -\frac{\sin((m_{st}+1)\pi/m_{st})}{\sin(\pi/m_{st})}\alpha_s - \frac{-\cos(\pi/m_{st})}{\langle \alpha_r, \beta_s\rangle}\frac{\sin(\pi)}{\sin(\pi/m_{st})}\alpha_r \\ &= \alpha_s,\end{aligned}$$

and by exactly the same calculation in V_1',

$$w_{\{r,s\}} r_s(\alpha_s') = \alpha_s'. \tag{1.2.7}$$

Now since

$$\alpha_t = w\alpha_s = w(w_{\{r,s\}} r_s)^{-1} \underbrace{(w_{\{r,s\}} r_s)(\alpha_s)}_{=\alpha_s \text{ by } (1.2.6)} = w(w_{\{r,s\}} r_s)^{-1}(\alpha_s)$$

and $l(w(w_{\{r,s\}} r_s)^{-1}) < l(w)$, the inductive hypothesis yields that

$$\alpha_t' = w(w_{\{r,s\}} r_s)^{-1}(\alpha_s'). \tag{1.2.8}$$

It follows that

$$w(\alpha_s') = w(w_{\{r,s\}} r_s)^{-1} \underbrace{(w_{\{r,s\}} r_s)(\alpha_s')}_{=\alpha_s' \text{ by } (1.2.7)} = w(w_{\{r,s\}} r_s)^{-1}(\alpha_s') \underbrace{= \alpha_t'}_{\text{by } (1.2.8)}$$

as required.

If $m_{st} = 2k+1$ is odd, then $w_{\{r,s\}} r_s = \underbrace{\ldots r_r r_s r_r}_{(m_{rs}-1) \text{ factors}} = (r_s r_r)^k$. Now the formulas in Lemma 1.1.6 yield that

(1.2.9)

$$\begin{aligned}(w_{\{r,s\}} r_s)\alpha_s &= (r_s r_r)^k \alpha_s \\ &= \frac{\sin(\pi)}{\sin(\pi/(m_{st}))}\alpha_s + \frac{-\cos(\pi/(m_{st}))}{\langle \alpha_r, \beta_s\rangle}\frac{\sin(2k\pi/(m_{st}))}{\sin(\pi/(m_{st}))}\alpha_r \\ &= \frac{-\cos(\pi/(m_{st}))}{\langle \alpha_r, \beta_s\rangle}\alpha_r,\end{aligned}$$

and by exactly the same calculation in V_1',

$$(w_{\{r,s\}}r_s)\alpha_s' = \frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s\rangle}\alpha_r'. \tag{1.2.10}$$

Now since

$$\begin{aligned} \alpha_t &= w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha_s \\ &= w(w_{\{r,s\}}r_s)^{-1}\left(\frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s\rangle}\alpha_r\right) \qquad \text{(by (1.2.9))} \end{aligned} \tag{1.2.11}$$

and $l(w(w_{\{r,s\}}r_s)^{-1}) < l(w)$, the inductive hypothesis yields that

$$\alpha_t' = w(w_{\{r,s\}}r_s)^{-1}\left(\frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s\rangle}\alpha_r'\right). \tag{1.2.12}$$

It follows that

$$\begin{aligned} w\alpha_s' &= w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha_s' \\ &= w(w_{\{r,s\}}r_s)^{-1}\left(\frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s\rangle}\alpha_r'\right) && \text{(by (1.2.10))} \\ &= \alpha_t' && \text{(by (1.2.12))} \end{aligned}$$

as required.

The other half of the proposition follows by similar arguments applied in V_2 and V_2'. □

While we are at it, we can also prove the following:

Proposition 1.2.17. *Suppose that $s, t \in S$. If $w(\alpha_s) = \nu\alpha_t$ for some $w \in W$ and some non-zero ν, then $w(\beta_s) = \frac{1}{\nu}\beta_t$.*

Proof. If $l(w) = 0$, then $s = t$ and $\nu = 1$, and there is nothing to prove. Thus we may assume that $l(w) \geq 1$ and proceed by an induction on $l(w)$. Choose $r \in S$, such that $l(wr_r) = l(w) - 1$. Again by Lemma 1.2.16 $\langle r_r, r_s\rangle$ is finite (and m_{rs} is finite too), and hence $w_{\{r,s\}}$ exists. The same rank 2 calculation as used in the proof of Proposition 1.2.9 above yields that

$$(w_{\{r,s\}}r_s)(\alpha_s) = \begin{cases} \alpha_s & \text{if } m_{rs} \text{ is even,} \\ \frac{-\cos(\pi/m_{rs})}{\langle \alpha_r, \beta_s\rangle}\alpha_r & \text{if } m_{rs} \text{ is odd.} \end{cases} \tag{1.2.13}$$

Similarly

$$(w_{\{r,s\}}r_s)(\beta_s) = \begin{cases} \beta_s & \text{if } m_{rs} \text{ is even,} \\ \frac{-\cos(\pi/m_{rs})}{\langle\alpha_s,\beta_r\rangle}\beta_r & \text{if } m_{rs} \text{ is odd.} \end{cases} \tag{1.2.14}$$

First suppose that m_{rs} is even. Since

$$\nu\alpha_t = w\alpha_s = w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)(\alpha_s) = \underbrace{w(w_{\{r,s\}}r_s)^{-1}(\alpha_s)}_{\text{by (1.2.13)}}$$

and $l(w(w_{\{r,s\}}r_s)^{-1}) < l(w)$, the inductive hypothesis yields that

$$\tfrac{1}{\nu}\beta_t = w(w_{\{r,s\}}r_s)^{-1}(\beta_s). \tag{1.2.15}$$

It follows that

$$w\beta_s = w(w_{\{r,s\}}r_s)^{-1}\underbrace{(w_{\{r,s\}}r_s)(\beta_s)}_{=\beta_s \text{ by (1.2.14)}} = w(w_{\{r,s\}}r_s)^{-1}(\beta_s) \underbrace{=\tfrac{1}{\nu}\beta_t}_{\text{by (1.2.15)}}$$

and hence the desired result follows by induction.

Next suppose that m_{rs} is odd. Since

$$\begin{aligned} \nu\alpha_t = w\alpha_s &= w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha_s \\ &= \underbrace{w(w_{\{r,s\}}r_s)^{-1}(\tfrac{-\cos(\pi/m_{rs})}{\langle\alpha_r,\beta_s\rangle}\alpha_r)}_{\text{by (1.2.14)}}, \end{aligned}$$

or, equivalently,

$$-\nu\tfrac{\cos(\pi/m_{rs})}{\langle\alpha_s,\beta_r\rangle}\alpha_t = w(w_{\{r,s\}}r_s)^{-1}\alpha_r,$$

and since $l(w(w_{\{r,s\}}r_s)^{-1}) < l(w)$, the inductive hypothesis yields that

$$-\tfrac{\cos(\pi/m_{rs})}{\nu\langle\alpha_r,\beta_s\rangle}\beta_t = w(w_{\{r,s\}}r_s)^{-1}\beta_r.$$

That is

$$\tfrac{1}{\nu}\beta_t = w(w_{\{r,s\}}r_s)^{-1}(\tfrac{-\cos(\pi/m_{rs})}{\langle\alpha_s,\beta_r\rangle}\beta_r), \tag{1.2.16}$$

and it follows that

$$\begin{aligned} w\beta_s &= w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\beta_s \\ &= \underbrace{w(w_{\{r,s\}}r_s)^{-1}\tfrac{-\cos(\pi/m_{rs})}{\langle\alpha_s,\beta_r\rangle}\beta_r}_{\text{by (1.2.14)}} \underbrace{=\tfrac{1}{\nu}\beta_t}_{\text{by (1.2.16)}} \end{aligned}$$

whence the desired result follows by induction. □

It follows immediately from Proposition 1.2.17 that there is a well-defined map $\Phi_1 \to \Phi_2$ such that $w\alpha_s \mapsto w\beta_s$ for all $s \in S$ and $w \in W$. This is clearly the unique W-equivariant map $\Phi_1 \to \Phi_2$ satisfying $\alpha_s \mapsto \beta_s$ for all $s \in S$.

Definition 1.2.18. Let $\phi : \Phi_1 \to \Phi_2$ be the W-equivariant map satisfying $\phi(\alpha_s) = \beta_s$ for all $s \in S$.

It follows readily from Proposition 1.2.17 and W-equivariance that ϕ is a bijection and the following holds:

Corollary 1.2.19. *Given $t \in S$ and a nonzero scalar ν, then $\nu\alpha_t \in \Phi_1$ if and only if $\frac{1}{\nu}\beta_t \in \Phi_2$. Furthermore, if $\nu\alpha_t \in \Phi_1$ then $\phi(\nu\alpha_t) = \frac{1}{\nu}\beta_t$.* □

In fact we can generalize the above, as follows.

Lemma 1.2.20. *Suppose that $\alpha \in \Phi_1$, and $\nu \neq 0$ such that $\nu\alpha \in \Phi_1$. Then*

$$\phi(\nu\alpha) = \tfrac{1}{\nu}\phi(\alpha).$$

Proof. Since $\alpha \in \Phi_1$, we may write $\alpha = w\alpha_t$ for some $w \in W$ and $t \in S$, and since $\nu\alpha \in \Phi_1$ it follows that $\nu\alpha_t = \nu w^{-1}\alpha = w^{-1}(\nu\alpha)$ is in Φ_1. Hence by Corollary 1.2.19 and the W-equivariance of ϕ,

$$\begin{aligned} \phi(\nu\alpha) &= \phi(\nu(w\alpha_t)) = \phi(w(\nu\alpha_t)) \\ &= w\phi(\nu\alpha_t) = w(\tfrac{1}{\nu}\phi(\alpha_t)) = \tfrac{1}{\nu}\phi(w\alpha_t) = \tfrac{1}{\nu}\phi(\alpha) \end{aligned}$$

as required. □

Equipped with Proposition 1.2.9, we may define the canonical coefficient of any simple root of Π_1 (respectively, of Π_2) in a given root of Φ_1 (respectively, of Φ_2). Let $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, and let $\alpha' = \phi_1(\alpha)$ and $\beta' = \phi_2(\beta)$ be the corresponding elements of Φ_1' and Φ_2'. Since

Π'_1 and Π'_2 are linearly independent, for each $t \in S$ there are uniquely determined real numbers $\mathrm{coeff}_{\alpha_t}(\alpha)$ and $\mathrm{coeff}_{\beta_t}(\beta)$ satisfying

$$\alpha' = \sum_{t \in S} \mathrm{coeff}_{\alpha_t}(\alpha)\alpha'_t,$$

and

$$\beta' = \sum_{t \in S} \mathrm{coeff}_{\beta_t}(\beta)\beta'_t.$$

Then Proposition 1.2.9 yields that

$$\alpha = \pi_1(\alpha') = \pi_1(\sum_{t \in S} \mathrm{coeff}_{\alpha_t}(\alpha)\alpha'_t) = \sum_{t \in S} \mathrm{coeff}_{\alpha_t}(\alpha)\alpha_t$$

and

$$\beta = \pi_2(\beta') = \pi_2(\sum_{t \in S} \mathrm{coeff}_{\beta_t}(\beta)\beta'_t) = \sum_{t \in S} \mathrm{coeff}_{\beta_t}(\beta)\beta_t.$$

Definition 1.2.21. Suppose that $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. For each $t \in S$, we call $\mathrm{coeff}_{\alpha_t}(\alpha)$ the *canonical coefficient* of α_t in the root α and $\mathrm{coeff}_{\beta_t}(\beta)$ the *canonical coefficient* of β_t in β.

1.3. The canonical coefficients and the depth of roots

In this section we study the canonical coefficients defined in section 1.2. It turns out that for each $s \in S$ and $\alpha \in \Phi_1$, the coefficient $\mathrm{coeff}_{\alpha_s}(\alpha)$ is closely related to the coefficient $\mathrm{coeff}_{\beta_s}(\phi(\beta))$, where ϕ is the W-equivariant map defined in Definition 1.2.18. Also we introduce the concept of *depth* of a root following the conventions of [**6**] and [**5**], and we give a characterisation of the depth of a root in terms of the length of the reflection corresponding to that root.

Definition 1.3.1. Given $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, define the *support* of α, written $\mathrm{supp}(\alpha)$, and the *support* of β, written $\mathrm{supp}(\beta)$, to be the sets:

$$\mathrm{supp}(\alpha) = \{\, \alpha_s \mid s \in S \text{ and } \mathrm{coeff}_{\alpha_s}(\alpha) \neq 0 \,\}$$

and

$$\mathrm{supp}(\beta) = \{\, \beta_s \mid s \in S \text{ and } \mathrm{coeff}_{\beta_s}(\beta) \neq 0 \,\}.$$

The following group-theoretic result is purely about Coxeter groups and is well known (for example, Section 2 of [**14**]). It is essential for Proposition 1.3.3 below and it is going to be used repeatedly in Section 2.3 later. For completeness, a proof is included here.

Lemma 1.3.2. *Let $I \subseteq S$ and $w \in W$. Choose $w' \in wW_I$ to be of minimal length. Then $l(w'v) = l(w') + l(v)$ for all $v \in W_I$. In particular, $w'\alpha_s \in \Phi_1^+$ whenever $s \in I$.*

Proof. We prove this result by an induction on $l(v)$, the case $l(v) = 0$ being trivial. So assume that $l(v) > 0$, and write $v = v'r_s$ for some $v' \in W_I$ and $s \in I$ such that $l(v') = l(v) - 1$. If we can show that $l(w'v) = l(w'v') + 1$ then the desired result will follow by induction. By Corollary 1.2.5, to prove that $l(w'v) = l(w'v') + 1$ it suffices to show that $(w'v')\alpha_s \in \Phi_1^+$.

Observe that, for any $t \in I$, the minimality of $l(w')$ in wW_I together with Corollary 1.2.5 implies that $l(w'r_t) = l(w') + 1$, and $w'\alpha_t \in \Phi_1^+$. Moreover, since $l(v'r_s) = l(v) = l(v') + 1$, Corollary 1.2.5 yields that $v'\alpha_s \in \Phi_1^+$, and so we may write $v'\alpha_s = \sum_{t \in I} \lambda_t \alpha_t$, where $\lambda_t \geq 0$ for all $t \in I$. Since $w'\alpha_t \in \Phi_1^+ \subseteq \mathrm{PLC}(\Pi_1)$ for all $t \in I$, it follows that $(w'v')(\alpha_s) = \sum_{t \in I} \lambda_t w'\alpha_t \in \Phi_1 \cap \mathrm{PLC}(\Pi_1) = \Phi_1^+$, as required. □

Corollary 1.2.19 says that if $t \in S$ and λ is any nonzero scalar then $\lambda\alpha_t$ is a root if and only if $\frac{1}{\lambda}\beta_t$ is root. Thus in the case of singleton support roots $\alpha = \lambda\alpha_t$ and $\phi(\alpha) = \frac{1}{\lambda}\beta_t$ ($t \in S$), for any $s \in S$ we have either $\mathrm{coeff}_{\alpha_s}(\alpha)\,\mathrm{coeff}_{\beta_s}(\phi(\alpha)) = 0$ (if $s \neq t$) or $\mathrm{coeff}_{\alpha_s}(\alpha)\,\mathrm{coeff}_{\beta_s}(\phi(\alpha)) = 1$ (if $s = t$). An extension to this fact is given in the next proposition.

Proposition 1.3.3. *Let $\alpha \in \Phi_1^+$. The for each $t \in S$ we have*

$$\mathrm{coeff}_{\alpha_t}(\alpha) > 0 \textit{ if and only if } \mathrm{coeff}_{\beta_t}(\phi(\alpha)) > 0$$

and in this case $\mathrm{coeff}_{\alpha_t}(\alpha)\,\mathrm{coeff}_{\beta_t}(\phi(\alpha)) \geq 1$.

Proof. We use induction on the length of $w \in W$ to prove that the statement holds whenever $\alpha = w\alpha_r$ for some $r \in S$. Note that $\alpha = w\alpha_r$

implies that $\phi(\alpha) = w\phi(\alpha_r) = w\beta_r$. If $l(w) = 0$ then the assertion is trivially true.

Suppose then that $l(w) > 0$, and let $s \in S$ and $w' \in W$ be such that $w = w'r_s$ and $l(w') = l(w) - 1$. Choose w'' to be a minimal length element of $w\langle r_r, r_s\rangle$. Then $l(w'') \leq l(wr_s) < l(w)$, and hence the inductive hypothesis applies to $w''\alpha_r$ as well as to $w''\alpha_s$, so that for all $t \in S$ either $\operatorname{coeff}_{\alpha_t}(w''\alpha_r)\operatorname{coeff}_{\beta_t}(w''\beta_r) \geq 1$ or else $\operatorname{coeff}_{\alpha_t}(w''\alpha_r)$ and $\operatorname{coeff}_{\beta_t}(w''\beta_r)$ are both zero, and either $\operatorname{coeff}_{\alpha_t}(w''\alpha_s)\operatorname{coeff}_{\beta_t}(w''\beta_s) \geq 1$ or else $\operatorname{coeff}_{\alpha_t}(w''\alpha_s)$ and $\operatorname{coeff}_{\beta_t}(w''\beta_s)$ are both zero.

Since $l(w''r_r), l(w''r_s) \geq l(w'')$ by the minimality of $l(w'')$ in the coset $w\langle r_r, r_s\rangle$, it follows from Corollary 1.2.5 that $w''\alpha_r$, $w''\beta_r$, $w''\alpha_s$ and $w\beta_s$ are all positive. Thus for all $t \in S$,

$$\operatorname{coeff}_{\alpha_t}(w''\alpha_r) \geq 0 \text{ and } \operatorname{coeff}_{\beta_t}(w''\beta_r) \geq 0 \tag{1.3.1}$$

and

$$\operatorname{coeff}_{\alpha_t}(w''\alpha_s) \geq 0 \text{ and } \operatorname{coeff}_{\beta_t}(w''\beta_s) \geq 0. \tag{1.3.2}$$

Let $u \in \langle r_r, r_s\rangle$ such that $w = w''u$. Since w'' is of minimal length in the coset $w\langle r_r, r_s\rangle$, it follows from Lemma 1.3.2 that $l(w) = l(w'') + l(u)$, and hence $N_1(u) \subseteq N_1(w)$ by Lemma 1.2.12. Since $w\alpha_r \in \Phi_1^+$, it follows that $u\alpha_r \in \Phi_1^+$ too. Hence Corollary 1.2.5 yields that $l(ur_r) > l(u)$, and, in particular, u has no reduced expression ending in r_r. This implies that $l(u) < m_{rs}$, and so Lemma 1.1.11 yields that

$$u\alpha_r = \lambda_1\alpha_r + \mu_1\alpha_s \quad \text{and} \quad u\beta_r = \lambda_2\beta_r + \mu_2\beta_s$$

for some some nonnegative scalars $\lambda_1, \lambda_2, \mu_1$ and μ_2. Observe that this implies

$$u\phi_1(\alpha_r) = \lambda_1\phi_1(\alpha_r) + \mu_1\phi_1(\alpha_s) \quad \text{and} \quad u\phi_2(\beta_r) = \lambda_2\phi_2(\beta_r) + \mu_2\phi_2(\beta_s)$$

Hence

$$\phi_1(w\alpha_r) = w''\phi_1(u\alpha_r) = \lambda_1 w''\phi_1(\alpha_r) + \mu_1 w''\phi_1(\alpha_s)$$

and

$$\phi_2(w\beta_r) = w''\phi_2(u\beta_r) = \lambda_2 w''\phi_2(\beta_r) + \mu_2 w''\phi_2(\beta_s).$$

It follows that for all $t \in S$,

$$\begin{aligned} \operatorname{coeff}_{\alpha_t}(w\alpha_r) &= \operatorname{coeff}_{\alpha_t}(w''(u\alpha_r)) \\ &= \lambda_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_r) + \mu_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_s), \end{aligned} \tag{1.3.3}$$

and likewise $\operatorname{coeff}_{\beta_t}(w\beta_r) = \lambda_2 \operatorname{coeff}_{\beta_t}(w''\beta_r) + \mu_2 \operatorname{coeff}_{\beta_t}(w''\beta_s)$.

A direct rank 2 calculation (as in Lemmas 1.1.6 and 1.1.8) yields explicit formulas for the coefficients λ_1, μ_1, λ_2 and μ_2, and we find that

$$\lambda_1 > 0 \text{ if and only if } \lambda_2 > 0 \text{ and } \mu_1 > 0 \text{ if and only if } \mu_2 > 0. \tag{1.3.4}$$

Moreover, if $m_{rs} = m < \infty$ then $\lambda_1\lambda_2$ and $\mu_1\mu_2$ belong to the set $\{\,(\frac{\sin k\theta}{\sin\theta})^2 \mid 1 \le k \le m-1\} \cup \{0\}$, where $\theta = \pi/m$, while if $m_{rs} = \infty$ then $\lambda_1\lambda_2$ and $\mu_1\mu_2$ belong to $\{\,(\frac{\sinh k\theta}{\sinh\theta})^2 \mid k \in \mathbb{N}\,\} \cup \{0\}$, where $\theta = \cosh^{-1}(\sqrt{\langle\alpha_r,\beta_s\rangle\langle\alpha_s,\beta_r\rangle})$. It follows in either case that if $\lambda_1\lambda_2$ or $\mu_1\mu_2$ is nonzero then it is at least 1.

Suppose that $\operatorname{coeff}_{\alpha_t}(w\alpha_r) > 0$. Then (1.3.3) shows that one of $\lambda_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_r)$ or $\mu_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_s)$ must be strictly positive. Now if $\lambda_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_r) > 0$ then $\lambda_1 > 0$ and $\operatorname{coeff}_{\alpha_t}(w''\alpha_r) > 0$, and since $l(w'') < l(w)$ the inductive hypothesis yields that $\operatorname{coeff}_{\beta_t}(w''\beta_r) > 0$. Thus in this case (1.3.4) yields that for all $t \in S$,

$$\begin{aligned} \operatorname{coeff}_{\beta_t}(w\beta_r) &= \lambda_2 \operatorname{coeff}_{\beta_t}(w''\beta_r) + \mu_2 \operatorname{coeff}_{\beta_t}(w''\beta_s) \\ &\ge \lambda_2 \operatorname{coeff}_{\beta_t}(w''\beta_r) \\ &> 0 \end{aligned}$$

since μ_2 and $\operatorname{coeff}_{\beta_t}(w''\beta_s)$ are both nonnegative. On the other hand, if $\mu_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_r) > 0$ a similar argument yields that

$$\operatorname{coeff}_{\beta_t}(w\beta_r) \ge \mu_2 \operatorname{coeff}_{\beta_t}(w''\beta_s) > 0,$$

so that $\operatorname{coeff}_{\alpha_t}(w\alpha_r) > 0$ implies $\operatorname{coeff}_{\beta_t}(w\beta_r) > 0$ in either case.

By symmetry, we can also deduce that $\operatorname{coeff}_{\beta_t}(w\beta_r) > 0$ implies $\operatorname{coeff}_{\alpha_t}(w\alpha_r) > 0$, completing the proof of the first part.

Suppose that $t \in S$ and $\operatorname{coeff}_{\alpha_t}(w\alpha_r)$ and $\operatorname{coeff}_{\beta_t}(w\beta_r)$ are positive. Observe from (1.3.3) that either $\lambda_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_r)$ or $\mu_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_s)$ must be positive, and consider first the case $\lambda_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_r) > 0$. Then $\lambda_1 > 0$, which implies $\lambda_1\lambda_2 \geq 1$ by the rank 2 calculation, and $\operatorname{coeff}_{\alpha_t}(w''\alpha_r) > 0$, which implies that $\operatorname{coeff}_{\alpha_t}(w''\alpha_r)\operatorname{coeff}_{\beta_t}(w''\beta_r) \geq 1$ by the inductive hypothesis. Hence

$$\begin{aligned}\operatorname{coeff}_{\alpha_t}(w\alpha_r)\operatorname{coeff}_{\beta_t}(w\beta_r) &\geq (\lambda_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_r))(\lambda_2 \operatorname{coeff}_{\beta_t}(w''\beta_r)) \\ &= (\lambda_1\lambda_2)(\operatorname{coeff}_{\alpha_t}(w''\alpha_r)\operatorname{coeff}_{\beta_t}(w''\beta_r)) \\ &\geq 1.\end{aligned}$$

On the other hand, if $\mu_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_s) > 0$ the same conclusion follows, since

$$\begin{aligned}\operatorname{coeff}_{\alpha_t}(w\alpha_r)\operatorname{coeff}_{\beta_t}(w\beta_r) &\geq (\mu_1 \operatorname{coeff}_{\alpha_t}(w''\alpha_s))(\mu_2 \operatorname{coeff}_{\beta_t}(w''\beta_s)) \\ &= (\mu_1\mu_2)(\operatorname{coeff}_{\alpha_t}(w''\alpha_s)\operatorname{coeff}_{\beta_t}(w''\beta_s)) \\ &\geq 1.\end{aligned}$$

by the rank 2 calculation and the inductive hypothesis, and this completes the proof of the second assertion of the proposition. □

It is readily seen from Proposition 1.3.3 that:

Corollary 1.3.4. *Let $\alpha \in \Phi_1$, then*

$$\alpha \in \Phi_1^+ \text{ if and only if } \phi(\alpha) \in \Phi_2^+,$$

and

$$\alpha \in \Phi_1^- \text{ if and only if } \phi(\alpha) \in \Phi_2^-.$$

Furthermore, $\operatorname{supp}(\phi(\alpha)) = \phi(\operatorname{supp}(\alpha))$.

Proof. Observe the following if and only if statements: $\alpha \in \Phi_1^+$ if and only if for some $t \in S$, $\operatorname{coeff}_{\alpha_t}(\alpha) > 0$, and by Proposition 1.3.3 this happens if and only $\operatorname{coeff}_{\beta_t}(\phi(\alpha)) > 0$ for some $t \in S$, and this in turn happens if and only if $\phi(\alpha) \in \Phi_2^+$.

Replace α by $-\alpha$, then the above yields that $\alpha \in \Phi_1^-$ if and only if $\phi(\alpha) \in \Phi_2^-$.

Finally, Proposition 1.3.3 yields that $\operatorname{supp}(\phi(\alpha)) = \phi(\operatorname{supp}(\alpha))$. □

Definition 1.3.5. (i) For each $\alpha \in \Phi_1^+$ and $\beta \in \Phi_2^+$ define the *depth* of α (written $\mathrm{dp}_1(\alpha)$) and the *depth* of β (written $\mathrm{dp}_2(\beta)$) to be

$$\mathrm{dp}_1(\alpha) = \min\{\, l(w) \mid\; w \in W \text{ and } w\alpha \in \Phi_1^- \,\},$$

and

$$\mathrm{dp}_2(\beta) = \min\{\, l(w) \mid\; w \in W \text{ and } w\beta \in \Phi_2^- \,\}.$$

(ii) For $\alpha_1, \alpha_2 \in \Phi_1^+$ (respectively $\beta_1, \beta_2 \in \Phi_2^+$), write $\alpha_1 \preceq_1 \alpha_2$ (resp. $\beta_1 \preceq_2 \beta_2$) if and only if there exists $w \in W$ (resp. $w' \in W$) such that $\alpha_2 = w\alpha_1$, and $\mathrm{dp}_1(\alpha_2) = \mathrm{dp}_1(\alpha_1) + l(w)$ (respectively $\beta_2 = w'\beta_1$, and $\mathrm{dp}_2(\beta_2) = \mathrm{dp}_2(\beta_1) + l(w')$). Further we write $\alpha_1 \prec_1 \alpha_2$ if $\alpha_1 \preceq_1 \alpha_2$ but $\alpha_1 \neq \alpha_2$, and we write $\beta_1 \prec_2 \beta_2$ if $\beta_1 \preceq_2 \beta_2$ but $\beta_1 \neq \beta_2$.

Lemma 1.3.6. (Lemma 1.6 of [**6**]) *$\preceq_1$ and $\preceq_2$ are partial orderings on Φ_1^+ and Φ_2^+ respectively.* □

The following result can be easily deduced from Definition 1.3.5.

Lemma 1.3.7. *Suppose that $\alpha_1, \alpha_2 \in \Phi_1^+$ and $\beta_1, \beta_2 \in \Phi_2^+$ such that $\alpha_1 = w\alpha_2$ and $\beta_1 = w'\beta_2$ for some $w, w' \in W$. Then*

$$|\,\mathrm{dp}_1(\alpha_1) - \mathrm{dp}_1(\alpha_2)| \leq l(w) \text{ and } |\,\mathrm{dp}_2(\beta_1) - \mathrm{dp}_2(\beta_2)| \leq l(w').$$

□

Lemma 1.3.8. *Suppose that $\alpha_1, \alpha_2 \in \Phi_1^+$, with $\alpha_1 \prec_1 \alpha_2$. Let $w \in W$ be of minimal length such that $\alpha_2 = w\alpha_1$. Let $w = r_{s_1} r_{s_2} \cdots r_{s_l}$, $(s_1, \cdots, s_l \in S)$ with $l = l(w)$. Then for all $i \in \{1, 2, \cdots, l-1\}$,*

$$(r_{s_{i+1}} \cdots r_{s_l})\alpha_1 \prec_1 (r_{s_i} r_{s_{i+1}} \cdots r_{s_l})\alpha_1.$$

Proof. Given these conditions, $\mathrm{dp}_1(\alpha_2) = \mathrm{dp}_1(\alpha_1) + l(w)$. Observe that $r_{s_{i+1}} \cdots r_{s_l}$ is a right hand segment of $r_{s_i} r_{s_{i+1}} \cdots r_{s_l}$, which in turn is a right hand segment of w, so Lemma 1.2.12 (iv) yields that

$$N_1(r_{s_{i+1}}\cdots r_{s_l}) \subset N_1(r_{s_i}r_{s_{i+1}}\cdots r_{s_l}) \subseteq N_1(w),$$

In particular, $(r_{s_i}r_{s_{i+1}}\cdots r_{s_l})\alpha_1 \in \Phi_1^+$ for all $i \in \{1,\cdots,l\}$ (since $w\alpha_1 \in \Phi_1^+$). We prove this lemma by way of contradiction. suppose the statement of this lemma is false, and choose $i \in \{1,\cdots,l-1\}$ such that $(r_{s_{i+1}}\cdots r_{s_l})\alpha_1 \not\prec (r_{s_i}r_{s_{i+1}}\cdots r_{s_l})\alpha_1$. Thus

$$\mathrm{dp}_1((r_{s_i}\cdots r_{s_l})\alpha_1) \leq \mathrm{dp}_1((r_{s_{i+1}}\cdots r_{s_l})\alpha_1). \tag{1.3.5}$$

Since $\alpha_2 = (r_{s_1}\ldots r_{s_{i-1}})(r_{s_i}\ldots r_{s_l}\alpha_1)$ it follows that

$$\begin{aligned}
\mathrm{dp}_1(\alpha_1) + l(w) &= \mathrm{dp}_1(\alpha_2) \\
&\leq (i-1) + \mathrm{dp}_1((r_{s_i}\cdots r_{s_l})\alpha_1) && \text{(by Lemma 1.3.7)} \\
&\leq (i-1) + \mathrm{dp}_1(r_{s_{i+1}}\cdots r_{s_l}\alpha_1) && \text{(by (1.3.5))} \\
&\leq (i-1) + (l-i) + \mathrm{dp}_1(\alpha_1) && \text{(by Lemma 1.3.7)} \\
&= l - 1 + \mathrm{dp}_1(\alpha_1)
\end{aligned}$$

which is absurd. Hence for all $i \in \{1,\cdots,l-1\}$,

$$(r_{s_{i+1}}\cdots r_{s_l})\alpha_1 \prec_1 (r_{s_i}\cdots r_{s_l})\alpha_1,$$

as requied. □

Remark 1.3.9. Lemma 1.3.7 has a natural analogue in Φ_2^+.

The following result is a generalisation of Lemma 1.7 of [**6**]. Essentially the same reasoning as that used in [**6**] can be used to prove it, but for completeness, a proof is included here.

Lemma 1.3.10. *Let $s \in S$, $\alpha \in \Phi_1^+ \setminus \mathbb{R}\{\alpha_s\}$, and $\beta \in \Phi_2^+ \setminus \mathbb{R}\{\beta_s\}$. Then*

$$\mathrm{dp}_1(r_s\alpha) = \begin{cases} \mathrm{dp}_1(\alpha) - 1 & \textit{if } \langle \alpha, \beta_s \rangle > 0, \\ \mathrm{dp}_1(\alpha) & \textit{if } \langle \alpha, \beta_s \rangle = 0, \\ \mathrm{dp}_1(\alpha) + 1 & \textit{if } \langle \alpha, \beta_s \rangle < 0; \end{cases}$$

and

$$\mathrm{dp}_2(r_s\beta) = \begin{cases} \mathrm{dp}_2(\beta) - 1 & \text{if } \langle \alpha_s, \beta \rangle > 0, \\ \mathrm{dp}_2(\beta) & \text{if } \langle \alpha_s, \beta \rangle = 0, \\ \mathrm{dp}_2(\beta) + 1 & \text{if } \langle \alpha_s, \beta \rangle < 0. \end{cases}$$

Proof. If $\langle \alpha, \beta_s \rangle = 0$ then $r_s\alpha = \alpha - 2\langle \alpha, \beta_s \rangle \alpha_s = \alpha$; hence trivially $\mathrm{dp}_1(r_s\alpha) = \mathrm{dp}_1(\alpha)$.

Suppose next that $\langle \alpha, \beta_s \rangle > 0$, By Lemma 1.3.7, it suffices to show that $\mathrm{dp}_1(r_s\alpha) < \mathrm{dp}_1(\alpha)$. To do so we construct a $w \in W$ with $w(r_s\alpha) \in \Phi_1^-$ and $l(w) < \mathrm{dp}_1(\alpha)$. Choose $v \in W$ such that $v\alpha \in \Phi_1^-$ and $l(v) = \mathrm{dp}_1(\alpha)$. If $v\alpha_s \in \Phi_1^-$ we set $w = vr_s$; then $l(w) = l(v) - 1$ (by Corollary 1.2.5) and $w(r_s\alpha) = v\alpha \in \Phi_1^-$, as required. Hence we may assume that $v\alpha_s \in \phi_1^+$. Now

$$v(r_s\alpha) = v(\alpha - 2\langle \alpha, \beta_s \rangle \alpha_s) = \underbrace{v\alpha}_{\in \Phi_1^-} - \underbrace{2\langle \alpha, \beta_s \rangle v\alpha_s}_{\in -\mathrm{PLC}(\Pi_1)} \in \Phi_1^-.$$

Furthermore, the condition $\alpha \in \Phi_1^+ \setminus \mathbb{R}\{\alpha_s\}$ implies that $v\alpha$ and $-2\langle \alpha, \beta_s \rangle v\alpha_s$ are not scalar multiples of each other, and hence it follows that there are at least two simple roots in the support of $v(r_s\alpha)$. Now choose $r \in S$, $w \in W$ with $v = r_r w$ and $l(v) = l(w) + 1$. Since $v(r_s\alpha)$ has at least two simple roots in its support, it follows that applying r_r will not change the sign of $v(r_s\alpha)$ (recall that $N_1(r_r) = \{\widehat{\alpha_r}\}$). Thus $w(r_s\alpha) = r_r(vr_s\alpha) \in \Phi_1^-$ and $l(w) < l(v) = \mathrm{dp}_1(\alpha)$.

Finally, suppose that $\langle \alpha, \beta_s \rangle < 0$. Then $\langle r_s\alpha, \beta_s \rangle = -\langle \alpha, \beta_s \rangle > 0$; so the preceding paragraph shows that

$$\mathrm{dp}_1(\alpha) = \mathrm{dp}_1(r_s(r_s\alpha)) = \mathrm{dp}_1(r_s\alpha) - 1.$$

As usual, the other half of this lemma follows by an exactly similar argument. □

Lemma 1.3.11. *Suppose that* $\alpha \in W$. *Then* $\mathrm{dp}_1(\alpha) = \mathrm{dp}_2(\phi(\alpha))$.

Proof. Let $w \in W$ be such that $w\alpha \in \Phi_1^-$ and $\mathrm{dp}_1(\alpha) = l(w)$. Then $\phi(w\alpha) \in \Phi_2^-$ by Corollary 1.3.4. Since ϕ is W-equivariant, it follows that $w(\phi(\alpha)) \in \Phi_2^-$. Hence $\mathrm{dp}_2(\phi(\alpha)) \leq l(w) = \mathrm{dp}_1(\alpha)$. By symmetry we also have $\mathrm{dp}_1(\alpha) \leq \mathrm{dp}_2(\phi(\alpha))$, whence equality. □

Lemma 1.3.12. *Suppose that $\alpha \in \Phi_1$ and $s \in S$. Then*

$$\langle \alpha, \beta_s \rangle > 0 \textit{ if and only if } \langle \alpha_s, \phi(\alpha) \rangle > 0;$$

and

$$\langle \alpha, \beta_s \rangle = 0 \textit{ if and only if } \langle \alpha_s, \phi(\alpha) \rangle = 0;$$

and

$$\langle \alpha, \beta_s \rangle < 0 \textit{ if and only if } \langle \alpha_s, \phi(\alpha) \rangle < 0.$$

Proof. This follows from Lemma 1.3.10 and Lemma 1.3.11. □

And we can generalize the above to:

Corollary 1.3.13. *Suppose that $\alpha_1, \alpha_2 \in \Phi_1$. Then*

$$\langle \alpha_1, \phi(\alpha_2) \rangle > 0 \quad \textit{if and only if} \quad \langle \alpha_2, \phi(\alpha_1) \rangle > 0;$$

and

$$\langle \alpha_1, \phi(\alpha_2) \rangle = 0 \quad \textit{if and only if} \quad \langle \alpha_2, \phi(\alpha_1) \rangle = 0;$$

and

$$\langle \alpha_1, \phi(\alpha_2) \rangle < 0 \quad \textit{if and only if} \quad \langle \alpha_2, \phi(\alpha_1) \rangle < 0.$$

Proof. Write $\alpha_2 = w\alpha_s$ for some $w \in W$ and $s \in S$. Since $\langle\, ,\, \rangle$ is W-invariant and ϕ is W-equivariant, thus

$$\langle \alpha_1, \phi(\alpha_2) \rangle = \langle \alpha_1, w\beta_s \rangle = \langle w^{-1}\alpha_1, \beta_s \rangle.$$

Observe that Lemma 1.3.12 yields that $\langle w^{-1}\alpha_1, \beta_s \rangle > 0$ if and only if $\langle \alpha_s, \phi(w^{-1}\alpha_1) \rangle > 0$ which in turn happens if and only if

$$\langle \alpha_s, \phi(w^{-1}\alpha_1) \rangle = \langle w\alpha_s, \phi(\alpha_1) \rangle = \langle \alpha_2, \phi(\alpha_1) \rangle > 0.$$

The rest of the desired result follows in a similar way. □

Definition 1.3.14. Suppose that $\alpha = w\alpha_s$, $\beta = \phi(\alpha)$, where $w \in W$ and $s \in S$. Then define $r_\alpha, r_\beta \in W$ by $r_\alpha = r_\beta \colon = wr_sw^{-1}$, and we call

r_α the *reflection* corresponding to α and r_β the *reflection* corresponding to β.

Remark 1.3.15. It is clear from Definition 1.3.14 and Lemma 1.2.1 that for all $x \in \Phi_1$ and $y \in \Phi_2$,

$$r_\alpha(x) = x - 2\langle x, \phi(\alpha)\rangle\alpha \quad \text{and} \quad r_\beta(y) = y - 2\langle \phi^{-1}(\beta), y\rangle\beta.$$

Observe that $r_x(x) = -x$ and $r_y(y) = -y$. Note that $r_{\alpha_s} = r_s$, $r_{\beta_t} = r_t$ for all $s, t \in S$. Clearly for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, $r_\alpha = r_{-\alpha}$, and $r_\beta = r_{-\beta}$. Furthermore, suppose that z_1, $z_2 \in \Phi_i$ $(i = 1, 2)$. Then $\widehat{z_1} = \widehat{z_2}$ if and only if $r_{z_1} = r_{z_2}$.

Lemma 1.3.16. *Suppose that $x \in \Phi_1$, $y \in \Phi_2$ and $w, v \in W$. Then $r_{wx} = wr_xw^{-1}$ and $r_{vy} = vr_yv^{-1}$.*

Proof. Observe that for any $v \in V_1$

$$\begin{aligned}
& wr_xw^{-1}(v) \\
&= w(w^{-1}v - 2\langle w^{-1}v, \phi(x)\rangle x) \\
&= v - 2\langle w^{-1}v, \phi(x)\rangle wx \\
&= v - 2\langle v, w\phi(x)\rangle wx && \text{(since } \langle\ ,\ \rangle \text{ is } W\text{-invariant)} \\
&= v - 2\langle v, \phi(wx)\rangle wx && \text{(since } \phi \text{ is } W\text{-equivariant)} \\
&= r_{wx}v.
\end{aligned}$$

Since $v \in V_1$ was arbitrary, it follows that $r_{wx} = wr_xw^{-1}$. Entirely similar argument shows that $r_{vy} = vr_yv^{-1}$. □

Set $T := \bigcup_{w \in W} wRw^{-1}$ and call it the *set of reflections* in W. For each $i \in \{1, 2\}$ since $\Phi_i = W\Pi_i$, it follows from Lemma 1.3.16 and the definition of T that there is a bijection between $\widehat{\Phi_i}$ and T via $\widehat{z} \leftrightarrow r_z$.

The following Proposition is a natural extension to Corollary 1.2.5:

Proposition 1.3.17. *For each $i \in \{1, 2\}$, let $w \in W$ and $x \in \Phi_i^+$. If $l(wr_x) > l(w)$ then $wx \in \Phi_i^+$. If $l(wr_x) < l(w)$ then $wx \in \Phi_i^-$.*

Proof. We prove the statement that $l(wr_x) > l(w)$ if and only if wx is positive in the case $x \in \Phi_1$ here, and again we stress that a similar argument also shows the desired result holds in Φ_2.

Observe that the second statement follows from the first, applied to wr_x in place of w: indeed if $l(wr_x) < l(w)$ then $l((wr_x)r_x) > l(wr_x)$ forcing $(wr_x)x = w(r_x x) = -wx \in \Phi_1^+$, that is, $wx \in \Phi_1^-$.

Now we prove the first statement in Φ_1. Proceed by induction on $l(w)$, the case $l(w) = 0$ being trivial. If $l(w) > 0$, there exists $s \in S$ with $l(r_s w) = l(w) - 1$. Then

$$l((r_s w)r_x) = l(r_s(wr_x)) \geq l(wr_x) - 1 > l(w) - 1 = l(r_s w).$$

Then the inductive hypothesis yields that $(r_s w)x \in \Phi_1^+$. Suppose for a contadiction that $wx \in \Phi_1^-$. Then $\widehat{wx} \in N_1(r_s)$ and Lemma 1.2.12 (i) yields that $wx = -\lambda\alpha_s$ for some $\lambda > 0$. But then $r_s wx = \lambda\alpha_s$ would imply $(r_s w)r_x(r_s w)^{-1} = r_s$ by Lemma 1.3.16 and Remark 1.3.15. But this yields that $wr_x = r_s w$ contradicting $l(wr_x) > l(w) > l(r_s w)$. As a result, wx must be positive. □

It is readily checked that for all $w \in W$ and $t \in T$, $l(wt) \cong l(w)$ modulo 2. In particular, the length of any such t must be odd, which in turn shows that $l(wt) \neq l(w)$ Combining this observation with Proposition 1.3.17 we have the following:

Corollary 1.3.18. *Let $w \in W$ and $x \in \Phi_i^+$ $(i \in \{1, 2\})$. Then*

$$\begin{cases} l(wr_x) > l(w) & \text{if and only if } wx \in \Phi_i^+, \\ l(wr_x) < l(w) & \text{if and only if } wx \in \Phi_i^-. \end{cases}$$

□

The next result gives a connection between the depth of a root and the length of the corresponding reflection.

Lemma 1.3.19. *Let $\alpha \in \Phi_1^+$ such that $\alpha = (r_{s_1} r_{s_2} \cdots r_{s_l})\alpha_s$, where $s, s_1, \cdots, s_l \in S$ and $\mathrm{dp}_1(\alpha) = l + 1$. Then $l(r_\alpha) = 2l + 1$.*

Proof. The proof is based on an induction on l. The case $l = 0$ ($\alpha = \alpha_s$) is trivial. Thus we may assume that $l > 1$. Observe that Lemma 1.3.7 and Lemma 1.3.8 yield that $\mathrm{dp}_1(r_{s_2} \ldots r_{s_l}\alpha_s) = l$. Then the inductive hypothesis yields that

$$l(r_{s_2} \ldots r_{s_l} r_s r_{s_l} \cdots r_{s_2}) = 2l - 1. \tag{1.3.6}$$

That is, $(r_{s_2} \ldots r_{s_l})\alpha_s \prec_1 (r_{s_1} r_{s_2} \ldots r_{s_l})\alpha_s$, and hence Lemma 1.3.10 gives

$$\langle r_{s_2} \ldots r_{s_l}\alpha_s, \beta_{s_1}\rangle < 0. \tag{1.3.7}$$

Lemma 1.3.12 then yields that $\langle \alpha_{s_1}, (r_{s_2} \ldots r_{s_l})\beta_s\rangle < 0$. Observe that $\mathrm{dp}_1(\alpha) = \mathrm{dp}_1(r_{s_1} r_{s_2} \ldots r_{s_l}\alpha_s) = l + 1$ implies that $l((r_{s_1} r_{s_2} \ldots r_{s_l}) = l$, in particular, $r_{s_2} \ldots r_{s_l}$ is a right hand segment of $r_{s_1} r_{s_2} \ldots r_{s_l}$. Since $(r_{s_1} r_{s_2} \ldots r_{s_l})\alpha_s \in \Phi_1^+$, it follows that $(r_{s_2} \ldots r_{s_l})\alpha_s \in \Phi_1^+$ too. Thus

$$\begin{aligned}(r_{s_2} \ldots r_{s_l} r_s r_{s_l} \ldots r_{s_2})\alpha_{s_1} &= r_{(r_{s_2} \cdots r_{s_l}\alpha_s)}\alpha_{s_1} \\ &= \alpha_{s_1} - \underbrace{2\langle \alpha_{s_1}, (r_{s_2} \cdots r_{s_l})\beta_s\rangle}_{<0} \underbrace{(r_{s_2} \cdots r_{s_l}\alpha_s)}_{\in \Phi_1^+} \\ &\in \Phi_1^+.\end{aligned} \tag{1.3.8}$$

Hence

$$\begin{aligned} & l(r_{s_1}(r_{s_2} \cdots r_{s_l} r_s r_{s_l} \cdots r_{s_2})) \\ &= l((r_{s_2} \cdots r_{s_l} r_s r_{s_l} \cdots r_{s_2}) r_{s_1}) \\ &= l((r_{s_2} \cdots r_{s_l} r_s r_{s_l} \cdots r_{s_2})) + 1 \qquad \text{(by Corollary 1.2.5).}\end{aligned}$$

Now we claim that

$$(r_{s_1} r_{s_2} \cdots r_{s_l} r_s r_{s_l} \cdots r_{s_2})\alpha_{s_1} \in \Phi_1^+. \tag{1.3.9}$$

If (1.3.9) were true, then Corollary 1.2.5 would yield that

$$\begin{aligned} l(r_{s_1} r_{s_2} \cdots r_{s_l} r_s r_{s_l} \cdots r_{s_2} r_{s_1}) &= l(r_{s_1} r_{s_2} \cdots r_{s_l} r_s r_{s_l} \cdots r_{s_2}) + 1 \\ &= l(r_{s_2} \cdots r_{s_l} r_s r_{s_l} \cdots r_{s_2}) + 2,\end{aligned}$$

this combining with (1.3.6) gives us $l(r_{s_1}\cdots r_{s_l}r_sr_{s_l}\cdots r_{s_1}) = 2l+1$. Thus the result follows by induction provided we can prove (1.3.9), which we shall prove now. Suppose that the opposite to (1.3.9) holds, then Lemma 1.2.12(i) together with (1.3.8) yields that

$$(r_{s_2}\cdots r_{s_l}r_sr_{s_l}\cdots r_{s_2})\alpha_{s_1} = \lambda\alpha_{s_1} \tag{1.3.10}$$

for some $\lambda > 0$. But this says precisely that

$$r_sr_{s_l}\dots r_{s_2}\alpha_{s_1} = \lambda r_{s_l}\dots r_{s_2}\alpha_{s_1}. \tag{1.3.11}$$

Observe that (1.3.6) forces $r_{s_l}\cdots r_{s_2}$ to be a right hand segment of $r_{s_2}\cdots r_{s_l}r_sr_{s_l}\cdots r_{s_2}$. Thus (1.3.8) above yields that $(r_{s_l}\cdots r_{s_2})\alpha_{s_1}\in\Phi_1^+$. Hence $r_s(r_{s_l}\cdots r_{s_2})\alpha_{s_1} = \lambda(r_{s_l}\cdots r_{s_2})\alpha_{s_1}\in\Phi_1^+$ too. This yields that $(r_{s_l}\dots r_{s_2})\alpha_{s_1}$ is not a positive scalar multiple of α_s by Lemma 1.2.12(i). Observe now

$$\begin{aligned}
&r_s(r_{s_l}\dots r_{s_2}\alpha_{s_1})\\
&= (r_{s_l}\dots r_{s_2})\alpha_{s_1} - 2\langle(r_{s_l}\dots r_{s_2})\alpha_{s_1},\beta_s\rangle\alpha_s\\
&= \lambda(r_{s_l}\dots r_{s_2})\alpha_{s_1}
\end{aligned}$$

forces $\langle r_{s_l}\cdots r_{s_2}\alpha_{s_1},\beta_s\rangle = 0$. Since $\langle\ ,\ \rangle$ is W-invariant, it follows that $\langle\alpha_{s_1}, r_{s_2}\cdots r_{s_l}\beta_s\rangle = 0$. Lemma 1.3.12 then yields $\langle r_{s_2}\cdots r_{s_l}\alpha_s,\beta_{s_1}\rangle = 0$, contradicting (1.3.7). Thus (1.3.9) follows, as required. □

CHAPTER 2

Special Topics in Non-Orthogonal Setting

2.1. Decompositions of Generic Root Systems

Suppose that U_1 and U_2 are vector spaces over the real field $\mathbb{R}$ and suppose that there exist a bilinear map $\langle\ ,\ \rangle' : U_1 \times U_2 \to \mathbb{R}$ and linearly independent sets $X_1 \subset U_1$ and $X_2 \subset U_2$ indexed by the same set S'' via $s \mapsto x_s \in X_1$ and $s \mapsto y_s \in X_2$ for all $s \in S''$. Furthermore, suppose that the following condition holds:

(D1) $\langle x_s, y_s\rangle' = 1$ for all $s \in S''$.

For each $s \in S''$, define $\rho_1(x_s)\colon U_1 \to U_1$ and $\rho_2(y_s)\colon U_2 \to U_2$ by

$$\rho_1(x_s)(u) := u - 2\langle u, y_s\rangle' x_s$$
$$\rho_2(y_s)(v) := v - 2\langle x_s, v\rangle' y_s$$

for all $u \in U_1$ and $v \in U_2$, and for each $i \in \{1, 2\}$, define

$$R''_i := \{\rho_i(x) \mid x \in X_i\};$$
$$W''_i := \langle R''_i\rangle;$$
$$\Phi''_i := W''_i X_i;$$
$$\Phi''^{+}_i := \Phi''_i \cap \mathrm{PLC}(X_i) \quad \text{and}$$
$$\Phi''^{-}_i := -\Phi''^{+}_i.$$

Proposition 2.1.1. *For each $i \in \{1, 2\}$, suppose that $\Phi''_i = \Phi''^{+}_i \uplus \Phi''^{-}_i$. Then the following conditions must be satisfied for all $s, t \in S''$:*

(D2) $\langle x_s, y_t\rangle' \le 0$ *and* $\langle x_t, y_s\rangle' \le 0$ *whenever* $s \ne t$;

(D3) $\langle x_s, y_t\rangle' = 0$ *if and only if* $\langle x_t, y_s\rangle' = 0$;

(D4) *either* $\langle x_s, y_t\rangle'\langle x_t, y_s\rangle' = \cos^2(\frac{\pi}{m_{st}})$ *for some integer* $m_{st} \ge 2$, *or else* $\langle x_s, y_t\rangle'\langle x_t, y_s\rangle' \ge 1$.

To prove this proposition we shall need a few technical results first. These results are essentially taken from [**1**], and for completeness, their proofs are included here.

Let $\mathscr{A}$ be a commutative $\mathbb{R}$-algebra, let $q^{\frac{1}{2}}$ and X be units of $\mathscr{A}$, and let $\gamma \in \mathbb{R}$. Define A, B to be 2 by 2 matrices over $\mathscr{A}$ given by:

$$A = \begin{pmatrix} -1 & 2\gamma q^{1/2} X \\ 0 & q \end{pmatrix} \qquad B = \begin{pmatrix} q & 0 \\ 2\gamma q^{1/2} X^{-1} & -1 \end{pmatrix}.$$

It is easily proved by induction on $n \in \mathbb{N}$ that

$$B(AB)^n = \begin{pmatrix} q^{n+1} p_{2n+1} & -q^{n+\frac{1}{2}} p_{2n} X \\ q^{n+\frac{1}{2}} p_{2n+2} X^{-1} & -q^n p_{2n+1} \end{pmatrix} \tag{2.1.1}$$

$$A(BA)^n = \begin{pmatrix} -q^n p_{2n+1} & q^{n+\frac{1}{2}} p_{2n+2} X \\ -q^{n+\frac{1}{2}} p_{2n} X^{-1} & q^{n+1} p_{2n+1} \end{pmatrix} \tag{2.1.2}$$

$$(BA)^n = \begin{pmatrix} -q^n p_{2n-1} & q^{n+\frac{1}{2}} p_{2n} X \\ -q^{n-\frac{1}{2}} p_{2n} X^{-1} & q^n p_{2n+1} \end{pmatrix} \tag{2.1.3}$$

and

$$(AB)^n = \begin{pmatrix} q^n p_{2n+1} & -q^{n-\frac{1}{2}} p_{2n} X \\ q^{n+\frac{1}{2}} p_{2n} X^{-1} & -q^n p_{2n-1} \end{pmatrix} \tag{2.1.4}$$

where $p_n \in \mathbb{R}$ (for $n \in \{-1\} \cup \mathbb{N}$) is defined recursively by

$$p_{-1} = -1, \qquad p_0 = 0, \qquad p_{n+1} = 2\gamma p_n - p_{n-1} \quad (n \in \mathbb{N}). \tag{2.1.5}$$

The solution of the recurrence (2.1.5) is

(2.1.6)

$$p_n = \begin{cases} n & \text{if } \gamma = 1 \\ (-1)^{n+1} n & \text{if } \gamma = -1 \\ \frac{1}{2\sqrt{\gamma^2-1}}[(\gamma + \sqrt{\gamma^2-1})^n - (\gamma - \sqrt{\gamma^2-1})^n] & \text{if } |\gamma| > 1 \\ \dfrac{\sin n\theta}{\sin\theta} \quad \text{where } \theta = \cos^{-1}\gamma & \text{if } |\gamma| < 1. \end{cases}$$

Note that in the case $|\gamma| \geq 1$ we may alternatively write

$$p_n = \frac{\sinh n\theta}{\sinh \theta} \qquad \text{where } \theta = \cosh^{-1} \gamma$$

(as in Lemma 1.1.8).

Lemma 2.1.2. (Dyer, [**1**, Lemma 2.2])

(i) *Conditions* (1′) *and* (2′) *below are equivalent:*

(1′) $p_n p_{n+1} \geq 0$ *for all* $n \in \mathbb{N}$*;*

(2′) $\gamma \in \{\cos(\pi/m) \mid m \in \mathbb{N},\ m \geq 2\} \cup [1, \infty)$.

(ii) *If* $\gamma = \cos \frac{k\pi}{m}$ *for some* $k, m \in \mathbb{N}$ *with* $0 < k < m$ *then the matrices* A *and* B *satisfy the equation* $ABA \cdots = BAB \cdots$, *where there are* m *factors on either side.*

(iii) *If* $q = 1$ *then the matrix* AB *has order* m *if* $\gamma = \cos \frac{k\pi}{m}$ *for some* $k, m \in \mathbb{N}$ *with* $0 < k < m$ *and* $\gcd(m, k) = 1$, *and has infinite order otherwise.*

Proof. (i): Assume that (1′) holds. Observe that (2.1.5) yields that $p_1 = 1$ and $p_2 = 2\gamma$; hence $\gamma \geq 0$. Since (2′) obviously holds if $\gamma \geq 1$, we may assume that $0 \leq \gamma < 1$. Choose θ so that $0 < \theta \leq \frac{\pi}{2}$ and $\cos \theta = \gamma$, and let m be the largest integer such that

$$0 < \theta < 2\theta < \cdots < m\theta \leq \pi,$$

noting that $m \geq 2$. Now if $m\theta \neq \pi$ then $\pi < (m+1)\theta < 2\pi$, and it follows that

$$p_m = \frac{\sin m\theta}{\sin \theta} > 0$$

whereas

$$p_{m+1} = \frac{\sin(m+1)\theta}{\sin \theta} < 0,$$

contradicting (1′). Hence $m\theta = \pi$, so that $\gamma = \cos \frac{\pi}{m}$ for some $m \geq 2$, whence (2′) holds. Thus (1′) implies (2′).

Conversely, if (2′) holds then it follows from (2.1.6) that (1′) holds. This completes the proof of (i).

(ii) Suppose first that m is even, and write $m = 2r$, so that our task is to prove that $(AB)^r = (BA)^r$. We have that $p_n = \frac{\sin(nk\pi/2r)}{\sin(k\pi/2r)}$, which gives $p_{2r+1} = \frac{\sin(k\pi+(k\pi/2r))}{\sin(k\pi/2r)} = (-1)^k$ and $p_{2r-1} = (-1)^{k+1}$ by a similar calculation, while $p_{2r} = 0$. Hence the required result follows immediately from (2.1.3) and (2.1.4).

If m is odd our task is to prove that $B(AB)^r = A(BA)^r$, where $m = 2r+1$. In this case we find that $p_{2r+1} = 0$ while $p_{2r+2} = (-1)^k$ and $p_{2r} = (-1)^{k+1}$, and the required result follows immediately from (2.1.1) and (2.1.2).

(iii) If $\gamma = \cos\frac{k\pi}{m}$ then it follows immediately from Part (ii) that $(AB)^m = 1$, since $A^2 = B^2 = 1$ when $q = 1$. Furthermore, if $0 < n < m$ then $p_n = \frac{\sin(nk\pi/m)}{\sin(k\pi/m)} \neq 0$, and it follows from (2.1.4) that $(AB)^n \neq 1$. Thus AB has order m. Similarly if $|\gamma| \geq 1$ then it follows from (2.1.6) that p_n is nonzero for all integers $n > 0$, giving $(AB)^n \neq 1$ for all such n, so that AB has infinite order. $\square$

Now we are ready to prove Proposition 2.1.1:

Proof of Proposition 2.1.1. We present a proof that $\Phi_1'' = \Phi_1''^{+} \uplus \Phi_1''^{-}$ implies conditions (D2), (D3) and (D4) and we stress that the same argument applies equally to Φ_2''.

Suppose that $\Phi_1'' = \Phi_1''^{+} \uplus \Phi_1''^{-}$. Let $s, t \in S''$ be distinct. Consider the actions of $\rho_1(x_s)\rho_1(x_t)$ and $\rho_1(x_t)\rho_1(x_s)$ in the $\langle\{\rho_1(x_s), \rho_1(x_t)\}\rangle$-invariant subspace $\mathbb{R}x_s + \mathbb{R}x_t$:

$$
\begin{aligned}
(\rho_1(x_t)\rho_1(x_s))(x_s) = \rho_1(x_t)(\ x_s - 2\langle x_s, y_s\rangle' x_s\ ) &= \rho_1(x_t)(-x_s) \\
&= -x_s + 2\langle x_s, y_t\rangle' x_t.
\end{aligned}
$$

Since $\Phi_1'' = \Phi_1''^{+} \uplus \Phi_1''^{-}$ and X_1 is linearly independent, it follows from above that $\langle x_s, y_t\rangle' \leq 0$. Similarly, by looking at $(\rho_1(x_s)\rho_1(x_t))(x_t)$ we may deduce that $\langle x_t, y_s\rangle' \leq 0$, whence (D2) holds.

Next, suppose that $s, t \in S''$ are distinct with $\langle x_s, y_t\rangle' = 0$. Consider

$$\begin{aligned} \rho_1(x_t)\rho_1(x_s)(x_t) &= \rho_1(x_t)(\ x_t - 2\langle x_t, y_s\rangle' x_s\ ) \\ &= -x_t - 2\langle x_t, y_s\rangle' x_s + 4\langle x_t, y_s\rangle'\langle x_s, y_t\rangle' x_t \\ &= -x_t - 2\langle x_t, y_s\rangle' x_s. \end{aligned}$$

Since $\Phi_1'' = \Phi_1''^{+} \uplus \Phi_1''^{-}$ and X_1 is linearly independent, it follows from above that $\langle x_t, y_s\rangle' \geq 0$ and upon combining with (D2), this in turn yields that $\langle x_t, y_s\rangle' = 0$. In a similar way, we may deduce that $\langle x_t, y_s\rangle' = 0$ implies that $\langle x_s, y_t\rangle' = 0$, thus proving (D3).

Observe that if $\langle x_s, y_t\rangle' = 0$ and $\langle x_t, y_s\rangle' = 0$ then

$$\langle x_s, y_t\rangle'\langle x_t, y_s\rangle' = \cos^2(\pi/2),$$

satisfying (D4). Thus to prove that (D4) holds, we may assume that $\langle x_s, y_t\rangle' \neq 0$ and $\langle x_t, y_s\rangle' \neq 0$. Now let $\mathscr{A}, \gamma, q, X, A$ and B be as defined above. We set

$$\begin{aligned} \mathscr{A} &= \mathbb{R}; \\ q &= 1; \\ \gamma &= \sqrt{\langle x_s, y_t\rangle'\langle x_t, y_s\rangle'}; \end{aligned}$$

and

$$X = \frac{-\langle x_t, y_s\rangle'}{\sqrt{\langle x_s, y_t\rangle'\langle x_t, y_s\rangle'}}.$$

Compared with the proofs of Lemma 1.1.6 and Lemma 1.1.8 we see that A and B are the matrices representing the actions of $\rho_1(x_s)$ and $\rho_1(x_t)$ respectively on the $\langle\{\rho_1(x_s), \rho_1(x_t)\}\rangle$-invariant subspace $\mathbb{R}x_s + \mathbb{R}x_t$. By (2.1.1) and (2.1.4) the condition

$$\langle\{\rho_1(x_s), \rho_1(x_t)\}\rangle x_s \cup \langle\{\rho_1(x_s), \rho_1(x_t)\}\rangle x_t \subseteq \Phi_1''^{+} \uplus \Phi_1''^{-}$$

is equivalent to $p_n p_{n+1} \geq 0$ for all $n \in \mathbb{N}$. By Lemma 2.1.2 above, the condition $p_n p_{n+1} \geq 0$ for all $n \in \mathbb{N}$ is, in turn, equivalent to

$$\langle x_s, y_t\rangle'\langle x_t, y_s\rangle' \in \{\cos^2\frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\} \cup [1, \infty),$$

whence (D4) holds.

□

Now we are ready for the main result of this section:

Theorem 2.1.3. *The following are equivalent:*

(1) *for each* $i \in \{1, 2\}$, $\Phi_i'' = \Phi_i''^{+} \uplus \Phi_i''^{-}$;
(2) $(S'', \operatorname{span} X_1, \operatorname{span} X_2, X_1, X_2, \langle\,,\,\rangle'\,)$ *is a Coxeter datum.*

Proof. Suppose that (1) holds. It follows from the definition of $\langle\,,\,\rangle'$ and Proposition 2.1.1 above that conditions (C1) to (C5) of the definition for a Coxeter datum are all satisfied in our present situation. Since X_1 and X_2 are linearly independent, it is clear that $0 \notin \mathrm{PLC}(X_1)$ and $0 \notin \mathrm{PLC}(X_2)$, showing that (C6) of the definition of a Coxeter datum is also satisfied. Thus (1) implies (2).

Suppose that (2) holds. Then (1) simply follows from Lemma 1.2.4 applied to the Coxeter datum $(S'', \operatorname{span} X_1, \operatorname{span} X_2, X_1, X_2, \langle\,,\,\rangle'\,)$. □

Proposition 2.1.4. *Suppose that conditions (D2), (D3) and (D4) of Proposition 2.1.1 are satisfied. Then* (W_1'', R_1'') *and* (W_2'', R_2'') *are isomorphic Coxeter systems. Furthermore if* $s, t \in S''$, $s \neq t$, *then*

$$\operatorname{ord}(\rho_1(x_s)\rho_1(x_t)) = \begin{cases} m & \textit{if } \langle x_s, y_t\rangle'\langle x_t, y_s\rangle' = \cos^2\frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\ \infty & \textit{if } \langle x_s, y_t\rangle'\langle x_t, y_s\rangle' \geq 1. \end{cases}$$

Proof. Keep all notation as in Chapter 1. We have already observed that $(S'', \operatorname{span} X_1, \operatorname{span} X_2, X_1, X_2, \langle\,,\,\rangle')$ is a Coxeter datum; then Theorems 1.1.4 and 1.1.15 yields that (W_1'', R_1'') and (W_2'', R_2'') are isomorphic Coxeter systems. Furthermore, applying Lemma 1.1.8 (ii) and Proposition 1.1.9 to this Coxeter datum, we duduce that for distinct $s, t \in S''$,

$$\operatorname{ord}(\rho_1(x_s)\rho_1(x_t)) = \begin{cases} m & \text{if } \langle x_s, y_t\rangle'\langle x_t, y_s\rangle' = \cos^2\frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\ \infty & \text{if } \langle x_s, y_t\rangle'\langle x_t, y_s\rangle' \geq 1. \end{cases}$$

□

2.2. Canonical Generators of Reflection Root Subsystems

Keep all notation as in Chapter 1. Recall that $T = \bigcup_{w \in W} wRw^{-1}$ is the *set of reflections* in W.

Suppose that W' is a subgroup of W. We say that W' is a *reflection subgroup* of W whenever $W' = \langle W \cap T \rangle$. For each $i \in \{1, 2\}$, if Φ_i' is a subset of Φ_i such that $r_x y \in \Phi_i'$ whenever $x, y \in \Phi_i'$, then we call Φ_i' a *root subsystem* of Φ_i.

If W' is a reflection subgroup of W, set

$$\Phi_i(W') := \{\, x \in \Phi_i \mid r_x \in W' \,\}.$$

Let $x, y \in \Phi_i(W')$. Then $r_x, r_y \in W' \cap T$. Since Lemma 1.3.16 yields that $r_{(r_x y)} = r_x r_y r_x$, it follows that $r_{(r_x y)} \in W' \cap T$ showing that $r_x y \in \Phi_i(W')$. Therefore $\Phi_i(W')$ is a root subsystem of Φ_i and we call $\Phi_i(W')$ the *root subsystem corresponding to* W'. It can be seen (for example, (1.4.2) of [**11**]) that the above correspondence gives a bijection between reflection subgroups $W' \subseteq W$ and root subsystem $\Phi_i(W') \subseteq \Phi_i$. Observe that $\Phi(W')$ is stable under the action of W', indeed:

Lemma 2.2.1. *Let W' be a reflection subgroup of W. Then for each $i \in \{1, 2\}$*

$$W'\Phi_i(W') = \Phi_i(W').$$

Proof. We prove that $W'\Phi_1(W') = \Phi_1(W')$ here and we stress that the other half follows in the same way. Let $w \in W'$. By definition, we have $w = t_1 t_2 \cdots t_n$ where $t_1, t_2, \ldots, t_n \in W' \cap T$. The definition of T yields that, for all $i \in \{1, 2, \ldots, n\}$,

$$t_i = w_i r_{s_i} w_i^{-1} = \underbrace{r_{(w_i \alpha_{s_i})}}_{\text{by Lemma 1.3.16}}$$

for some $w_i \in W$ and $s_i \in S$. Since $r_{(w_i \alpha_{s_i})} \in W'$, it follows that $w_i \alpha_{s_i} \in \Phi_1(W')$. Now let $x \in \Phi_1(W')$ be arbitrary. It follows from the definition of a root subsystem that $t_n x = r_{w_n \alpha_{s_n}} x \in \Phi_1(W')$.

This in turn yields that $t_{n-1}t_nx \in \Phi_1(W')$ and so on. In particular, $wx = t_1 \cdots t_n x \in \Phi_1(W')$. Since $x \in \Phi_1(W')$ is arbitrary, it follows that $w\Phi_1(W') \subseteq \Phi_1(W')$. Finally, replacing w by w^{-1} we see that $\Phi_1(W') \subseteq w\Phi_1(W')$. □

Definition 2.2.2. Let W' be a reflection subgroup of W. For each $i \in \{1, 2\}$, set

$$\Delta_i(W') := \{\, x \in \Phi_i^+ \mid N_i(r_x) \cap \widehat{\Phi_i(W')} = \{\widehat{x}\}\,\}.$$

We call $\Delta_i(W')$ the *canonical roots* or *canonical generators* of $\Phi_i(W')$.

The key result in this section is a criterion for a set of positive roots in $\Phi_i(W')$, for some reflection subgroup W' of W, to be the set of canonical roots of $\Phi_i(W')$. Note that at this stage it is not entirely obvious that $\Phi_i(W')$ is generated by $\Delta_i(W')$, and we shall prove this fact in (iii) of Lemma 2.2.6 below. We acknowledge that the works presented here closely follow those in Chapter 3 of [**1**].

Remark 2.2.3. (1) Note that for each $\alpha \in \Phi_1$, $r_\alpha = r_{\phi(\alpha)}$ (where ϕ is as in Definition 1.2.18), and it follows that for any reflection subgroup W' of W, $\phi(\Phi_1(W')) = \Phi_2(W')$.

(2) For $\alpha_1, \alpha_2 \in \Phi_1^+$, Corollary 1.3.4 yields that $r_{\alpha_1}\alpha \in \Phi_1^-$ if and only if $r_{\phi(\alpha_1)}\phi(\alpha_2) \in \Phi_2^-$. Thus $N_2(r_{\phi(\alpha_1)}) = \phi(N_1(r_{\alpha_1}))$.

These observations lead to:

Lemma 2.2.4. *For any reflection subgroup W' of W, ϕ restricts to a bijection*

$$\Delta_1(W') \leftrightarrow \Delta_2(W').$$

□

Immediately from this observation we deduce that, for any reflection subgroup W' of W,

$$\{\, r_\alpha \in T \mid \alpha \in \Delta_1(W')\,\} = \{\, r_\beta \in T \mid \beta \in \Delta_2(W')\,\}$$

and we shall give a set like this a special name:

Definition 2.2.5. Let W' be a reflection subgroup of W. Set

$$S(W') := \{\, r_x \in T \mid x \in \Delta_i(W')\} \qquad \text{for any } i \in \{1, 2\}\,.$$

Observe that the definitions of $N_1(w)$, $N_2(w)$ ($w \in W$) and Proposition 1.3.17 yield that

$$S(W') = \{\, t \in W' \cap T \mid \text{if } t' \in W' \cap T \text{ with } l(tt') < l(t) \text{ then } t = t' \,\}.$$

Thus our $S(W')$ here defines the same set of reflections in W' as the set denoted by $S(W')$ in [**1**] (see 1.6 and Theorem 1.8 of [**1**]). Hence we may apply Theorem 1.8 of [**1**] directly:

Lemma 2.2.6. *Let W' be a reflection subgroup of W.*

(i) (Lemma (1.7) (ii) [**1**]) *If $t \in W' \cap T$, then there exist $m \in \mathbb{N}$ and $t_0, \cdots, t_m \in S(W')$ such that $t = t_m \cdots t_1 t_0 t_1 \cdots t_m$.*
(ii) (Thorem (1.8) (i) [**1**]) *$(W', S(W'))$ is a Coxeter system.*
(iii) *For each $i \in \{1, 2\}$, let $x \in \Pi_i \setminus \Phi_i(W')$. Then*

$$\Delta_i(r_x W' r_x) = r_x \Delta_i(W').$$

(iv) *For each $i \in \{1, 2\}$, $\Phi_i(W') = W' \Delta_i(W')$.*

Proof. Only (iii) and (iv) need to be proved here.

(iii): It is readily checked that $r\Phi_i(W') = \Phi_i(rW'r)$ for all $r \in T$. Since $x \in \Pi_i \setminus \Phi_i(W')$, it follows that $r_x \in R \setminus W'$. Let $y \in \Delta_i(W')$ be arbitrary. Then

$$\begin{aligned}
&N_i(r_{(r_xy)}) \cap \Phi_i(\widehat{r_xW'}r_x)\\
&= N_i(r_xr_yr_x) \cap \Phi_i(\widehat{r_xW'}r_x)\\
&\qquad\qquad\text{(by Lemma 1.3.16)}\\
&= (r_xN_i(r_xr_y) \dot{+} N_i(r_x)) \cap \Phi_i(\widehat{r_xW'}r_x)\\
&\qquad\qquad\text{(by Lemma 1.2.12 (iii))}\\
&= (r_xr_yN_i(r_x) \dot{+} r_xN_i(r_y) \dot{+} N_i(r_x)) \cap \Phi_i(\widehat{r_xW'}r_x)\\
&\qquad\qquad\text{(again by Lemma 1.2.12 (iii))}\\
&= r_x((r_yN_i(r_x) \dot{+} N_i(r_y) \dot{+} N_i(r_x)) \cap \widehat{\Phi_i(W')})\\
&= r_x((r_y\{\widehat{x}\} \dot{+} N_i(r_y) \dot{+} \{\widehat{x}\}) \cap \widehat{\Phi_i(W')})\\
&\qquad\qquad\text{(by Lemma 1.2.12 (i))}\\
&= r_x(N_i(r_y) \cap \widehat{\Phi_i(W')})\\
&\qquad\qquad\text{(since } \widehat{x}, r_y\widehat{x} \notin \widehat{\Phi_i(W')})\\
&= \{\widehat{r_xy}\}.
\end{aligned}$$

Hence $r_xy \in \Delta_i(r_xW'r_x)$. This proves that $r_x\Delta_i(W') \subseteq \Delta_i(r_xW'r_x)$. But $x \in \Pi_i \backslash r_x\Phi_i(W')$, so the above yields that $r_x\Delta_i(r_xW'r_x) \subseteq \Delta_i(W')$ proving the desired result.

(iv): Since $\Delta_i(W') \subseteq \Phi_i(W')$ for each $i \in \{1, 2\}$ and $\Phi_i(W')$ is a root subsystem it follows that $r\Delta_i(W') \subseteq \Phi_i(W')$ for all $r \in S(W')$. Then part (ii) above yields that $W'\Delta_i(W') \subseteq \Phi_i(W')$.

Conversely if $x \in \Phi_i(W')$ then $r_x \in W' \cap T$. By (i) above there are $x_0, x_1, \cdots, x_m \in \Delta_i(W')$ ($m \in \mathbb{N}$) such that

$$r_x = r_{x_m} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_m}.$$

Thus Lemma 1.3.16 yields that $\lambda x = r_{(r_{x_m} \cdots r_{x_1})}(x_0) \in W'\Phi_i(W')$ for some (nonzero) scalar λ. Since $\frac{1}{\lambda}x_0 = (r_{(r_{x_m} \cdots r_{x_1})})^{-1}(x) \in \Phi_i$, it follows that $\frac{1}{\lambda}x_0 \in \Delta_i(W')$ and hence $x = r_{(r_{x_m} \cdots r_{x_1})}(\frac{1}{\lambda}x_0) \in W'\Delta_i(W')$ as required. $\square$

Definition 2.2.7. Let W' be a reflection subgroup of W, and let $l' : W' \to \mathbb{N}$ be the length function on $(W', S(W'))$ defined by

$$l'(w) = \min\{\, n \in \mathbb{N} \mid w = r_1 \cdots r_n, \text{ where } r_i \in S(W')\,\}.$$

If $w = r_1 \cdots r_n \in W'$ ($r_i \in S(W')$) and $n = l'(w)$ then $r_1 \cdots r_n$ is called a *reduced expression* for w (with respect to $S(W')$).

Lemma 2.2.8. *Let W' be a reflection subgroup. For each $i \in \{1, 2\}$,*

(i) $N_i(r_x) \cap \widehat{\Phi_i(W')} = \{\widehat{x}\}$ *for all* $x \in \Delta_i(W')$;

(ii) *for all* $w_1 \in W$ *and* $w_2 \in W'$

$$N_i(w_1w_2) \cap \widehat{\Phi_i(W')} = w_2^{-1}(N_i(w_1) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(w_2) \cap \widehat{\Phi_i(W')}).$$

Proof. (i) is just the definition of $\Delta_i(W')$.

(ii) Lemma 1.2.12(iii) yields that $N_i(w_1w_2) = w_2^{-1}N_i(w_1) \dotplus N_i(w_2)$, and hence

$$N_i(w_1w_2) \cap \widehat{\Phi_i(W')} = (w_2^{-1}N_i(w_1) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(w_2) \cap \widehat{\Phi_i(W')}).$$

Since $w_2 \in W'$ it follows from Lemma 2.2.1 that $w_2^{-1}\widehat{\Phi_i(W')} = \widehat{\Phi_i(W')}$. Thus $w_2^{-1}N_i(w_1) \cap \widehat{\Phi_i(W')} = w_2^{-1}(N_i(w_1) \cap \widehat{\Phi_i(W')})$ giving us

$$N_i(w_1w_2) \cap \widehat{\Phi_i(W')} = w_2^{-1}(N_i(w_1) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(w_2) \cap \widehat{\Phi_i(W')}).$$

□

Lemma 2.2.9. *Let W' be a reflection subgroup. For each $i \in \{1, 2\}$ and all $w \in W'$, we have*

(i) $|N_i(w) \cap \widehat{\Phi_i(W')}| = l'(w)$. *Furthermore, if $w = r_{x_1} \cdots r_{x_n}$ is reduced with respect to $(W', S(W'))$ then*

$$N_i(w) \cap \widehat{\Phi_i(W')} = \{\widehat{y_1}, \cdots \widehat{y_n}\}$$

where $y_j = (r_{x_n} \cdots r_{x_{j+1}})x_j$ for all $j = 1, \cdots, n$.

(i) $N_i(w) \cap \widehat{\Phi_i(W')} = \{\widehat{x} \in \widehat{\Phi_i(W')} \mid l'(wr_x) < l'(w)\}$.

Proof. (i): For each $j \in \{1, \cdots, n\}$, set $t_j = r_{x_n} \cdots r_{x_{j+1}} r_{x_j} r_{x_{j+1}} \cdots r_{x_n}$, that is, $t_j = r_{y_j}$. If $t_j = t_k$ where $j > k$ then

$$\begin{aligned} w &= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_n} t_k \\ &= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_n} t_j \\ &= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_{j-1}} r_{x_{j+1}} \cdots r_{x_n} \end{aligned}$$

contradicting $l'(w) = n$. Hence the t_j's are all distinct and consequently all the $\widehat{y_j}$'s are all distinct. Now by repeated application of Lemma 2.2.8 (ii), for each $i \in \{1, 2\}$ we have

$$\begin{aligned} &N_i(w) \cap \widehat{\Phi_i(W')} \\ &= (N_i(r_{x_n} \cap \widehat{\Phi_i(W')}) \dotplus r_{x_n}(N_i(r_{n-1}) \cap \widehat{\Phi_i(W')}) \dotplus \cdots \\ &\qquad\qquad \dotplus r_{x_n} \cdots r_{x_2}(N_i(r_{x_1}) \cap \widehat{\Phi_i(W')}) \\ &= \{\widehat{y_n}\} \dotplus \{\widehat{y_{n-1}}\} \dotplus \cdots \dotplus \{\widehat{y_1}\} \\ &= \{\, \widehat{y_1}, \cdots, \widehat{y_n} \,\} \end{aligned}$$

and consequently $|N_i(w) \cap \widehat{\Phi_i(W')}| = l'(w)$.

(ii): Let $w = r_{x_1} \cdots r_{x_n}$ be a reduced expression for $w \in W'$ with respect to $S(W')$ $(x_j \in \Delta_i(W'))$. Then (i) yields that for each $i \in \{1, 2\}$

$$N_i(w) \cap \widehat{\Phi_i(W')} = \{\, \widehat{y_1}, \cdots, \widehat{y_n} \,\}$$

where $y_j = (r_{x_n} \cdots r_{x_{j+1}}) x_j$, for all $j \in \{1, \cdots, n\}$. Now for each such j,

$$w r_{y_j} = w r_{x_n} \cdots r_{x_{j+1}} r_{x_j} r_{x_{j+1}} \cdots r_{x_n} = r_{x_1} \cdots r_{x_{j-1}} r_{x_{j+1}} \cdots r_{x_n}$$

and so $l'(w r_{y_j}) \leq n - 1 < l'(w)$. Hence if $\widehat{x} \in N_i(w) \cap \widehat{\Phi_i(W')}$, then $l'(w r_x) < l'(w)$.

Conversely suppose that $x \in \Phi_i(W') \cap \Phi_i^+$ and $\widehat{x} \notin N_i(w)$. Then $\widehat{x} \notin r_x(N_i(w) \cap \widehat{\Phi_i(W')})$. But $\widehat{x} \in N_i(r_x) \cap \widehat{\Phi_i(W')}$, so

$$\widehat{x} \in r_x(N_i(w) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(r_x) \cap \widehat{\Phi_i(W')}) = N_i(w r_x) \cap \widehat{\Phi_i(W')}$$

and by what has just been proved, this implies that

$$l'(w) = l'((w r_x) r_x) < l'(w r_x).$$

Therefore if $x \in \Phi_i(W') \cap \Phi_i^+$ such that $l'(wr_x) < l'(w)$, then we must have $\widehat{x} \in N_i(w) \cap \widehat{\Phi_i(W')}$. $\square$

Lemma 2.2.10. ((Lemma 3.2) of [**1**]) *Let W' be a reflection subgroup of W. For each $i \in \{1, 2\}$, let $x, y \in \Delta_i(W')$ such that $r_x \neq r_y$. Let $n = \mathrm{ord}(r_x r_y)$. Then for $0 \leq m < n$*

$$\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}} x \in \Phi_i^+ \qquad \text{and} \qquad \underbrace{\cdots r_x r_y r_x}_{m \text{ factors}} y \in \Phi_i^+.$$

Proof. Since $x, y \in \Delta_i(W')$ it follows that

$$l'((\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}})r_x) = m + 1 > m = l'(\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}})$$

whenever $m < n$. Then Lemma 2.2.9 (ii) yields that

$$\widehat{x} \notin N_i(\underbrace{(\cdots r_y r_x r_y)}_{m \text{ factors}}) \cap \widehat{\Phi_i(W')}.$$

Therefore $\underbrace{(\cdots r_y r_x r_y)}_{m \text{ factors}} x \in \Phi_i^+$.

By symmetry $\underbrace{(\cdots r_x r_y r_x)}_{m \text{ factors}} y \in \Phi_i^+$ too. $\square$

Lemma 2.2.11. ((Lemma 3.3) of [**1**]) *Let W' be a reflection subgroup of W. For each $i \in \{1, 2\}$, let $x, y \in \Delta_i(W')$ with $r_x \neq r_y$. Let $n = \mathrm{ord}(r_x r_y)$ and write*

$$\underbrace{(\cdots r_y r_x r_y)}_{m \text{ factors}} x = c_m x + d_m y \quad \text{and} \quad \underbrace{(\cdots r_x r_y r_x)}_{m \text{ factors}} y = c'_m x + d'_m y.$$

Then $c_m \geq 0$, $d_m \geq 0$, $c'_m \geq 0$ and $d'_m \geq 0$ whenever $m < n$.

Proof. By symmetry, it will suffice to prove that $d_m \geq 0$ and $d'_m \geq 0$. The proof of this will be based on an induction on $l(r_x)$.

Suppose first that $l(r_x) = 1$. Then $\lambda x \in \Pi_i$ for some $\lambda > 0$. Write $y = \sum_{z \in \Pi_i} \lambda_z z$ where $\lambda_z \geq 0$ for all $z \in \Pi_i$. In fact, $\lambda_{z_0} > 0$ for some $z_0 \in \Pi_i \setminus \{x\}$, since otherwise we would have $y \in \mathbb{R}x$ and so $r_x = r_y$.

Now for $0 \leq m < n$, Lemma 2.2.10 yields that

$$(\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}})x = c_m x + \sum_{z \in \Pi_i} d_m \lambda_z z.$$

Observe that the coefficient of z_0 in this is $d_m \lambda_{z_0} \geq 0$. Since $\lambda_{z_0} > 0$, it follows that $d_m \geq 0$. Similarly $d'_m \geq 0$.

Suppose inductively now that the result is true for reflection subgroups W'' of W and x', $y' \in \Delta_i(W'')$ with $r_x \neq r_y$ and $l(r_{x'}) < l(r_x)$ where $l(r_x) \geq 3$. It is well know that there exists $r_z \in R$ (so $z \in \Pi_i$) that $l(r_z r_x r_z) = l(r_x) - 2$. Then $l(r_x r_z) < l(r_x)$, and thus $\widehat{z} \in N_i(r_x)$. But since $x \in \Delta_i(W')$ and $x \neq z$ (since $l(r_x) \geq 3$), it follows that $r_z \notin W'$. Let $W'' = r_z W' r_z$. Lemma 2.2.6 (iii) yields that $\Delta_i(W'') = r_z \Delta_i(W')$ and therefore $r_z x$, $r_z y \in \Delta_i(W'')$. Now

$$r_{(r_z x)} = r_z r_x r_z \qquad \text{and} \qquad r_{(r_z y)} = r_z r_y r_z \tag{2.2.1}$$

and hence $\operatorname{ord}(r_{(r_z x)} r_{(r_z y)}) = \operatorname{ord}(r_x r_y) = n$. Since $l(r_{(r_z x)}) = l(r_x) - 2$, the inductive hypothesis gives

$$(\underbrace{\cdots r_{(r_z y)} r_{(r_z x)} r_{(r_z y)}}_{m \text{ factors}})(r_z x) = c_m (r_z x) + d_m (r_z y)$$

and

$$(\underbrace{\cdots r_{(r_z x)} r_{(r_z y)} r_{(r_z x)}}_{m \text{ factors}})(r_z y) = c'_m (r_z x) + d'_m (r_z y)$$

where d_m, $d'_m \geq 0$ for $0 \leq m < n$. By (2.2.1) the result follows on applying r_z to both sides of the last two equations. □

Proposition 2.2.12. *Let W' be a reflection subgroup of W. Suppose that $x, y \in \Delta_1(W')$ with $r_x \neq r_y$. Let $n = \operatorname{ord}(r_x r_y) \in \{\infty\} \cup \mathbb{N}$. Then*

$$\langle x, \phi(y) \rangle \leq 0$$

and

$$\begin{cases} \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle = \cos^2 \frac{\pi}{n} & (n \in \mathbb{N}, n \geq 2) \\ \langle x, \phi(y) \rangle \langle y, \phi(x) \rangle \in [1, \infty) & (n = \infty) \end{cases}$$

Proof. Observe that since $r_{\phi(x)} = r_x \neq r_y = r_{\phi(y)}$, it follows that $\{x, y\}$ and $\{\phi(x), \phi(y)\}$ are both linearly independent. Therefore we may apply last section to the present situation, that is,

$$X_1 = \{x, y\} \quad X_2 = \{\phi(x), \phi(y)\} \quad \psi = \phi \quad \langle\,,\,\rangle' = \langle\,,\,\rangle$$

and

$$R_1'' = R_2'' = \{r_x, r_y\} \quad W_1'' = W_2'' = \langle\{r_x, r_y\}\rangle$$

Consequently $\Phi_1'' = W_1''X_1 = \langle\{r_x, r_y\}\rangle\{x, y\}$. Observe that the elements of Ψ are $\pm(\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}})x$ and $\pm(\underbrace{\cdots r_x r_y r_x}_{m \text{ factors}})y$ $(0 \leq m < \operatorname{ord}(r_x r_y))$. Lemma 2.2.11 then yields that

$$\Psi = \Psi^+ \uplus \Psi^-.$$

Therefore Proposition 2.1.1 yields that

$$\begin{cases} \langle x, \phi(y)\rangle\langle y, \phi(x)\rangle = \cos^2\frac{\pi}{n} & (n \in \mathbb{N}, n \geq 2) \\ \langle x, \phi(y)\rangle\langle y, \phi(x)\rangle \in [1, \infty) & (n = \infty). \end{cases}$$

□

Let Δ_1 be a subset of Φ_1^+ satisfying the following conditions:

(1) $r_x \neq r_y$ for all $x, y \in \Delta_1$, $x \neq y$.
(2) $\langle x, \phi(y)\rangle \leq 0$ for all $x, y \in \Delta_1$, $x \neq y$.
(3) $\langle x, \phi(y)\rangle\langle y, \phi(x)\rangle \in \{\cos^2\frac{\pi}{m} \mid m \in \mathbb{N}, m \geq 2\} \cup [1, \infty)$, for all $x, y \in \Delta_1$.

Let $W' = \langle r_x \mid x \in \Delta_1\rangle$. The rest of this section is devoted to showing that

$$\{\lambda x \in \Phi_1^+ \mid \lambda > 0, x \in \Delta_1\} = \Delta_1(W').$$

Most of the arguments to be used in proving this are adapted from [**1**].

Let Δ_1' be a subset of Δ_1.

Let U be a vector space over $\mathbb{R}$ on a basis $\Pi := \{e_x \mid x \in \Delta_1'\}$. Let U' be a vector space over $\mathbb{R}$ on a basis $\Pi' := \{f_{\phi(x)} \mid x \in \Delta_1'\}$.

Define a bilinear map $\langle\,,\,\rangle : U \times U' \to \mathbb{R}$ by requiring

$$\langle e_x, f_{\phi(y)}\rangle = \langle x, \phi(y)\rangle \qquad \text{for all } x, y \in \Delta_1'.$$

For each $x \in \Delta_1'$, define a linear transformation $\rho_x : U \to U$ by

$$\rho_x(u) := u - 2\langle u, f_{\phi(x)}\rangle e_x \qquad \text{for all } u \in U.$$

Let

$$\begin{aligned} R'' &= \{\, \rho_x \mid x \in \Delta_1' \,\}; \\ W'' &= \langle R'' \rangle; \\ \Psi &= W''\Pi; \\ \Psi^+ &= \Psi \cap \mathrm{PLC}(\pi); \\ \Psi^- &= -\Psi^+. \end{aligned}$$

Observe that $(\Delta_1', U, U', \Pi, \Pi', \langle\,,\,\rangle)$ is a Coxeter datum, and it follows by Theorem 1.1.4 (as well as Proposition 2.1.4 above) that (W'', R'') is a Coxeter system. Let $l'' : W'' \to \mathbb{N}$ be the length function on (W'', R''). Corollary 1.2.5 applied to this Coxeter datum yields that:

Proposition 2.2.13. *Suppose that $w'' \in W''$ and $x \in \Delta_1'$. Then*

$$l''(w''\rho_x) = \begin{cases} l''(w'') - 1 & \text{if } w''e_x \in \Psi^- \\ l''(w'') + 1 & \text{if } w''e_x \in \Psi^+. \end{cases} \quad \square$$

Note that by Proposition 2.1.4, for $x, y \in \Delta_1$ with $x \neq y$ we have:

$$\begin{aligned} &\mathrm{ord}(\rho_x, \rho_y) \\ &= \begin{cases} m & \langle e_x, f_{\phi(y)}\rangle\langle e_y, f_{\phi(x)}\rangle = \cos^2\frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\ \infty & \langle e_x, f_{\phi(y)}\rangle\langle e_y, f_{\phi(x)}\rangle \geq 1 \end{cases} \\ &= \begin{cases} m & \langle x, \phi(y)\rangle\langle y, \phi(x)\rangle = \cos^2\frac{\pi}{m}, m \in \mathbb{N}, m \geq 2 \\ \infty & \langle x, \phi(y)\rangle\langle y, \phi(x)\rangle \geq 1 \end{cases} \\ &= \mathrm{ord}(r_x r_y) \qquad \text{(by Lemma 1.1.8 (ii) and Proposition 1.1.9)} \end{aligned}$$

Since $W'' = \langle R'' \rangle$ the above yield that there exists a homomorphism

$$\theta : W'' \to W' \quad \text{such that } \theta(\rho_x) = r_x, \quad x \in \Delta_1'.$$

Next let $L : U \to V_1$ be the $\mathbb{R}$-linear map such that $L(e_x) = x$, for all $x \in \Delta_1'$. We claim that

$$L(w''u) = \theta(w'')L(u) \qquad w'' \in W'', u \in U. \tag{2.2.2}$$

To prove (2.2.2) first observe that if $x, y \in \Delta_1'$, then

$$\begin{aligned} &L(\rho_x(e_y)) \\ &= L(e_y - 2\langle e_y, f_{\phi(x)}\rangle e_x) \\ &= y - 2\langle e_y, f_{\phi(x)}\rangle x \\ &= y - 2\langle y, \phi(x)\rangle x \\ &= r_x y \\ &= \theta(\rho_x)L(e_y). \end{aligned}$$

By linearity, this gives that for all $x \in \Delta_1'$,

$$L(\rho_x(u)) = \theta(\rho_x)L(U) \qquad \text{for all } u \in U.$$

Since $W'' = \langle \rho_x \mid x \in \Delta_1' \rangle$ and θ is a homomorphism, the claim (2.2.2) therefore follows by an induction on the length of w'' in (W'', R'').

The following is an adaptation of Lemma 3.6 of [**1**] into our present situation.

Proposition 2.2.14. *With the above notation,*

$$\Delta_1(W') \subseteq \{ \lambda z \in \Phi_1^+ \mid \lambda > 0, z \in \Delta_1 \}.$$

Proof. Take $\Delta_1' = \Delta_1$. Since

$$\theta(R'') = \{ r_x \mid x \in \Delta_i \} \qquad \text{and} \qquad W' = \langle r_x \mid x \in \Delta_i \rangle$$

it follows that θ is surjective.

Let $x \in \Delta_1(W')$. Choose $w \in W''$ with $\theta(w) = r_x \in W'$. Now let $\rho_{x_1} \cdots \rho_{x_n}$ $(x_i \in \Delta_1')$ be a reduced expression for w in (W'', R''). Note that $n \geq 1$. Now $l''(w\rho_{x_n}) < l''(w)$, so by Proposition 2.2.13 we have $we_{x_n} \in \Psi^-$, say

$$we_{x_n} = -\sum_{y \in \Delta'_1} c_y e_y \qquad \text{where } c_y \geq 0 \text{ for all } y \in \Delta'_1.$$

Hence

$$r_x(x_n) = \theta(w)L(e_{x_n}) = L(we_{x_n}) = L(-\sum_{y \in \Delta'_1} c_y e_y) = -\sum_{y \in \Delta'_1} c_y y.$$

Clearly $r_x(x_n) \in \Phi_1$ and because each $y \in \Delta_1$ can be expressed as a nonnegative linear combination of elements of Π_1, so it follows that $r_x(x_n) \in \Phi_1^-$. Since $x_n \in \Delta_1 \subseteq \Phi_1(W') \subseteq \Phi_1^+$ so it follows that

$$\widehat{x_n} \in N_1(r_x) \cap \widehat{\Phi_1(W')}. \tag{2.2.3}$$

But $x \in \Delta_1(W')$ and so (2.2.3) above yields that $x = \lambda x_n$ for some positive scalar λ. Since $x \in \Delta_1(W')$ was arbitrary it follows that

$$\Delta_1(W') \subseteq \{ \lambda z \in \Phi_1^+ \mid \lambda > 0, z \in \Delta_1 \}.$$

□

The following is a generalization of Proposition 3.7 of [**1**].

Proposition 2.2.15. *With the above notation*

$$\Delta_1(W') = \{ \lambda z \in \Phi_1^+ \mid \lambda > 0, z \in \Delta_1 \}.$$

Proof. Take $\Delta'_1 = \Delta_1(W')$; this is possible by Proposition 2.2.14 and Proposition 2.2.12.

Since $(W', S(W'))$ is a Coxeter system we know that

$$\theta : (W'', R'') \to (W', S(W'))$$

is a Coxeter system isomorphism. In particular

$$l'(\theta(w'')) = l''(w'') \qquad \text{for all } w'' \in W'' \tag{2.2.4}$$

where l' is the length function on $(W', S(W'))$.

Let $x \in \Delta_1$. Then $r_x \in W' \cap T$. By Lemma 2.2.6 (i), there exist $x_0, \cdots, x_n \in \Delta_1(W') = \Delta'_1$ such that

$$r_x = r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_n}.$$

Since $r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_0} r_{x_0} r_{x_1} \cdots r_{x_n} = r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_n}$, there is no loss of generality in assuming that $l'(r_{x_n} \cdots r_{x_1} r_{x_0}) > l'(r_{x_n} \cdots r_{x_1})$. Then (2.2.4) yields that

$$l''(\rho_{x_n} \cdots \rho_{x_1} \rho_{x_0}) > l''(\rho_{x_n} \cdots \rho_{x_1}),$$

and thus by Proposition 2.2.13 $y := \rho_{x_n} \cdots \rho_{x_1}(e_{x_0}) \in \Psi^+$, say

$$y = \sum_{z \in \Delta'_1} c_z e_z \qquad \text{where } c_z \geq 0 \text{ for all } z \in \Delta'_1.$$

Now we have $L(y) = r_{x_n} \cdots r_{x_1}(x_0) = \sum_{z \in \Delta'_1} c_z z \in \Phi_1^+$. Write $y' = L(y)$. Then

$$r_{y'} = r_{x_n} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_n} = r_x.$$

Since $x, y' \in \Phi_1^+$ the above yields that $x = \mu y' = \sum_{z \in \Delta'_1} (\mu c_z) z$ for some positive scalar μ.

Now suppose that $x \notin \Delta'_1(W') = \Delta'_1$. Then by the definition of Δ_1,

$$\langle x, \phi(z) \rangle \leq 0 \qquad \text{for all } z \in \Delta'_1, \text{ since } x \neq z.$$

By Corollay 1.3.13 this imples that $\langle z, \phi(x) \rangle \leq 0$. But now

$$1 = \langle x, \phi(x) \rangle = \langle \sum_{z \in \Delta'_1} (\mu c_z) z, \phi(x) \rangle = \sum_{z \in \Delta'_1} (\mu c_z) \langle z, \phi(x) \rangle \leq 0.$$

This contradiction shows that the assumption $x \notin \Delta'_1(W')$ is false. Since $x \in \Delta_1$ was arbitrary it follows that $\Delta_1 \subseteq \Delta_1(W')$. Observe that if $\alpha \in \Delta_1(W')$ then $\lambda\alpha \in \Delta_1(W')$ whenever $\lambda\alpha \in \Phi_1^+$. Thus

$$\{ \lambda x \in \Phi_1^+ \mid \lambda > 0, x \in \Delta_1 \} \subseteq \Delta_1(W'). \tag{2.2.5}$$

Finally, (2.2.5) and Proposition 2.2.14 together yield that

$$\{ \lambda x \in \Phi_1^+ \mid \lambda > 0, x \in \Delta_1 \} \subseteq \Delta_1(W').$$

□

2.3. Comparison with the Standard Geometric Realisation of Coxeter Groups

In this section we study the connections between the non-orthogonal geometric realization studied in Chapter 1 with the classical *Tits representation* (the standard orthogonal geometric realisation, in the sense of [**10**] and [**12**]) of Coxeter groups. We pay special attention to a comparisons between the canonical coefficients defined in section 1.2 and their natural counterparts in the classical theory.

Definition 2.3.1. Let V be a vector space over the real field $\mathbb{R}$, and let $(\ ,\ ): V \times V \to \mathbb{R}$ be a symmetric bilinear form. let $\Pi := \{\, \gamma_s \mid s \in S \,\}$ be a set of linearly independent vectors satisfying the the following conditions:

(C1') $(\gamma_s, \gamma_s) = 1$, for all $s \in S$;
(C2') $(\gamma_s, \gamma_t) \leq 0$, for all $s, t \in S$, $s \neq t$;
(C3') $(\gamma_s, \gamma_t)^2 = \langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle$, for all $s, t \in S$.

Let $\rho: S \to \mathrm{GL}(V)$ be defined by

$$\rho(s)(v) \colon v - 2(v, \gamma_s)\gamma_s$$

for all $v \in V$.

Again it is readily checked that $\mathscr{C}'' = (S, V, V, \Pi, \Pi, (\ ,\ ))$ is a Coxeter datum with Coxeter parameters m_{st}, $s, t \in S$. Let W be the associated abstract Coxeter group. Then by Theorem 1.1.4, there is an isomorphism

$$f : W \to \langle\, \rho(s) \mid s \in S \,\rangle \quad \text{satisfying } f(r_s) = \rho(s) \text{ for all } s \in S.$$

We refer to such V as the associated (to the non-orthogonal geometric realization) *Standard geometric realisation (Tits representation)* of W.

Definition 2.3.2. For all $w \in W$ and $v \in V$, we write wv in place of $f(w)v$, and let the *root system* of W in V be denoted by

$$\Phi := W\Pi = \{\, w\gamma_s \mid w \in W, s \in S \,\},$$

and we call the elements of Φ the set of *roots* of W in V. Given $\gamma \in \Phi$, and $s \in S$, define $\operatorname{coeff}_{\gamma_s} \in \mathbb{R}$ by requiring that $\gamma = \sum_{s\in S} \operatorname{coeff}_{\gamma_s}(\gamma)\gamma_s$. Define $\Phi^+ := \Phi \cap \mathrm{PLC}(\Pi)$, and call elements of Φ^+ the set of *positive roots* in V; define $\Phi^- = -\Phi^+$, and call the elements of Φ^- the set of *negative roots* in V.

For $\gamma \in \Phi^+$, define the *depth* of γ (denoted by $\mathrm{dp}(\gamma)$) to be

$$\mathrm{dp}(\gamma) = \min\{\, m \in \mathbb{N} \mid w \in W \text{ such that } w\gamma \in \Phi^- \text{ and } l(w) = m \,\}.$$

For $\gamma_1, \gamma_2 \in \Phi^+$, write $\gamma_1 \preceq \gamma_2$ if there exists some $w \in W$ such that $\gamma_2 = w\gamma_1$, and $\mathrm{dp}(\gamma_2) = \mathrm{dp}(\gamma_1) + l(w)$. Write $\gamma_1 \prec \gamma_2$ if $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. If $\gamma_1 \preceq \gamma_2$, we say that γ_1 *precedes* γ_2.

For each $\gamma \in \Phi$, define the *reflection corresponding to* γ, $r_\gamma \colon V \to V$ to be the linear transformation given by

$$r_\gamma(v) = v - 2(v,\gamma)\gamma, \text{ for all } v \in V.$$

Observe that the above defintions give a Standard Tits representation of W in the sense of [**12**] and [**10**]. It can be readily checked that $(\ ,\ )$ is W-invariant, that is, $(x,y) = (wx, wy)$ for all x, $y \in V$ and $w \in W$. Suppose that $x \in \Phi$ and $\lambda x \in \Phi$ for some scalar λ. Then $x = w\gamma_s$ for some $w \in W$ and $s \in S$, and it follows from what we have just noted that

$$(x,x) = (w\gamma_s, w\gamma_s) = (\gamma_s, \gamma_s) = 1.$$

Then $\lambda^2 = \lambda^2(x,x) = (\lambda x, \lambda x) = (w\gamma_s, w\gamma_s) = (\gamma_s,\gamma_s) = 1$ that is, $\lambda = \pm 1$. Thus we conclude that in Φ the only non trivial scalar multiple of a root is its negative.

Adapting the results obtained in Section 1.1 and Section 1.3 into the associated standard realization, we have:

Lemma 2.3.3. (i) *(Lemma 1.2.4)* $\Phi = \Phi^+ \uplus \Phi^-$.

(ii) *(Lemma 1.2.13) W is finite if and only if Φ is finite.*

(iii) *(Lemma 1.2.1) The symmetric bilinear form $(,)$ is W-invariant, that is, for all $u, v \in V$ and for all $w \in W$, $(u,v) = (wu, wv)$.*

(iv) *(Lemma 1.2.5) If $w \in W$ and $s \in S$, then*

$$l(wr_s) = \begin{cases} l(w) + 1 & \text{if } w\gamma_s \in \Phi^+, \\ l(w) - 1 & \text{if } w\gamma_s \in \Phi^-. \end{cases}$$

(v) *(Lemma 1.7 of [**6**]) Let $s \in S$ and $\gamma \in \Phi^+ \setminus \gamma_s$. Then*

$$\mathrm{dp}(r_s\gamma) = \begin{cases} \mathrm{dp}(\gamma) - 1 & \text{if } (\gamma, \gamma_s) > 0, \\ \mathrm{dp}(\gamma) & \text{if } (\gamma, \gamma_s) = 0, \\ \mathrm{dp}(\gamma) + 1 & \text{if } (\gamma, \gamma_s) < 0. \end{cases}$$

(vi) *(Proposition 2.1 [**5**]) Let $\gamma \in \Phi$, and $r \in S$. Then $\mathrm{coeff}_{\gamma_r}(\gamma) > 0$ implies that $\mathrm{coeff}_{\gamma_r}(\gamma) \geq 1$. Furthermore, if $0 < \mathrm{coeff}_{\gamma_r}(\gamma) < 2$, then either $\mathrm{coeff}_{\gamma_r}(\gamma) = 1$ or $\mathrm{coeff}_{\gamma_r}(\gamma) = 2\cos(\frac{\pi}{m_{r_1 r_2}})$, where r_1, $r_2 \in S$ with $4 \leq m_{r_1 r_2} < \infty$.*

□

By Lemma 2.3.3(i), for any given $\gamma \in \Phi$, $\mathrm{coeff}_{\gamma_s}(\gamma)$ are of the same sign for all $s \in S$ (either all non-negative or all non-positive). The *support* $\mathrm{supp}(\gamma)$ of γ is the set of all $\gamma_s \in \Pi$ with $\mathrm{coeff}_{\gamma_s}(\gamma) \neq 0$.

The following result establishes the connection between the non-orthogonal geometric realisation of W and the Tits representation defined above.

Proposition 2.3.4. *There are W-equivariant maps $f_1 \colon \Phi_1 \to \Phi$, and $f_2 \colon \Phi_2 \to \Phi$, satisfying*

$$f_1(\alpha_s) = \gamma_s = f_2(\beta_s)$$

for all $s \in S$.

Proof. This result follows from similar arguments to those used to prove Proposition 1.2.9 and Proposition 1.2.17. □

Remark 2.3.5. Unlike in Proposition 1.2.9, we stress that these W-equivariant maps f_1 and f_2 are not injective in general.

Lemma 2.3.6. *Suppose that* $\alpha \in \Phi_1$. *Then* $\alpha \in \Phi_1^+$ *implies that* $f_1(\alpha) \in \Phi^+$, *and* $\alpha \in \Phi_1^-$ *implies that* $f_1(\alpha) \in \Phi^-$.

Proof. Let $\alpha \in \Phi_1$. Then $\alpha = w\alpha_r$, for some $w \in W$, and $r \in S$. If $\alpha \in \Phi_1^+$, then Corollary 1.2.5 yields that $l(wr_r) = l(w) + 1$, in which case Lemma 2.3.3 (iv) and Proposition 2.3.4 yield that

$$f_1(\alpha) = f_1(w\alpha_r) = wf_1(\alpha_r) = w\gamma_r \in \Phi^+.$$

Likewise we see that $\alpha \in \Phi_1^-$ implies that $f_1(\alpha) \in \Phi^-$. □

Lemma 2.3.7. *Suppose that* $\alpha \in \Phi_1^+$. *Then* $\mathrm{dp}_1(\alpha) = \mathrm{dp}(f_1(\alpha))$, *that is, depth is* W*-invariant.*

Proof. Let $w \in W$ be such that $w\alpha \in \Phi_1^-$, and $\mathrm{dp}_1(\alpha) = l(w)$. Then Lemma 2.3.6 and Proposition 2.3.4 yield $f_1(w\alpha) = wf_1(\alpha) \in \Phi^-$, and so $\mathrm{dp}(f_1(\alpha)) \leq l(w) = \mathrm{dp}_1(\alpha)$. By symmetry $\mathrm{dp}_1(\alpha) \leq \mathrm{dp}(f_1(\alpha))$ as well, whence equality. □

Corollary 2.3.8. *Suppose that* $\alpha \in \Phi_1$, *and* $s \in S$. *Then*

$$(f_1(\alpha), \gamma_s) > 0 \text{ if and only if } \langle \alpha, \beta_s \rangle > 0,$$

and

$$(f_1(\alpha), \gamma_s) = 0 \text{ if and only if } \langle \alpha, \beta_s \rangle = 0,$$

and

$$(f_1(\alpha), \gamma_s) < 0 \text{ if and only if } \langle \alpha, \beta_s \rangle < 0.$$

Proof. Follows from Lemma 2.3.7 and Lemma 2.3.3 (v). □

The above can be immediately generalized to the following:

Corollary 2.3.9. *Suppose that* $\alpha_1, \alpha_2 \in \Phi_1$. *Then*

$$(f_1(\alpha_1), f_1(\alpha_2)) > 0 \quad \text{if and only if} \quad \langle \alpha_1, \phi(\alpha_2) \rangle > 0,$$

and

$$(f_1(\alpha_1), f_1(\alpha_2)) = 0 \quad \text{if and only if} \quad \langle \alpha_1, \phi(\alpha_2) \rangle = 0,$$

and

$$(f_1(\alpha_1), f_1(\alpha_2)) < 0 \quad \textit{if and only if} \quad \langle \alpha_1, \phi(\alpha_2)\rangle < 0.$$

Proof. Apply the same argument used in the proof of Corollary 1.3.13 to Corollary 2.3.8 and the desired result follows.

□

Proposition 2.3.10. *For each $\alpha \in \Phi_1$, and for each $r \in S$,*

$$\mathrm{coeff}_{\alpha_r}(\alpha)\,\mathrm{coeff}_{\beta_r}(\phi(\alpha)) \geq (\mathrm{coeff}_{\gamma_r}(f_1(\alpha)))^2.$$

Proof. Replace α by $-\alpha$ if necessary, we may assume that $\alpha \in \Phi_1^+$ and furthermore, we may write $\alpha = w\alpha_s$, where $w \in W$ and $s \in S$. The proof is based on an induction on $l(w)$. If $l(w) = 0$, then the result clearly holds. Thus we may assume that $l(w) \geq 1$, and choose $t \in S$ such that $l(wr_t) = l(w)-1$. Then we may write $w = w_1w_2$, where w_2 is an alternating product of r_s and r_t, ending in r_t, and w_1 is of minimal length in the coset $w\langle r_s, r_t\rangle$. Thus Lemma 1.3.2 yields that

$$l(w) = l(w_1) + l(w_2), \qquad l(w_1r_s) > (w_1), \qquad \text{and} \qquad l(w_1r_t) > l(w_1).$$

Consequently Corollary 1.2.5 yields that $w_1\alpha_s \in \Phi_1^+$ and $w_1\alpha_t \in \Phi_1^+$. Now a rank 2 calculation yields that $w_2\alpha_s = p\alpha_s + \lambda q\alpha_t$, where λ is a positive constant and $pq \geq 0$. If $p, q < 0$ then

$$\alpha = w\alpha_s = w_1w_2\alpha_s = w_1(p\alpha_s + \lambda q\alpha_t) = pw_1\alpha_s + \lambda q w_1\alpha_t \in \Phi_1^-$$

contradicting the assumption that $\alpha \in \Phi_1^+$. Therefore $p, q \geq 0$. Now a direct rank 2 calculation shows that

$$w_2\gamma_s = p\gamma_s + q\gamma_t, \quad w_2\alpha_s = p\alpha_s + \lambda q\alpha_t, \quad \text{and} \quad w_2\beta_s = p\beta_s + \tfrac{q}{\lambda}\beta_t.$$

Recall the W-equivariant maps ϕ_1 and ϕ_2 from Proposition 1.2.9 and we see that the above yields

$$w_2\alpha'_s = \phi_1(w_2\alpha_s) = p\alpha'_s + \lambda q\alpha'_t \quad \text{and} \quad w_1\beta'_s = \phi_2(w_2\beta_s) = p\beta'_s + \tfrac{q}{\lambda}\beta'_t.$$

Now we set

$$x := \mathrm{coeff}_{\alpha_r}(\alpha)\ ,\ x' := \mathrm{coeff}_{\beta_r}(\phi(\alpha))\ ,\ x'' := \mathrm{coeff}_{\gamma_r}(f_1(\alpha));$$

and

$$y := \operatorname{coeff}_{\alpha_r}(w_1\alpha_s)\ ,\ y' := \operatorname{coeff}_{\beta_r}(w_1\beta_s)\ ,\ y'' := \operatorname{coeff}_{\gamma_r}(w_1\gamma_s);$$

and

$$z := \operatorname{coeff}_{\alpha_r}(w_1\alpha_t)\ ,\ z' := \operatorname{coeff}_{\beta_r}(w_1\beta_t)\ ,\ z'' := \operatorname{coeff}_{\gamma_r}(w_1\gamma_t).$$

Since $l(w_1) < l(w)$, it follows from the inductive hypotheis that

$$yy' \geq (y'')^2 \qquad \text{and} \qquad zz' \geq (z'')^2.$$

Now

$$\begin{aligned} xx' - (x'')^2 &= (py + \lambda qz)(py' + \tfrac{1}{\lambda}qz') - (py'' + qz'')^2 \\ &= p^2(yy' - y''^2) + q^2(zz' - z''^2) + pq(\tfrac{1}{\lambda}yz' + \lambda zy' - 2y''z''). \end{aligned}$$

From the inductive hypotheis the first two summands are nonnegative. It follows from the inductive hypotheis and the geometric mean and arithmetic mean inequality applied to the terms $\frac{1}{\lambda}yz'$ and $\lambda y'z$ (indeed $\frac{1}{\lambda}yz' + \lambda y'z \geq 2\sqrt{yy'zz'} \underbrace{\geq 2y''z''}_{\text{by the inductive hypothesis}}$) that the third summand $pq(\frac{1}{\lambda}yz' + \lambda y'z - 2y''z'')$ is also nonnegative, whence $xx' - x''^2 \geq 0$ and the desired result follows by induction. $\square$

Proposition 2.3.11. *Suppose that* $\alpha_1, \alpha_2 \in \Phi_1$. *Then*

$$\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq (f_1(\alpha_1), f_1(\alpha_2))^2. \tag{2.3.1}$$

Proof. Since both $\langle\ ,\ \rangle$ and $(\ ,\ )$ are W-invariant, and ϕ, f_1 are W-equivariant, we may replace α_1, and α_2 by $u\alpha_1$ and $u\alpha_2$ for a suitable $u \in W$ such that $\alpha_2 = \alpha_s$ for some $s \in S$. Furthermore, replace α_1 by $-\alpha_1$ if needs be, we may assume that $\alpha_1 \in \Phi_1^+$. We proceed by an induction on the depth of α_1.

If $\mathrm{dp}_1(\alpha_1) = 1$, then $\alpha_1 = \lambda\alpha_r$, where $r \in S$ and λ is a positive constant. Then Corollay 1.2.19 yields that $\phi(\alpha_1) = \frac{1}{\lambda}\beta_r$, and hence

$$\begin{aligned} \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle &= \lambda\langle \alpha_r, \beta_s\rangle\tfrac{1}{\lambda}\langle \alpha_s, \beta_r\rangle = \langle \alpha_r, \beta_s\rangle\langle \alpha_s, \beta_r\rangle \\ &= (\gamma_r, \gamma_s)^2 \quad \text{(by the definition of } (\ ,\ )) \\ &= (f_1(\alpha_1), f_1(\alpha_2)). \end{aligned}$$

Thus we may assume that $\mathrm{dp}_1(\alpha_1) > 1$. Next if $\langle \alpha_1, \beta_s \rangle > 0$, then Lemma 1.3.10 yields that $r_s\alpha_1 \prec_1 \alpha_1$, and hence

$$\begin{aligned}
&\langle \alpha_1, \beta_s \rangle \langle \alpha_s, \phi(\alpha_1) \rangle \\
&= \langle r_s\alpha_1, r_s\beta_s \rangle \langle r_s\alpha_s, \phi(r_s\alpha_1) \rangle = (-\langle r_s\alpha_1, \beta_s \rangle)(-\langle \alpha_s, \phi(r_s\alpha_1) \rangle) \\
&\geq (f_1(r_s\alpha_1), \gamma_s)^2 \quad \text{(by the inductive hypothesis)} \\
&= (f_1(\alpha_1), -\gamma_s)^2 \\
&= (f_1(\alpha_1), \gamma_s)^2
\end{aligned}$$

as required. Thus we may further assume that $\langle \alpha_1, \beta_s \rangle \leq 0$.

Next choose $t \in S$ such that $r_t\alpha_1 \prec_1 \alpha_1$. Then Lemma 1.3.10 yields that $\langle \alpha_1, \beta_t \rangle > 0$ and, in particular, $s \neq t$. Let $w \in W$ be a maximal length alternating product of r_s and r_t ending in r_t such that $\mathrm{dp}_1(w\alpha_1) = \mathrm{dp}_1(\alpha_1) - l(w)$. If $w = r_{s_1} \cdots r_{s_l}$ is a reduced expression for w, then Lemma 1.3.8 yields that $r_{s_l}\alpha_1 \prec_1 \alpha_1$, that is, $\langle \alpha_1, \beta_{s_l} \rangle > 0$. Therefore $s \neq s_l$, and w has no reduced expression ending in r_s. Furthermore, we observe that $\mathrm{dp}_1(w\alpha_1) \lneqq \mathrm{dp}_1(\alpha_1)$, and so the inductive hypotheis yields that

$$\langle w\alpha_1, \beta_s \rangle \langle \alpha_s, \phi(w\alpha_1) \rangle \geq (\gamma_s, f_1(w\alpha_1))^2 \tag{2.3.2}$$

and

$$\langle w\alpha_1, \beta_t \rangle \langle \alpha_t, \phi(w\alpha_1) \rangle \geq (\gamma_t, f_1(w\alpha_1))^2. \tag{2.3.3}$$

Since w has no reduced expression ending in r_s, it follows that w is a product of r_s and r_t with strictly fewer than m_{st} factors. Thus by Lemma 1.1.11 (or equally a direct rank 2 calculation) there are nonnegative constants p, q and positive constant λ such that

$$w\gamma_s = p\gamma_s + q\gamma_t \quad \text{and} \quad w\alpha_s = p\alpha_s + \lambda q\alpha_t \quad \text{and} \quad w\beta_s = p\beta_s + \tfrac{1}{\lambda} q\beta_t.$$

Thus

$$\begin{aligned}
&\langle \alpha_1, \beta_s\rangle\langle \alpha_s, \phi(\alpha_1)\rangle - (\gamma_s, f_1(\alpha_1))^2 \\
&= \langle w\alpha_1, w\beta_s\rangle\langle w\alpha_s, \phi(w\alpha_1)\rangle - (w\gamma_s, f_1(w\alpha_1))^2 \\
&\qquad\qquad (\text{ since } \langle\ ,\ \rangle \text{ and } (\ ,\ ) \text{ are } W\text{-invariant}) \\
&= \langle w\alpha_1, p\beta_s + \tfrac{q}{\lambda}\beta_t\rangle\langle p\alpha_s + \lambda q\alpha_t, \phi(w\alpha_1)\rangle - (f_1(w\alpha_1), p\gamma_s + q\gamma_t)^2 \\
&= \underbrace{p^2(\langle w\alpha_1, \beta_s\rangle\langle \alpha_s, \phi(w\alpha_1)\rangle - (f_1(w\alpha_1), \gamma_s)^2)}_{\text{A}} \\
&\qquad + \underbrace{q^2(\langle w\alpha_1, \beta_t\rangle\langle \alpha_t, \phi(w\alpha_1)\rangle - (f_1(w\alpha_1), \gamma_s)^2)}_{\text{B}} \ + \ C.
\end{aligned}$$

where

$$\begin{aligned}
C = pq(\ &\tfrac{1}{\lambda}\langle w\alpha_1, \beta_t\rangle\langle \alpha_s, \phi(w\alpha_1)\rangle + \lambda\langle w\alpha_1, \beta_s\rangle\langle \alpha_t, \phi(w\alpha_1)\rangle \\
&- 2(f_1(w\alpha_1), \gamma_s)(f_1(w\alpha_1), \gamma_t)\ ).
\end{aligned}$$

It follows from (2.3.2) and (2.3.3) that A and B are both nonnegative. It follows from the geometric mean and arithmetic mean ineqaulity that

$$\begin{aligned}
&\tfrac{1}{\lambda}\langle w\alpha_1, \beta_t\rangle\langle \alpha_s, \phi(w\alpha_1)\rangle + \lambda\langle w\alpha_1, \beta_s\rangle\langle \alpha_t, \phi(w\alpha_1)\rangle \\
&\qquad \geq 2\sqrt{\langle w\alpha_1, \beta_s\rangle\langle \alpha_s, \phi(w\alpha_1)\rangle\langle w\alpha_1, \beta_t\rangle\langle \alpha_t, \phi(w\alpha_1)\rangle} \\
&\qquad \geq 2(f_1(w\alpha_1), \gamma_s)(f_1(w\alpha_1), \gamma_t) \qquad \text{(by (2.3.2) and (2.3.3))},
\end{aligned}$$

that is, $C \geq 0$ as well. Therefore $\langle \alpha_1, \beta_s\rangle\langle \alpha_s, \phi(\alpha_1)\rangle \geq (\gamma_s, f_1(\alpha_1))^2$, and the desired result follows by induction. □

Lemma 2.3.12. *Suppose that $\alpha \in \Phi_1$, and $t \in S$. Then* $\mathrm{coeff}_{\alpha_t}(\alpha) = 0$ *if and only if* $\mathrm{coeff}_{\gamma_t}(f_1(\alpha)) = 0$.

Proof. By Proposition 2.3.10 we only need to show that the condition $\mathrm{coeff}_{\gamma_t}(f_1(\alpha)) = 0$ implies that $\mathrm{coeff}_{\alpha_t}(\alpha) = 0$. Replacing α by $-\alpha$ if needs be, we may assume that $\alpha \in \Phi_1^+$. We may write $\alpha = w\alpha_r$ for some $w \in W$ and $r \in S$. If $l(w) = 0$ then there is nothing to prove. Thus we may assume that $l(w) \geq 1$ and proceed by an induction on $l(w)$. Choose $s \in S$ such that $l(wr_s) = l(w) - 1$, and write $w = w_1 w_2$ where

$w_1 \in w\langle r_r, r_s\rangle$ is of minimal length, and w_2 is an alternating product of r_r and r_s ending in r_s. By Lemma 1.3.2, $l(w) = l(w_1) + l(w_2)$. Since w_2 is a right segment of w, and since $\alpha(= w\alpha_r)$ is positive, it follows that $w_2\alpha_r \in \Phi_1^+$. Again a rank 2 calculation shows that

$$w_2\gamma_r = p\gamma_r + q\gamma_s, \quad w_2\alpha_r = p\alpha_r + \lambda q\alpha_s \text{ and } w_2\beta_r = p\beta_r + \tfrac{q}{\lambda}\beta_s$$

for some non-negative constants p, q and positive constant λ. Again Proposition 1.2.9 yields that

$$w_2\alpha'_r = \phi_1(w_2\alpha_r) = p\alpha'_r + \lambda q\alpha'_s \quad \text{and} \quad w_2\beta'_r = \phi_2(w_2\beta_r) = p\beta'_r + \tfrac{q}{\lambda}\alpha'_s.$$

Then

$$\begin{aligned} 0 &= \mathrm{coeff}_{\gamma_t}(f_1(\alpha)) = \mathrm{coeff}_{\gamma_t}(w\gamma_r) = \mathrm{coeff}_{\gamma_t}(w_1(p\gamma_r + q\gamma_s)) \\ &= p\,\mathrm{coeff}_{\gamma_t}(w_1\gamma_r) + q\,\mathrm{coeff}_{\gamma_t}(w_1\gamma_s). \end{aligned}$$

By Lemma 1.3.2 and Lemma 2.3.3 (iv), $w_1\gamma_r$ and $w_1\gamma_s$ are both positive, and hence

$$p\,\mathrm{coeff}_{\gamma_t}(w_1\gamma_r) = q\,\mathrm{coeff}_{\gamma_t}(w_1\gamma_s) = 0.$$

Then the inductive hypothesis yields that

$$p\,\mathrm{coeff}_{\alpha_t}(w_1\alpha_r) = q\,\mathrm{coeff}_{\alpha_t}(w_1\alpha_s) = 0$$

and therefore

$$\mathrm{coeff}_{\alpha_t}(\alpha) = \mathrm{coeff}_{\alpha_t}(w\alpha_r) = p\,\mathrm{coeff}_{\alpha_t}(w_1\alpha_r) + \lambda q\,\mathrm{coeff}_{\alpha_t}(w_1\alpha_s) = 0$$

as required. □

Combining the above with Lemma 2.3.6 we immediately have:

Corollary 2.3.13. *Suppose that $\alpha \in \Phi_1$, and $t \in S$. Then*

$$\mathrm{coeff}_{\alpha_t}(\alpha) = 0 \text{ if and only if } \mathrm{coeff}_{\gamma_t}(f_1(\alpha)) = 0,$$

and

$$\mathrm{coeff}_{\alpha_t}(\alpha) > 0 \text{ if and only if } \mathrm{coeff}_{\gamma_t}(f_1(\alpha)) > 0,$$

and

$$\mathrm{coeff}_{\alpha_t}(\alpha) < 0 \text{ if and only if } \mathrm{coeff}_{\gamma_t}(f_1(\alpha)) < 0. \quad \square$$

Proposition 2.3.14. *Suppose that* $\alpha \in \Phi_1^+$ *and* $r \in S$. *Then*

$$\operatorname{coeff}_{\alpha_r}(\alpha) \operatorname{coeff}_{\beta_r}(\phi(\alpha)) = 1 \textit{ if and only if } \operatorname{coeff}_{\gamma_r}(f_1(\alpha)) = 1.$$

Proof. Suppose that $\operatorname{coeff}_{\alpha_r}(\alpha) \operatorname{coeff}_{\beta_r}(\phi(\alpha)) = 1$. Proposition 2.3.10 and Lemma 2.3.3(vi) combined yield that either $\operatorname{coeff}_{\gamma_r}(f_1(\alpha)) = 1$ or $\operatorname{coeff}_{\gamma_r}(f_1(\alpha)) = 0$. Observe that we can rule out the latter for it contradicts Corollary 2.3.13.

Next for the converse implication, suppose that $\operatorname{coeff}_{\gamma_r}(f_1(\alpha)) = 1$. As usual we write $\alpha = w\alpha_s$, $w \in W$ and $s \in S$. If $l(w) = 0$, then there is nothing to prove. Thus we may assume that $l(w) > 0$ and proceed by induction. Again we choose $t \in S$ such that $l(wr_t) = l(w) - 1$, and write $w = w_1 w_2$, where $w_1 \in w\langle r_s, r_t\rangle$ is of minimal length, and w_2 is an alternating product of r_s and r_t ending in r_t. Lemma 1.3.2 then yields that, $l(w) = l(w_1) + l(w_2)$. Since w_2 is a right hand segment of w, then the condition $w\alpha_s \in \Phi_1^+$ yields that $w_2\alpha_s \in \Phi_1^+$, which in turn yields that $w_2\gamma_s \in \Phi^+$ by Corollary 2.3.13. Hence, direct rank 2 calculations yield that there are nonnegative constants p, q and positive constant λ such that

$$w_2\gamma_s = p\gamma_s + q\gamma_t \ ; \ w_2\alpha_s = p\alpha_s + \lambda q\alpha_t \ ; \ w_2\beta_s = p\beta_s + \tfrac{1}{\lambda} q\beta_t.$$

Again Proposition 1.2.9 yields that

$$w_2\alpha'_s = \phi_1(w_2\alpha_s) = p\alpha'_s + \lambda q\alpha'_t \quad \text{and} \quad w_2\beta'_s = \phi_2(w_2\beta_s) = p\beta'_s + \tfrac{q}{\lambda}\beta'_t.$$

And consequently

$$f_1(\alpha) = w\gamma_s = w_1(p\gamma_s + q\gamma_t) = pw_1\gamma_s + qw_1\gamma_t.$$

Hence

$$1 = p \operatorname{coeff}_{\gamma_r}(w_1\gamma_s) + q \operatorname{coeff}_{\gamma_r}(w_1\gamma_t). \tag{2.3.4}$$

It follows from Lemma 1.3.2 and Lemma 2.3.3 (iv) combined that $w_1\gamma_s \in \Phi^+$ and $w_1\gamma_t \in \Phi_1^+$. Thus Lemma 2.3.3(vi) yields that precisely one of the following is the case:

$$\text{coeff}_{\gamma_r}(w_1\gamma_s) = p = 1, \text{ and } q\,\text{coeff}_{\gamma_r}(w_1\gamma_t) = 0$$

or

$$\text{coeff}_{\gamma_r}(w_1\gamma_t) = q = 1, \text{ and } p\,\text{coeff}_{\gamma_r}(w_1\gamma_s) = 0.$$

Since $l(w_1) < l(w)$, the inductive hypothesis and Corollary 2.3.13 above imply either

$$p^2\,\text{coeff}_{\alpha_r}(w_1\alpha_s)\,\text{coeff}_{\beta_r}(w_1\beta_s) = 1, \text{ and } q\,\text{coeff}_{\alpha_r}(w_1\alpha_t) = 0$$

or

$$q^2\,\text{coeff}_{\alpha_r}(w_1\alpha_t)\,\text{coeff}_{\beta_r}(w_1\beta_t) = 1, \text{ and } p\,\text{coeff}_{\alpha_r}(w_1\alpha_s) = 0.$$

Therefore

$$\begin{aligned}
&\text{coeff}_{\alpha_r}(\alpha)\,\text{coeff}_{\beta_r}(\phi(\alpha)) \\
&\quad = \text{coeff}_{\alpha_r}(pw_1\alpha_s + \lambda q w_1\alpha_t)\,\text{coeff}_{\beta_r}(pw_1\beta_s + \tfrac{1}{\lambda}qw_1\beta_t) \\
&\quad = (p\,\text{coeff}_{\alpha_r}(w_1\alpha_s) + \lambda q\,\text{coeff}_{\alpha_r}(w_1\alpha_t))\cdot \\
&\qquad\qquad\qquad (p\,\text{coeff}_{\beta_r}(w_1\beta_s) + \tfrac{q}{\lambda}\,\text{coeff}_{\beta_r}(w_1\beta_t)) \\
&\quad = p^2\,\text{coeff}_{\alpha_r}(w_1\alpha_s)\,\text{coeff}_{\beta_r}(w_1\beta_s) + \tfrac{1}{\lambda}pq\,\text{coeff}_{\alpha_r}(w_1\alpha_s)\,\text{coeff}_{\beta_r}(w_1\beta_t) \\
&\qquad + \lambda pq\,\text{coeff}_{\alpha_r}(w_1\alpha_t)\,\text{coeff}_{\beta_r}(w_1\beta_s) + q^2\,\text{coeff}_{\alpha_r}(w_1\alpha_t)\,\text{coeff}_{\beta_r}(w_1\beta_t) \\
&\quad = 1
\end{aligned}$$

because precisely one summand is nonzero and is equal to 1. □

Proposition 2.3.15. *Suppose that $\alpha \in \Phi_1^+$ and $r \in S$, such that*

$$1 \le \text{coeff}_{\alpha_r}(\alpha)\,\text{coeff}_{\beta_r}(\phi(\alpha)) < 4.$$

Then either

$$\text{coeff}_{\alpha_r}(\alpha)\,\text{coeff}_{\beta_r}(\phi(\alpha)) = 1$$

or

$$\text{coeff}_{\alpha_r}(\alpha)\,\text{coeff}_{\beta_r}(\phi(\alpha)) = 4\cos^2(\frac{\pi}{m}),$$

where $m = m_{r_1r_2}$, for some $r_1, r_2 \in S$ and $4 \le m < \infty$.

Proof. First we write $\alpha = w\alpha_s$ for some $w \in W$ and $s \in S$. If $l(w) = 0$, then there is nothing to prove. Thus we may assume that $l(w) > 0$ and proceed by induction again. Similarly as in Propostion 2.3.14, we choose $t \in S$ such that $l(wr_t) = l(w) - 1$, and let $w_1 \in w\langle r_s, r_t\rangle$ be of minimal length, and $w_2 \in \langle r_s, r_t\rangle$ such that $w = w_1w_2$. Again as in the proof of Proposition 2.3.14, there are nonnegative constants p, q and positive constant λ such that

$$w_2\gamma_s = p\gamma_s + q\gamma_t \ ; \ w_2\alpha_s = p\alpha_s + \lambda q\alpha_t \ ; \ w_2\beta_s = p\beta_s + \tfrac{1}{\lambda}q\beta_t.$$

Again Proposition 1.2.9 yields that

$$w_2\alpha'_s = \phi_1(w_2\alpha_s) = p\alpha'_s + \lambda q\alpha'_t \quad \text{and} \quad w_2\beta'_s = \phi_2(w_2\beta_s) = p\beta'_s + \tfrac{q}{\lambda}\beta'_t.$$

And consequently

$$\begin{aligned}
&\operatorname{coeff}_{\alpha_r}(\alpha)\operatorname{coeff}_{\beta_r}(\phi(\alpha))\\
&= \operatorname{coeff}_{\alpha_r}(w_1(p\alpha_s + \lambda q\alpha_t))\operatorname{coeff}_{\beta_r}(w_1(p\beta_s + \tfrac{q}{\lambda}\beta_t))\\
&= (p\operatorname{coeff}_{\alpha_r}(w_1\alpha_s) + \lambda q\operatorname{coeff}_{\alpha_r}(w_1\alpha_t))\\
&\qquad\qquad (p\operatorname{coeff}_{\beta_r}(w_1\beta_s) + \tfrac{q}{\lambda}\operatorname{coeff}_{\beta_r}(w_1\beta_t))\\
&= p^2\operatorname{coeff}_{\alpha_r}(w_1\alpha_s)\operatorname{coeff}_{\beta_r}(w_1\beta_s) + \tfrac{1}{\lambda}pq\operatorname{coeff}_{\alpha_r}(w_1\alpha_s)\operatorname{coeff}_{\beta_r}(w_1\beta_t)\\
&+ \lambda pq\operatorname{coeff}_{\alpha_r}(w_1\alpha_t)\operatorname{coeff}_{\beta_r}(w_1\beta_s) + q^2\operatorname{coeff}_{\alpha_r}(w_1\alpha_t)\operatorname{coeff}_{\beta_r}(w_1\beta_t).
\end{aligned}$$

Suppose that $p, q > 0$, $\operatorname{coeff}_{\alpha_r}(w_1\alpha_s) > 0$ and $\operatorname{coeff}_{\alpha_r}(w_1\alpha_t) > 0$. Since $p = \operatorname{coeff}_{\gamma_s}(w_2\gamma_s)$ and $q = \operatorname{coeff}_{\gamma_t}(w_2\gamma_s)$, it follows from Lemma 2.3.3(vi) and Proposition 1.3.3 combined that

$$p^2\operatorname{coeff}_{\alpha_r}(w_1\alpha_s)\operatorname{coeff}_{\beta_r}(w_1\beta_s) \geq 1 \tag{2.3.5}$$

and

$$q^2\operatorname{coeff}_{\alpha_r}(w_1\alpha_t)\operatorname{coeff}_{\beta_r}(w_1\beta_t) \geq 1 \tag{2.3.6}$$

From these a geometric mean and arithmetic mean inequality argument yields that

(2.3.7)

$$\tfrac{1}{\lambda}pq\operatorname{coeff}_{\alpha_r}(w_1\alpha_s)\operatorname{coeff}_{\beta_r}(w_1\beta_t) + \lambda pq\operatorname{coeff}_{\alpha_r}(w_1\alpha_t)\operatorname{coeff}_{\beta_r}(w_1\beta_s) \geq 2.$$

But (2.3.5), (2.3.6) and (2.3.7) together will give that

$$\operatorname{coeff}_{\alpha_r}(\alpha)\operatorname{coeff}_{\beta_r}(\phi(\alpha)) \geq 4.$$

contradicting our assumption. Therefore at least one of the terms p, q, $\operatorname{coeff}_{\alpha_r}(w_1\alpha_s)$ or $\operatorname{coeff}_{\alpha_r}(w_1\alpha_t)$ must be zero. Then Proposition 1.3.3 yields that either

(2.3.8)
$$1 \leq \operatorname{coeff}_{\alpha_r}(\alpha)\operatorname{coeff}_{\beta_r}(\phi(\alpha)) = p^2\operatorname{coeff}_{\alpha_r}(w_1\alpha_s)\operatorname{coeff}_{\beta_r}(w_1\beta_s) < 4,$$

or

(2.3.9)
$$1 \leq \operatorname{coeff}_{\alpha_r}(\alpha)\operatorname{coeff}_{\beta_r}(\phi(\alpha)) = q^2\operatorname{coeff}_{\alpha_r}(w_1\alpha_t)\operatorname{coeff}_{\beta_r}(w_1\beta_t) < 4.$$

If (2.3.8) is the case, then Proposition 2.3.10 yields that

$$1 \leq p\operatorname{coeff}_{\gamma_r}(w_1\gamma_s) < 2; \tag{2.3.10}$$

whereas if (2.3.9) is the case, then Proposition 2.3.10 yields that

$$1 \leq q\operatorname{coeff}_{\gamma_r}(w_1\gamma_t) < 2. \tag{2.3.11}$$

Suppose that (2.3.8) is the case. Then Lemma 2.3.3(vi) yields that at least one of p or $\operatorname{coeff}_{\gamma_r}(w_1\gamma_s)$ must be 1. If $p = 1$, since $l(w_1) < l(w)$, it follows from the inductive hypotheis that

$$\operatorname{coeff}_{\alpha_r}(\alpha)\operatorname{coeff}_{\beta_r}(\phi(\alpha)) = \operatorname{coeff}_{\alpha_r}(w_1a\alpha_s)\operatorname{coeff}_{\beta_r}(w_1\beta_s)$$

$$= \begin{cases} 1 \\ \text{or} \\ 4\cos^2(\frac{\pi}{m}), \end{cases}$$

where m is an integer of the required form. On the other hand if $\operatorname{coeff}_{\gamma_r}(w_1\gamma_s) = 1$, then by the last Proposition

$$\operatorname{coeff}_{\alpha_r}(w_1\alpha_s)\operatorname{coeff}_{\beta_r}(w_1\beta_s) = 1.$$

Hence $\operatorname{coeff}_{\alpha_r}(\alpha)\operatorname{coeff}_{\beta_r}(\phi(\alpha)) = p^2$. By Lemma 2.3.3(vi), p^2 is either 1 or $4\cos^2(\frac{\pi}{m'})$, where m' is an integer of the required form.

Similarly, if (2.3.9) is the case, then

$$\operatorname{coeff}_{\alpha_r}(\alpha)\operatorname{coeff}_{\beta_r}(\phi(\alpha)) = \begin{cases} 1 \\ \text{or} \\ 4\cos^2(\frac{\pi}{m''}), \end{cases}$$

where m'' is an integer of the required form, and this completes our proof. □

The following is a well known result due to Dyer. A proof of this result can be found in [**6**] (Propositon 1.4) and [**13**](Proposition 4.5.4).

Proposition 2.3.16. *Let $\gamma_1, \gamma_2 \in \Phi^+$. Then the (dihedral) subgroup of W generated by r_{γ_1} and r_{γ_2} is finite if and only if $|(\gamma_1, \gamma_2)| < 1$.* □

Combining Proposition 2.3.11 and the above, we can immediately deduce the following:

Corollary 2.3.17. *Let $\alpha_1, \alpha_2 \in \Phi_1$ with $r_{\alpha_1} \neq r_{\alpha_2}$. The subgroup of W generated by r_{α_1} and r_{α_2} is finite if $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle < 1$.* □

We prove the converse to Corollary 2.3.17 at the end of Chapter 3 Section 3.5.

Lemma 2.3.18. (Lemma 2.4 (i) of [**1**]) *Suppose that $\gamma_1, \gamma_2 \in \Phi^+$ are distinct. Then*

$$\langle\{r_{\gamma_1}, r_{\gamma_2}\}\rangle\{\gamma_1, \gamma_2\} \subseteq \mathrm{PLC}(\{\gamma_1, \gamma_2\}) \uplus -\mathrm{PLC}(\{\gamma_1, \gamma_2\})$$

if and only if

$$(\gamma_1, \gamma_2) \in (-\infty, -1] \cup \{\, -\cos\frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2\,\}.$$

□

Proof. We apply the results obtained in Section 2.2 here. We set $\{\gamma_1, \gamma_2\} = X_1 = X_2$ and let ψ to be the identity map on $\{\gamma_1, \gamma_2\}$. Consequently $(\,,\,) = \langle\,,\,\rangle'$ and hence Proposition 2.1.1 yields that

$$\langle\{r_{\gamma_1}, r_{\gamma_2}\}\rangle\{\gamma_1, \gamma_2\} \subseteq \mathrm{PLC}(\{\gamma_1, \gamma_2\}) \uplus -\mathrm{PLC}(\{\gamma_1, \gamma_2\})$$

if and only if

$$(\gamma_1, \gamma_2) \in (-\infty, -1] \cup \{ -\cos\frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2 \},$$

as required. □

Proposition 2.3.19. *Suppose that $\alpha_1, \alpha_2 \in \Phi_1^+$ such that $r_{\alpha_1} \neq r_{\alpha_2}$. Then*

$$\begin{cases} \langle \alpha_1, \phi(\alpha_2)\rangle \leq 0 \qquad \textit{and} \\ \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \in \{ \cos^2\frac{\pi}{m} \mid m \in \mathbb{N} \textit{ and } m \geq 2 \} \cup [1, \infty) \end{cases}$$

if and only if

$$(f_1(\alpha_1), f_1(\alpha_2)) \in (-\infty, -1] \cup \{ -\cos\frac{\pi}{m} \mid m \in \mathbb{N} \textit{ and } m \geq 2 \}.$$

Proof. Since $r_{f_1(\alpha_1)} = r_{\alpha_1} \neq r_{\alpha_2} = r_{f_1(\alpha_2)}$, it follows that both $\{\alpha_1, \alpha_2\}$ and $\{f_1(\alpha_1), f_1(\alpha_2)\}$ are linearly independent. Now set

$$X_1 = \{\alpha_1, \alpha_2\}, \qquad W_1'' = \langle \{r_{\alpha_1}, r_{\alpha_2}\}\rangle, \qquad \Phi_1'' = W_1''X_1$$

and let ψ be the restriction of ϕ on Φ_1''. Recall that by Lemma 2.3.6 we have $f_1(\Phi_1^+) = \Phi^+$ and $f_1^{-1}(\Phi^+) = \Phi_1^+$. Thus Proposition 2.1.1 yields that

$$\begin{cases} \langle \alpha_1, \phi(\alpha_2)\rangle \leq 0 \qquad \text{and} \\ \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \in \{ \cos^2\frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2 \} \cup [1, \infty) \end{cases}$$

if and only if $\Phi_1'' = \Phi_1''^{+} \uplus \Phi_1''^{-}$, and this happens if and only if

$$\langle \{r_{f_1(\alpha_1)}, r_{f_1(\alpha_2)}\}\rangle \{f_1(\alpha_1), f_1(\alpha_2)\} \subseteq \mathrm{PLC}(\{f_1(\alpha_1), f_1(\alpha_2)\}) \\ \uplus - \mathrm{PLC}(\{f_1(\alpha_1), f_1(\alpha_2)\})$$

and by Lemma 2.3.18 above this happens if and only if

$$(f_1(\alpha_1), f_1(\alpha_2)) \in (-\infty, -1] \cup \{ -\cos\frac{\pi}{m} \mid m \in \mathbb{N} \text{ and } m \geq 2 \}.$$

□

2.4. Dual Spaces of V_1, V_2 and the Tits cone

Throughout this section we assume that S is finite.

For any real vector space Y, we define $Y^* = \mathrm{Hom}(Y, \mathbb{R})$, and call it the *algebric dual space* of Y. For each $i \in \{1, 2\}$, let C_i be a convex cone in V_i and we define

$$C_i^* = \{\, f \in V_i^* \mid f(v) \geq 0 \text{ for all } v \in C_i \,\}.$$

Similarly for each convex cone F_i in V_i^* we define

$$F_I^* = \{\, v \in V_i \mid g(v) \geq 0 \text{ for all } g \in F_i \,\}.$$

It is readily checked that such C_i^* and F_i^* are themselves convex cones.

Lemma 2.4.1. *(i) For each $i \in \{1, 2\}$ the condition $0 \notin \mathrm{PLC}(\Pi_i)$ implies the existence of an $f_i \in V_i^*$ such that $f_i(x) > 0$ for all $x \in \Pi_i$.*

(ii) For each $i \in \{1, 2\}$, $(C_i^*)^* = \overline{C_i}$ for each convex cone C_i in V_i (where $\overline{C_i}$ is the topological closure of C_i with respect to the standard topology on $V_i \cong \mathbb{R}^{d_i}$ with $d_i = \dim(\mathrm{span}(V_i))$).

Proof. (i) For each $i \in \{1, 2\}$, set

$$X_i = \{\, \sum_{x \in \Pi_i} \lambda_x x \mid \lambda_x \geq 0 \text{ for all } x \text{ and } \sum_{x \in \Pi_i} \lambda_x = 1 \,\}.$$

Let $(\ ,\ )_i$ be a positive definite inner product on V_i and let $\|\ \|_i$ be the associated norm. Since X_i is topologically closed it follows that there exists a $v_i \in X_i$ with $\|v_i\|_i = (v_i, v_i)_i$ being minimal. Now let x_i be an arbitrary element of X_i and consider the function

$$g_i(t) \colon = \|(1-t)v_i + tx_i\|_i^2 \qquad (t \in [0, 1]).$$

Since X_i is convex, the choice of v_i yields that g_i attains a minimum at $t = 0$. Hence $g_i'(0) \geq 0$. But $g_i'(0) = 2(v_i, x_i - v_i)_i$; hence $(v_i, x_i)_i \geq (v_i, v_i)_i > 0$. It then follows that the linear functional f_i defined by $f_i(x) = (x, v_i)_i$ takes strictly positive values on Π_i completing the proof of (i).

(ii) Clearly for each convex cone $C_i \subset V_i$, $C_i \subseteq (C_i^*)^*$, furthermore, since $(C_i^*)^*$ is topologically closed it follows that $\overline{C_i} \subseteq (C_i^*)^*$. Thus it only remains to prove that $(C_i^*)^* \subseteq \overline{C_i}$. Suppose for a contradiction that there exists $v_i \in (C_i^*)^* \setminus \overline{C_i}$. Let $(\ ,\ )_i$ be a positive definite inner product on V_i and let $\|\ \|_i$ be the associated norm. Let $u_i \in \overline{C_i}$ be chosen so that $\|u_i - v_i\|_i$ is minimal. As in the proof of the (i) above, let $x_i \in \overline{C_i}$ be arbitrary, and consider the function

$$h_i(t) \colon = \|(1-t)u_i + tx_i - v_i\|_i^2 \qquad (t \in [0,1]).$$

Since C_i is a convex cone it is clear that $\overline{C_i}$ is also a convex cone. Then it follows that h_i attains a minimum at $t = 0$. Thus

$$0 \leq h_i'(0) = 2(x_i - u_i, u_i - v_i)_i. \tag{2.4.1}$$

By specializing x_i to, say, $2u_i$ and $\frac{1}{2}u_i$ we see from (2.4.1) that

$$0 = (u_i, u_i - v_i)_i. \tag{2.4.2}$$

Consequently (2.4.1) yields that

$$0 \leq (x_i, u_i - v_i)_i. \tag{2.4.3}$$

Hence the linear functional f_i defined by $f_i(x) = (x, u_i - v_i)_i$ is in C_i^*. Now since $v_i \in (C_i^*)^*$ it follows that $0 \leq f_i(v_i) = (v_i, u_i - v_i)_i$ and observe that this and (2.4.2) together yield that $(u_i - v_i, u_i - v_i)_i \leq 0$. Since $(\ ,\ )$ is positive definite we have $v_i = u_i$ contradicting that $v_i \notin \overline{C_i}$. □

There is a natural action of W on V_i^* $(i = 1, 2)$ as follows: if $w \in W$ and $f_i \in V_i^*$ then $wf_i \in V_i^*$ is defined by

$$(wf_i)(v_i) = f_i(w^{-1}v_i), \qquad \text{for all } v_i \in V_i.$$

For each $i \in \{1, 2\}$, set $P_i := \mathrm{PLC}(\Pi_i) \cup \{0\}$. Then

$$P_i^* = \{f \in V_i^* \mid f(x) \geq 0 \text{ for all } x \in \Pi_i\}.$$

Generalizing the concept of a *Tits cone* as defined in [**18**] and in section 5.13 of [**12**], we define the *Tits cone* in this non-orthogonal setting to be

$$U_i := \bigcup_{w \in W} wP_i^*.$$

Observe that from these definitions we immediately have

(2.4.4)

$$\begin{aligned} U_i^* &= \{\, v \in V_i \mid (wf)v \geq 0 \text{ for all } f \in P_i^* \text{ and } w \in W \,\} \\ &= \{\, v \in V_i \mid f(w^{-1}v) \geq 0 \text{ for all } f \in P_i^* \text{ and } w \in W \,\} \\ &= \bigcap_{w \in W} \{\, v \in V_i \mid f(w^{-1}v) \geq 0 \text{ for all } f \in P_i^* \,\} \\ &= \bigcap_{w \in W} \{\, wv \in V_i \mid f(v) \geq 0 \text{ for all } f \in P_i^* \,\} \\ &= \bigcap_{w \in W} \{\, wv \in V_i \mid v \in (P_i^*)^* \,\} \\ &= \bigcap_{w \in W} w\overline{P_i} \qquad \text{(by Lemma 2.4.1(ii), since } P_i \text{ is a convex cone)} \\ &= \bigcap_{w \in W} wP_i \qquad \text{(since } P_i \text{ is closed).} \end{aligned}$$

Observe that then Lemma 2.4.1(i) yields that there are $f_1 \in V_i^*$ and $f_2 \in V_2^*$ such that $f_1(x) \geq 1$ for all $x \in \Pi_1$ and $f_2(y) \geq 1$ for all $y \in \Pi_2$. For the next result we fix one such pair of linear functionals $\{f_1, f_2\}$.

Proposition 2.4.2. *Let $v_1 \in U_1^*$ and $v_2 \in U_2^*$. Then $\langle v_1, v_2 \rangle \leq 0$.*

Proof. Suppose for a contradiction that there exists $v_1 \in U_1^*$ and $v_2 \in U_2^*$ such that $\langle v_1, v_2 \rangle > 0$. Replace v_2 by a positive scalar multiple of itself if needed, we may assume that $\langle v_1, v_2 \rangle = 1$. Let

$$\mathscr{B} = \{\, x \in U_2^* \mid f_2(x) \leq f_2(v_2) \text{ and } \langle z, x \rangle \geq 1 \\ \text{for some } z \in U_1^* \text{ with } f_1(z) \leq f_1(v_1) \,\}.$$

Observe that $v_2 \in \mathscr{B}$, so $\mathscr{B} \neq \emptyset$.

Put $\epsilon = \frac{2}{nf_1(v_1)}$ where $n = |S|$. We shall show that for any given $x \in \mathscr{B}$ there exists $y \in \mathscr{B}$ such that $f_2(y) \leq f_2(x) - \epsilon$.

Given $x \in \mathscr{B}$, let $z = \sum_{a \in \Pi_1} \lambda_a a \in U_1^*$ ($\lambda_a \geq 0$ for all $a \in \Pi_1$) be such that $\langle z, x \rangle \geq 1$ and $f_1(z) \leq f_1(v_1)$. Observe that since

$\langle z, x\rangle = \sum_{a\in\Pi_1} \lambda_a\langle a, x\rangle \geq 1$ it follows that there exists some $a_0 \in \Pi_1$ such that $\lambda_{a_0}\langle a_0, x\rangle \geq \frac{1}{n}$, that is,

$$\langle a_0, x\rangle \geq \frac{1}{\lambda_{a_0} n} \geq \frac{1}{nf_1(v_1)} = \frac{\epsilon}{2}$$

because $\lambda_{a_0} \leq f_1(z) \leq f_1(v_1)$. Now let $y = r_{\phi(a_0)}x$. It is clear from (2.4.4) that U_2^* is W-invariant. Now since $x \in U_2^*$, it follows that $y \in U_2^*$ too. Thus

$$\begin{aligned} f_2(y) &= f_2(x - 2\langle a_0, x\rangle\phi(a_0)) \\ &= f_2(x) - 2\langle a_0, x\rangle f_2(\phi(a_0)) \\ &\leq f_2(x) - \epsilon f_2(\phi(a_0)) \\ &\leq f_2(x) - \epsilon \\ &< f_2(v_2). \end{aligned}$$

We claim that $y \in \mathscr{B}$. To prove this we need to find some $t \in U_1^*$ such that $\langle t, y\rangle \geq 1$ and $f_1(t) \leq f_1(v_1)$.

First consider the case that $\langle z, \phi(a_0)\rangle \geq 0$. Put $t = r_{a_0}z$. Then $t \in U_1^*$ (since z is and U_1^* is W-invariant) and

$$\langle t, y\rangle = \langle r_{a_0}z, r_{\phi(a_0)}x\rangle = \langle z, x\rangle \geq 1$$

furthermore,

$$f_1(t) = f_1(z) - 2\langle z, \phi(a_0)f_1(a_0) \leq f_1(z) \leq f_1(v_1)$$

as required. Hence $y \in \mathscr{B}$ when $\langle z, \phi(a_0)\rangle \geq 0$.

Next if $\langle z, \phi(a_0)\rangle < 0$ then $t = z$ will do; indeed

$$\begin{aligned} \langle z, y\rangle &= \langle z, x - 2\langle a_0, x\rangle\phi(a_0)\rangle \\ &= \langle z, x\rangle - 2 \underbrace{\langle a_0, x\rangle}_{(\geq \frac{1}{\lambda_{a_0} n} > 0)} \langle z, \phi(a_0)\rangle \\ &\geq \langle z, x\rangle \\ &\geq 1. \end{aligned}$$

Furthermore, by our construction $z \in U_1^*$ and $f_1(z) \leq f_1(v_1)$. Hence when $\langle z, \phi(a_0)\rangle < 0$, $y \in \mathscr{B}$ as well and this establishes the claim.

Starting with $x = v_2$, a finite number of iterations of the above process will produce a $y \in \mathscr{B}$ with $f_2(y)$ being negative, contradicting the fact that $U_2^* \subseteq P_2$ and $f_2(P_2) \subseteq (0, \infty)$. □

The following is a well-known result:

Lemma 2.4.3. ([**13**] 4.5.3) *Let H be a finite subgroup of W. Then there exists $w \in W$ and $J \subseteq S$ such that W_J is a finite parabolic subgroup and $wHw^{-1} \subseteq W_J$.* □

Lemma 2.4.4. *The set*

$$\{\, \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_2)\rangle \mid \alpha_1, \alpha_2 \in \Phi_1 \text{ and } \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_2)\rangle < 1 \,\}$$

is finite.

Proof. Let $\alpha_1, \alpha_2 \in \Phi_1$ such that $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_2)\rangle < 1$. By Corollary 2.3.17 we know that $H := \langle \{ r_{\alpha_1}, r_{\alpha_2} \} \rangle$ is a finite dihedral subgroup of W. Thus Lemma 2.4.3 yields that there are $w \in W$ and $J \subseteq S$ such that W_J is finite and $wHw^{-1} \subseteq W_J$. In particular, there are $x, y \in \Phi_1(W_J)$ such that $\alpha_1 = wx$ and $\alpha_2 = wy$. Now

$$\begin{aligned} &\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_2)\rangle \\ &= \langle wx, w\phi(y)\rangle\langle wy, w\phi(x)\rangle \\ &= \langle x, \phi(y)\rangle\langle y, \phi(x)\rangle \\ &\in \{\, \langle a, \phi(b)\rangle\langle b, \phi(a)\rangle \mid a, b \in \Phi_1 \text{ and } r_a, r_b \in \bigcup_{I \subseteq S, W_I \text{finite}} W_I \}. \end{aligned}$$

Since S is a finite set it follows that the last set on the right is finite proving the desired result. □

Immediately from the above we have:

Corollary 2.4.5. *There is a positive number θ such that*

$$\theta < \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_2)\rangle \qquad \textit{for all } \alpha_1, \alpha_2 \in \Phi_1 \textit{ with } 0 \neq \langle \alpha_1, \phi(\alpha_2)\rangle.$$

□

Chapter 3

The Dominance Hierarchy of Root Systems of Coxeter Groups

3.1. Introduction

If x and y are roots in the root system with respect to the *standard (Tits) geometric realization* of a Coxeter group W, we say that x *dominates* y (written $x \operatorname{dom}_W y$) if wy is a negative root whenever wx is a negative root. We call a positive root x *elementary* if it does not dominate any positive root other than x itself. The set of all elementary roots is denoted by $\mathscr{E}$. It has been proven by B. Brink and R. B. Howlett [Math. Ann. **296** (1993), 179–190] that $\mathscr{E}$ is finite if (and only) if W is a finite rank Coxeter group. Amongst other things, this finiteness property enabled Brink and Howlett to establish the automaticity of all finite rank Coxeter groups. Later Brink has also given a complete description of the set $\mathscr{E}$ for arbitrary finite rank Coxeter groups in [J. Algebra **206** (1998), 371–412]. But until the present, a systematic study of the dominance behaviour among non-elementary positive roots still remains to be completed. In this chapter we answer a collection of questions concerning dominance between such non-elementary positive roots. In particular, we show that for any finite rank Coxeter group and for any non-negative integer n, the set of roots dominating precisely n other positive roots is finite. We then give both upper and lower bounds to the sizes of all such sets as well as an inductive algorithm to compute all such sets. In this chapter, for any given positive root x, we obtain explicitly the set of positive roots dominated by x, if there are any.

In Section 3.4, we study some cones closely related to the Tits cone and their connection with the dominance concept.

We stress that all notations used in this chapter are exactly the same as those used in previous chapters.

Let $N : W \to \Phi^+$ be the function defined by

$$N(w) = \{x \in \Phi^+ \mid wx \in \Phi^-\}.$$

It is a well-known fact that

$$N(w) = \{\, x \in \Phi^+ \mid l(wr_x) < l(w) \,\} \qquad \text{and} \qquad l(w) = |N(w)|$$

(for example, Proposition 5.7 of [**12**] or Proposition 4.4.6 of [**13**]). If Φ' is a subset of Φ such that whenever $x, y \in \Phi'$, $r_x y \in \Phi'$, then we call Φ' a *root subsystem* of Φ. Now if W' is a reflection subgroup of W, then we set

$$\Phi(W') = \{\, a \in \Phi \mid r_a \in W' \,\}.$$

Observe that $\Phi(W')$ is a root subsystem of Φ. We call $\Phi(W')$ the *root subsystem corresponding to* W'.

Recall that Lemma 2.2.4 yields $f_1(\Delta_1(W')) = f_2(\Delta_2(W'))$ for all reflection subgroups W' of W. This observation enables us to define the *canonical roots* of W' in Φ.

Definition 3.1.1. For any reflection subgroup W' of W, let

$$\Delta(W') = f_1(\Delta_1(W')) = f_2(\Delta_2(W')).$$

Remark 3.1.2. Recall that $T := \bigcup_{w \in W} wRw^{-1}$ denotes the set of reflections in W. Since for all $\alpha \in \Phi_1$ we have $r_\alpha = r_{f_1(\alpha)}$, it follows that

$$S(W') = \{\, r_x \in T \mid x \in \Delta(W') \,\}.$$

With this justification, we call $\Delta(W')$ the *canonical roots* for $\Phi(W')$.

Proposition 3.1.3. (i) *Suppose that W' is a reflection subgroup of W. Suppose that a and $b \in \triangle(W')$ with $a \neq b$. Then*

$$(a, b) \in \{\, -\cos(\pi/n) \mid n \in \mathbb{N}, n \geq 2 \,\} \cup (-\infty, -1].$$

(ii) *Let* $X \subseteq \Phi^+$ *be a set such that whenever* $a, b \in X$ $a \neq b$ *the condition*

$$(a, b) \in \{ -\cos(\pi/n) \mid n \in \mathbb{N}, n \geq 2 \} \cup (-\infty, -1]$$

is satisfied. If we set $W' = \langle r_x \mid x \in X \rangle$*, then* $X = \Delta(W')$*.*

Proof. (i) Follows from Proposition 2.2.12 and Proposition 2.3.19.

(ii) Follows from Proposition 2.2.15 and Proposition 2.3.19. □

3.2. Rank 2 Root Subsystems

Suppose that $\Phi(\langle r_a, r_b \rangle)$ is an infinite rank 2 root system with canonical roots a and b. In this section we classify the root subsystems of $\Phi(\langle r_a, r_b \rangle)$.

Since $\Phi(\langle r_a, r_b \rangle)$ is infinite, it follows from Lemma 2.3.3 (ii) that $\langle r_a, r_b \rangle$ (the dihedral subgroup generated by r_a and r_b) must be infinite. Thus it follows from Proposition 4.5.4 of [**13**] that $(a, b) \leq -1$. Let $\theta = \cosh^{-1}(-(a, b))$, and for each integer i, set $c_i := \frac{\sinh(i\theta)}{\sinh \theta}$. Then

$$r_a a = -a, \quad r_b b = -b,$$

$$(r_a r_b)a = \frac{\sinh(3\theta)}{\sinh \theta} a + \frac{\sinh(2\theta)}{\sinh \theta} b = c_3 a + c_2 b$$

and

$$(r_b r_a)b = \frac{\sinh(2\theta)}{\sinh \theta} a + \frac{\sinh(3\theta)}{\sinh \theta} b = c_2 a + c_3 b.$$

An induction shows that for all integer i,

$$\text{(3.2.1)} \qquad \begin{cases} (r_a r_b)^i a &= c_{2i+1} a + c_{2i} b; \\ r_b (r_a r_b)^i a &= c_{2i+1} a + c_{2i+2} b; \\ (r_b r_a)^i b &= c_{2i} a + c_{2i+1} b; \\ r_a (r_b r_a)^i b &= c_{2i+2} a + c_{2i+1} b. \end{cases}$$

These calculations lead to the following well known result:

Lemma 3.2.1.

$$\Phi(\langle r_a, r_b \rangle) = \{ c_i a + c_{i \pm 1} b \mid i \in \mathbb{Z} \}.$$

□

Observe from Lemma 3.2.1 and (3.2.1) above that $w(c_n a + c_{n\pm1} b) = a$ for some $w \in \langle r_a, r_b \rangle$ if and only if n is odd whereas $w'(c_m b + c_{m\pm1} a) = b$ for some $w' \in \langle r_a, r_b \rangle$ if and only if m is odd. Furthermore, if i, j are integers such that $w(c_i a + c_{i\pm1} b) = c_j a + c_{j\pm1} b$ for some $w \in \langle r_a, r_b \rangle$, then $i \equiv j \pmod 2$.

Proposition 3.2.2. *Suppose that Φ' is a root subsystem of $\Phi(\langle r_a, r_b \rangle)$. Then Φ' is at most a rank 2 root subsystem.*

Proof. Suppose for a contradiction that there are at least three canonical generators x, y and z for the subsystem Φ'. By Lemma 3.2.1, there are three integers m, n and p such that

$$x = c_m a + c_{m\pm1} b;$$

$$y = c_n a + c_{n\pm1} b;$$

and

$$z = c_p a + c_{p\pm1} b.$$

If either $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n+1} b$ or $x = c_m a + c_{m-1} b$ and $y = c_n a + c_{n-1} b$, then $(x, y) = \cosh((m-n)\theta) \geq 1$, contradicting Proposition 3.1.3 (i). Without loss of generality, we may assume that $x = c_m a + c_{m+1} b$, and $y = c_n a + c_{n-1} b$. Now if $z = c_p a + c_{p+1} b$, then $(x, z) = \cosh((m-p)\theta) \geq 1$, contradicting Proposition 3.1.3 (i); on the other hand if $z = c_p a + c_{p-1} b$, then $(z, y) = \cosh((n-p)\theta) \geq 1$, again contradicting Proposition 3.1.3 (i). Therefore Φ' has at most two canonical generators, that is, Φ' is at most rank 2.

□

Suppose that $x = c_m a + c_{m\pm1} b$ and $y = c_n a + c_{n\pm1} b$ $(m, n \in \mathbb{Z})$, are roots in $\Phi(\langle r_a, r_b \rangle)$. To classify the rank 2 root subsystems of the form $\Phi(\langle r_x, r_y \rangle)$, it suffices to compute explicitly the canonical roots of $\Phi(\langle r_x, r_y \rangle)$.

Lemma 3.2.3. *Suppose that $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n-1} b$ are positive roots in $\Phi(\langle r_a, r_b \rangle)$ (that is, m is a non-negative integer* *and n is a positive integer). Then x and y are the canonical roots for $\Phi(\langle r_x, r_y\rangle)$. In particular, $\Phi(\langle r_x, r_y\rangle)$ is infinite in size.*

Proof. By Proposition 3.1.3 (i), to show that x and y are the canonical roots for $\Phi_{\langle r_x, r_y\rangle}$, it is enough to show that $(x, y) \leq -1$. Indeed, $(x, y) = -\cosh((n+m)\theta) \leq -1$. Thus Proposition 4.5.4 of [**13**] yields that the dihedral subgroup $\langle\, r_x, r_y\,\rangle$ is infinite, and consequently Lemma 2.3.3 (ii) yields that $\Phi(\langle r_x, r_y\rangle)$ is infinite in size.

□

Next we compute the canonical roots of $\Phi(\langle r_x, r_y\rangle)$, where x is of the form $c_m a + c_{m+1} b$ and y is of the form $c_n a + c_{n+1} b$, $m \neq n$.

Proposition 3.2.4. *Suppose that $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n+1}$ $(m, n \in \mathbb{Z},\ m \neq n)$, are roots in $\Phi(\langle r_x, r_y\rangle)$.*

(i) $\Phi(\langle r_x, r_y\rangle) = \{\, \pm(c_{k(m-n)-m} a + c_{k(m-n)-m-1} b) \mid k \in \mathbb{Z}\,\}$. *In particular this root subsystem is infinite in size.*

(ii) *The canonical roots of $\Phi(\langle r_x, r_y\rangle)$ are of the form $c_i a + c_{i-1} b$ and $c_j a + c_{j+1} b$, where*

$$i = \min\{\, k(m-n) - m \mid k \in \mathbb{Z} \text{ and } k(m-n) - m > 0\,\}$$

and

$$j = \min\{\, k(m-n) + m \mid k \in \mathbb{Z} \text{ and } k(m-n) + m \geq 0\,\}.$$

Proof. (i) Clearly $\Phi(\langle r_x, r_y\rangle)$ consists of all the roots of the form

$$(r_x r_y)^l x, \qquad r_y (r_x r_y)^l x, \qquad (r_y r_x)^l y \qquad \text{and} \qquad r_x (r_y r_x)^l y,$$

where l ranges over $\mathbb{Z}$. Depending on the choice of m and n, we have the following three cases to consider:

(1) Both m and n are even.

(2) Exactly one of m and n is even.

(3) Both m and n are odd.

First suppose that (1) is the case. Then equation (3.2.1) yields that $x = (r_b r_a)^{m/2} b$, and $y = (r_b r_a)^{n/2} b$. Consequently

$$r_x = r_b (r_a r_b)^m = (r_b r_a)^m r_b \ \text{ and } \ r_y = r_b (r_a r_b)^n = (r_b r_a)^n r_b.$$

Thus $r_x r_y = (r_b r_a)^{m-n}$ and $r_y r_x = (r_b r_a)^{n-m}$. Hence (3.2.1) yields that for any integer l,

$$\begin{aligned}(r_x r_y)^l x &= (r_b r_a)^{l(m-n)} (r_b r_a)^{m/2} b \\ &= c_{2l(m-n)+m} a + c_{2l(m-n)+m+1} b\end{aligned}$$

and

$$\begin{aligned}r_y (r_x r_y)^l x &= r_b (r_a r_b)^n (r_b r_a)^{l(m-n)} (r_b r_a)^{m/2} b \\ &= r_b (r_b r_a)^{l(m-n)-n+m/2} b \\ &= r_a (r_b r_a)^{l(m-n)-n+m/2-1} b \\ &= c_{2l(m-n)+m-2n} a + c_{2l(m-n)+m-2n-1} b.\end{aligned}$$

By symmetry, we duduce that $(r_y r_x)^l y = c_{2l(n-m)+n} a + c_{2l(n-m)+n+1} b$ and $r_x (r_y r_x)^l y = c_{2l(n-m)+n-2m} a + c_{2l(n-m)+n-2m-1} b$.

Next suppose that (2) is the case. We may assume, without loss of generality, that m is even. Then $x = (r_b r_a)^{m/2} b$ and $y = r_b (r_a r_b)^{(n-1)/2} a$. Similar rank 2 calculations as in (1) above yield that for any integer l,

$$\begin{aligned}(r_x r_y)^l x &= c_{2l(m-n)+m} a + c_{2l(m-n)+m+1} b, \\ r_y (r_x r_y)^l x &= c_{2l(m-n)+m-2n} a + c_{2l(m-n)+m-2n-1} b, \\ (r_y r_x)^l y &= c_{2l(n-m)+n} a + c_{2l(n-m)+n+1} b, \\ r_x (r_y r_x)^l y &= c_{2l(n-m)+n-2m} a + c_{2l(n-m)+n-2m-1} b.\end{aligned}$$

Now suppose that (3) is the case. Then $x = r_b (r_a r_b)^{(m-1)/2} a$ and $y = r_b (r_a r_b)^{(n-1)/2} a$. Again, a rank 2 calculation yields that for every integer l,

$$\begin{aligned}(r_x r_y)^l x &= c_{2l(m-n)+m} a + c_{2l(m-n)+m+1} b, \\ r_y (r_x r_y)^l x &= c_{2l(m-n)+m-2n} a + c_{2l(m-n)+m-2n-1} b, \\ (r_y r_x)^l y &= c_{2l(n-m)+n} a + c_{2l(n-m)+n+1} b, \\ r_x (r_y r_x)^l y &= c_{2l(n-m)+n-2m} a + c_{2l(n-m)+n-2m-1} b.\end{aligned}$$

Thus we see that $\Phi(\langle r_x r_y \rangle)$ consists of all roots of the form

$$c_{2l(m-n)+m}a + c_{2l(m-n)+m+1}b, \tag{3.2.2}$$

$$c_{2l(n-m)+n}a + c_{2l(n-m)+n+1}b \tag{3.2.3}$$

and

$$c_{2l(m-n)+m-2n}a + c_{2l(m-n)+m-2n-1}b, \tag{3.2.4}$$

$$c_{2l(n-m)+n-2m}a + c_{2l(n-m)+n-2m-1}b, \tag{3.2.5}$$

where l ranges over $\mathbb{Z}$.

Now let us consider all roots of the form (3.2.4) and (3.2.5). Observe that $2l(m-n)+m-2n = 2(l+1)(m-n)-m$ is of the form of an even multiple of $(m-n)$ minus m, whereas on the other hand $2l(n-m)+n-2m = -(2l+1)(m-n)-m$ is of the form of an odd multiple of $(m-n)$ minus m. Thus we conclude that the set of roots of the form (3.2.4) and (3.2.5) is exactly $\{\, c_{k(m-n)-m}a + c_{k(m-n)-m-1}b \mid k \in \mathbb{Z} \,\}$. Similarly we could also see that the set of roots of the form (3.2.2) and (3.2.3) is exactly $\{\, c_{k(m-n)+m}a + c_{k(m-n)+m+1}b \mid k \in \mathbb{Z} \,\}$. Finally we observe that for each integer l, $c_{-l} = \frac{\sinh(-l\theta)}{\sinh\theta} = -c_l$. Thus

$$c_{k(m-n)-m}a + c_{k(m-n)-m-1}b = -(c_{-k(m-n)+m}a + c_{-k(m-n)+m+1}b).$$

Therefore

$$\Phi(\langle r_x, r_y \rangle) = \{\, \pm(c_{k(m-n)-m}a + c_{k(m-n)-m-1}b) \mid k \in \mathbb{Z} \,\}.$$

Observe that, in particular, the root subsystem $\Phi(\langle r_x, r_y \rangle)$ is infinite in size. This completes the proof of (i).

(ii) Let α and β be the canonical roots for $\Phi(\langle r_x, r_y \rangle)$. Then $\alpha = c_i a + c_{i-1}b$ and $\beta = c_j a + c_{j+1}b$ for some positive integer i and some nonnegative integer j. Indeed, by Lemma 3.2.1, the only other possiblities are either $\alpha = c_i a + c_{i+1}b$ and $\beta = c_j a + c_{j+1}b$ or $\alpha = c_i a + c_{i-1}b$ and $\beta = c_i a + c_{j-1}b$. But then $(\alpha, \beta) = \cosh((i-j)\theta) \geq 1$, contradicting

Proposition 3.1.3 (i). Now by (i) above

$$\Phi(\langle r_x, r_y\rangle)$$
$$= \{\, c_{k(m-n)-m}a + c_{k(m-n)-m-1}b,\; c_{k(m-n)+m}a + c_{k(m-n)+m+1}b \mid k \in \mathbb{Z}\,\}.$$

Thus i must be of the form

$$k(m-n) - m,\;\; k \in \mathbb{Z},$$

and j must be of the form

$$k'(m-n) + m,\;\; k' \in \mathbb{Z}.$$

Since α and β are the canonical roots for $\Phi(\langle r_x, r_y\rangle)$, it follows that i and j must be as small as possible subject to the requirement that both α and β are positive roots. Therefore we conclude that i must be the least positive integer of the form $k(m-n)-m$, where k is an integer; and j must be the least nonnegative integer of the form $k'(m-n)+m$, where k' is an integer.

□

Finally we look at the root subsystem generated by $x = c_{m+1}a + c_m b$ and $y = c_{n+1}a + c_n b$.

Proposition 3.2.5. *Suppose that $x = c_{m+1}a + c_m b$ and $y = c_{n+1}a + c_n b$ are roots in $\Phi(\langle r_x, r_y\rangle)$.*

(i) $\Phi(\langle r_x, r_y\rangle) = \{\, \pm(c_{k(m-n)-m-1}a + c_{k(m-n)-m}b) \mid k \in \mathbb{Z}\,\}$. *In particular, this root subsystem is infinite in size.*

(ii) *The canonical roots for $\Phi_{\langle r_x, r_y\rangle}$ are $c_i a + c_{i-1}b$ and $c_j a + c_{j+1}b$, where*

$$i = \min\{\, k(m-n) + m \mid k \in \mathbb{Z} \text{ and } k(m-n)+m > 0\,\}$$

and

$$j = \min\{\, k(m-n) - m \mid k \in \mathbb{Z} \text{ and } k(m-n)-m \geq 0\,\}.$$

Proof. Follows from Proposition 3.2.4 with the roles of a and b interchanged. □

Combining Lemma 3.2.2, Lemma 3.2.3, Proposition 3.2.4 and Proposition 3.2.5, we immediately deduce that:

Corollary 3.2.6. *Suppose that* Φ' *is a root subsystem of* $\Phi(\langle r_a, r_b \rangle)$*. Then either* $|\Phi'| = 2$ *or* Φ' *is an infinite rank* 2 *subsystem.* □

3.3. The Dominance Hierarchy

Definition 3.3.1. (i) For x and $y \in \Phi$, we say that x *dominates* y, written $x \operatorname{dom}_W y$ if

$$\{ w \in W \mid w \cdot x \in \Phi^- \} \subseteq \{ w \in W \mid w \cdot y \in \Phi^- \}.$$

(ii) For each $x \in \Phi^+$, set

$$D(x) = \{ y \in \Phi^+ \mid y \neq x \text{ and } x \operatorname{dom}_W y \},$$

and for each $n \in \mathbb{N}$, define

$$D_n = \{ x \in \Phi^+ \bigm| |D(x)| = n \}$$

(the set of positive roots x that dominate exactly n other positive roots).

In [**6**] and [**5**] dominance is only defined on Φ^+, and it is found in [**6**] that it is a partial order on Φ^+. Here we have generalized the notion of dominance to the whole of Φ. It can be readily seen that this generalized dominance is a partial order on Φ. It also turns out that the geometric characterization of dominance remains the same as we extend the definition to cover all of Φ. It is clear from the above definition that

$$\Phi^+ = \biguplus_{n \in \mathbb{N}} D_n.$$

In this chapter we study the above decomposition. We already knew a good deal about D_0 from [**6**] and [**5**]: if W is finite then $D_0 = \Phi^+$, whereas if W is an infinite Coxeter group of finite rank, then $|D_0| < \infty$. Observe that in the latter case $\biguplus_{n \in \mathbb{N}, n \geq 1} D_n$ will be an infinite set. One major result of this chapter (Theorem 3.3.9 and Corollary 3.3.10 below) is that if S is finite then D_n is finite for all natural numbers n. We also give an upper bound for $|D_n|$. But first we need a few elementary results.

Lemma 3.3.2. (i) If x and $y \in \Phi^+$, then $x \operatorname{dom}_W y$ if and only if $(x, y) \geq 1$ and $\operatorname{dp}(x) \geq \operatorname{dp}(y)$.
(ii) Dominance is W-invariant: if $x \operatorname{dom}_W y$ then for any $w \in W$, $wx \operatorname{dom}_W wy$.
(iii) Suppose that $x, y \in \Phi$, and $x \operatorname{dom}_W y$. Then $-y \operatorname{dom}_W -x$.
(iv) Suppose that $x \in \Phi^+$ and $y \in \Phi^-$. Then $x \operatorname{dom}_W y$ if and only if $(x, y) \geq 1$.
(v) Let $x, y \in \Phi$. Then there is dominance between x and y if and only if $(x, y) \geq 1$.

Proof. (i) See [**6**, Lemma 2.3].
(ii) Clear from the definition of dominance.
(iii) Suppose for a contradiction that there exists $w \in W$ such that $w(-y) \in \Phi^-$ and $w(-x) \in \Phi^+$. Then $w(y) \in \Phi^+$ yet $w(x) \in \Phi^-$, contradicting the assumption that $x \operatorname{dom}_W y$.
(iv) Suppose that $x \operatorname{dom}_W y$. Since dominance is W-invariant, it follows that $r_y x \operatorname{dom}_W r_y y$. Because $r_y y = -y \in \Phi^+$, so $r_y x \in \Phi^+$. Thus part (i) yields that $(r_y x, r_y y) \geq 1$. Since $(\ ,\ )$ is W-invariant, it follows that $(x, y) \geq 1$.

Conversely, suppose that $x \in \Phi^+$, $y \in \Phi^-$ with $(x, y) \geq 1$. Then clearly $r_y x = x - 2(x, y)y \in \Phi^+$. Thus $r_y x$ and $r_y y = -y$ are both positive. Then it follows from part (i) that there is dominance between $r_y x$ and $r_y y$. Since dominance is W-invariant, it follows that there is dominance between x and y. Finally, since $x \in \Phi^+$ and $y \in \Phi^-$, it is clear that $x \operatorname{dom}_W y$.
(v) Suppose that $x, y \in \Phi^-$. Then part (i) yields that there is dominance between $-x$ and $-y$ if and only if $(-x, -y) = (x, y) \geq 1$. This combined with part (i) and part (iv) above yields the desired result.

□

Lemma 3.3.3. *Suppose that $x, y \in \Phi$ with $x \neq y$, $x \operatorname{dom}_W y$ and $y \in D_0$. Then $r_y x \in \Phi^+$.*

Proof. Suppose for a contradiction that $r_y x \in \Phi^-$. Lemma 3.3.2 (ii) then yields that $r_y x \operatorname{dom}_W r_y y = -y$. Now Lemma 3.3.2 (iii) yields that $y \operatorname{dom}_W -r_y x \in \Phi^+$. Since $y \in D_0$, this forces $-r_y x = y$, contradicting $x \neq y$. □

Proposition 3.3.4. *Suppose that $x, y \in \Phi$ are distinct with $x \operatorname{dom}_W y$.*
(i) *Let a, b be the canonical roots for the root subsystem $\Phi(\langle r_x, r_y \rangle)$. Then there is a $w \in \langle r_x, r_y \rangle$ such that either*

$$wx = a \text{ and } wy = -b \qquad \text{or} \qquad wx = b \text{ and } wy = -a.$$

In particular, $(a, b) = -(x, y)$.
(ii) *$x \operatorname{dom}_W y$ if and only if $x \operatorname{dom}_{\langle r_x, r_y \rangle} y$.*
(iii) *$(r_y x, x) \leq -1$ and $(r_y x, y) \leq -1$, and in particular, $r_y x$ cannot dominate either x or y.*

Proof. (i) Since $|(x, y)| \geq 1$ it follows from Proposition 2.3.16 that $\langle r_a, r_b \rangle = \langle r_x, r_y \rangle$ is infinite, and hence $(a, b) \leq -1$ by Proposition 3.1.3. Thus $(a, -b) \geq 1$, and $a \operatorname{dom}_W -b$, and similarly $b \operatorname{dom}_W -a$. Using the W-invariance of dominance it follows readily that there are two dominance chains in the root subsystem $\Phi(\langle r_a, r_b \rangle)$, as follows:

$$\cdots \operatorname{dom}_W\ r_a r_b r_a(b)\ \operatorname{dom}_W\ r_a r_b(a)\ \operatorname{dom}_W r_a(b)\ \operatorname{dom}_W\ a \\ \operatorname{dom}_W\ (-b)\ \operatorname{dom}_W\ r_b(-a)\ \operatorname{dom}_W\ r_b r_a(-b)\ \operatorname{dom}_W\ \cdots \tag{3.3.1}$$

and

$$\cdots\ \operatorname{dom}_W\ r_b r_a r_b(a)\ \operatorname{dom}_W\ r_b r_a(b)\ \operatorname{dom}_W\ r_b(a)\ \operatorname{dom}_W\ b \\ \operatorname{dom}_W(-a)\ \operatorname{dom}_W\ r_a(-b)\ \operatorname{dom}_W\ r_a r_b(-a)\ \operatorname{dom}_W\ \cdots . \tag{3.3.2}$$

Observe that each element of $\Phi(\langle r_a, r_b \rangle)$ lies in exactly one of the above chains, and the negative of any element of one of these chains lies in the other. Thus x', $y' \in \Phi(\langle r_a, r_b \rangle)$ are in the same chain if and only if $(x', y') \geq 1$ and in different chains if and only if $(x', y') \leq -1$.

From (3.3.1) we see that the roots dominated by a are all negative, and from (3.3.2) we see that the roots dominated by b are all negative. Since we may choose $w \in \langle r_a, r_b \rangle$ such that $wx = a$ or $wx = b$, and since $wx \operatorname{dom}_W wy$, it follows that either

$$wx = a \quad \text{and} \quad wy \in \Phi(\langle r_a, r_b \rangle) \cap \Phi^- \tag{3.3.3}$$

or

$$wx = b \quad \text{and} \quad wy \in \Phi(\langle r_a, r_b \rangle) \cap \Phi^-. \tag{3.3.4}$$

Suppose that $wx = a$. Then $(a, -wy) = (wx, -wy) = -(x, y) \leq -1$ and $-wy \in \Phi(\langle r_x, r_y \rangle) \cap \Phi^+$. Hence it follows from Proposition 3.1.3 (ii) that $\{a, -wy\}$ is the set of canonical roots for $\Phi(\langle r_x, r_y \rangle)$, forcing $-wy = b$. Similarly, in the case $wx = b$ we may conclude that $wy = -a$. Finally, observe that in either case, $(a, b) = -(wx, wy) = -(x, y)$.

(ii) First suppose that $x \operatorname{dom}_W y$. Then

$$\{\, w \in W \mid wx \in \Phi^- \,\} \subseteq \{\, w \in W \mid wy \in \Phi^- \,\},$$

and taking the intersection with $\langle r_x, r_y \rangle$ gives

$$\{\, w \in \langle r_x, r_y \rangle \mid wx \in \Phi^- \,\} \subseteq \{\, w \in \langle r_x, r_y \rangle \mid wy \in \Phi^- \,\},$$

which shows that $x \operatorname{dom}_{\langle r_x, r_y \rangle} y$.

Conversely, suppose that $x \operatorname{dom}_{\langle r_x, r_y \rangle} y$. By Lemma 3.3.2 (v) applied with W replaced by $\langle r_x, r_y \rangle$ we see that $(x, y)' \geq 1$ where $(\ ,\ )'$ is the restriction of $(\ ,\ )$ to the subspace spanned by x and y. Thus $(x, y) \geq 1$,applying Lemma 3.3.2 (v) again yields that either $x \operatorname{dom}_W y$ or $y \operatorname{dom}_W x$. But the latter alternative would imply that $y \operatorname{dom}_{\langle r_x, r_y \rangle} x$, by the first part of this proof, contrary to the fact that $x \operatorname{dom}_{\langle r_x, r_y \rangle} y$ and $x \neq y$.

(iii) Since $x \operatorname{dom}_W y$, Lemma 3.3.2 (v) yields that $(x, y) \geq 1$. Then $(r_y x, y) = (x, -y) \leq -1$ and hence there is no dominance between $r_y x$ and y. Also $(r_y x, x) = (x, x) - 2(x, y)^2 \leq -1$ and consequently there is no dominance between x and $r_y x$, proving (iii).

□

Remark 3.3.5. Observe from the proof of Proposition 3.3.4 (ii), that if $x, y \in \Phi^+$ are distinct, and $x \operatorname{dom}_W y$, then the depth of x relative to $(W', S(W'))$ is greater than the depth of y relative to $(W', S(W'))$.

Lemma 3.3.6. *Suppose that x, $y \in \Phi$ are distinct with $x \operatorname{dom}_W y$. Let a and b be the canonical roots for $\Phi(\langle r_x, r_y \rangle)$. Then either*

$$x = c_m a + c_{m+1} b \qquad \text{and} \qquad y = c_{m-1} a + c_m b$$

or

$$x = c_m a + c_{m-1} b \qquad \text{and} \qquad y = c_{m-1} a + c_{m-2} b$$

for some integer m.

Proof. Since $x, y \in \Phi(\langle r_a, r_b \rangle)$, Lemma 3.2.1 yields that

$$x = c_m a + c_{m\pm1} b \qquad \text{and} \qquad y = c_n a + c_{n\pm1} b$$

for some integers m and n. If either $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n-1} b$ or $x = c_m a + c_{m-1} b$ and $y = c_n a + c_{n+1} b$, then

$$(x, y) = -\cosh((n+m)\theta) \leq -1$$

contradicting $x \operatorname{dom}_W y$. Therefore there are only two possibilities

$$\begin{cases} x &= c_m a + c_{m+1} b \\ y &= c_n a + c_{n+1} b \end{cases} \tag{3.3.5}$$

or

$$\begin{cases} x &= c_{m+1} a + c_m b \\ y &= c_{n+1} a + c_n b. \end{cases} \tag{3.3.6}$$

First suppose that (3.3.5) is the case. Since a and b are the canonical roots for $\Phi(\langle r_a, r_b \rangle) = \Phi(\langle r_x, r_y \rangle)$, it follows from Proposition 3.2.4 (ii) that there are integers k_1 and k_2 such that

$$1 = k_1(m-n) - m \qquad \text{and} \qquad 0 = k_2(m-n) + m.$$

But then $k_1 + k_2 = \frac{1+m}{m-n} + \frac{-m}{m-n} = \frac{1}{m-n} \in \mathbb{Z}$. Clearly this is only possible when $m - n = \pm 1$. On the other hand, since $x \operatorname{dom}_W y$, it is readily seen that $m > n$, giving us

$$x = c_m a + c_{m+1} b \qquad \text{and} \qquad y = c_{m-1} a + c_m b.$$

If (3.3.6) is the case then similar reasoning as above yields that

$$x = c_m a + c_{m-1} b \qquad \text{and} \qquad y = c_{m-1} a + c_{m-2} b.$$

□

Remark 3.3.7. Let x, y, a and b be as in Proposition 3.3.4 (i) and Lemma 3.3.6 above. Then in fact x and y are consecutive terms in precisely one of the dominance chains (3.3.1) or (3.3.2).

Theorem 3.3.8. $D_1 \subseteq \{\, r_a b \mid a, b \in D_0 \,\}$. *Furthermore, if* $|S| < \infty$ *then* $|D_1| \leq |D_0|^2 - |D_0|$.

Proof. Suppose that $x \in D_1$ and $\{y\} = D(x)$. Clearly $y \in D_0$. By Lemma 3.3.3, we know that $r_y x \in \Phi^+$. Thus to prove Theorem 3.3.8, we only need to show that $r_y x \in D_0$.

Suppose for a contradiction that $r_y x \in \Phi^+ \setminus D_0$. Then there exists $z \in \Phi^+ \setminus \{r_y x\}$ with $r_y x \operatorname{dom}_W z$. Since dominance is W-invariant, it follows that $x \operatorname{dom}_W r_y z$. Because $z \in \Phi^+$, so clearly $r_y z \neq y$. Hence $r_y z \in \Phi^-$, because $D(x) = \{y\}$. Thus $(z, y) > 0$. Then

$$\begin{aligned} 1 \leq (r_y x, z) &= (x - 2(x, y)y, z) \\ &= (x, z) - 2(x, y)(y, z). \end{aligned}$$

Since we know that $(x, y) \geq 1$ ($x \operatorname{dom}_W y$) and $(z, y) > 0$, it follows that $1 \leq (x, z)$. Therefore Lemma 3.3.2 (v) yields that either $x \operatorname{dom}_W z$ or $z \operatorname{dom}_W x$. Suppose that $z \operatorname{dom}_W x$. Then $r_y x \operatorname{dom}_W z \operatorname{dom}_W x$, contradicting Proposition 3.3.4 (iii). On the other hand, if $x \operatorname{dom} z$, then our construction forces $z = y$. But then $r_y x \operatorname{dom} y$, again contradicting Proposition 3.3.4 (iii)

Thus $r_y x \in D_0$, as required. Since $x \in D_1$ was arbitrary, it follows that

$$D_1 \subseteq \{\, r_a b \mid a, b \in D_0 \,\}.$$

Finally, since D_1 does not contain elements of the form $r_a a$, $a \in D_0$, it follows that

$$D_1 \subseteq \{\, r_a b \mid a, b \in D_0 \,\} \setminus -D_0.$$

In the case that $|S| < \infty$, we have $|D_0| < \infty$ (by [**6**, Theorem 2.8]), so it follows from the above that $|D_1| \leq |D_0|^2 - |D_0|$. □

The above treatment of D_1 can be generalised to D_n for arbitrary $n \in \mathbb{N}$. Indeed we have:

Theorem 3.3.9. *For $n \in \mathbb{N}$,*

$$D_n \subseteq \{\, r_a b \mid a \in D_0, b \in \bigcup_{m \leq n-1} D_m \,\}.$$

Proof. The case $n = 1$ has been covered by Theorem 3.3.8, so we may assume that $n > 1$.

Let $x \in D_n$, and suppose that $D(x) = \{\, y_1, y_2, \ldots, y_n \,\}$, with y_n being minimal with respect to dominance. Observe that $y_n \in D_0$. Then Lemma 3.3.3 yields that $r_{y_n} x \in \Phi^+$. Hence either $r_{y_n} x \in D_0$ or $r_{y_n} x \in \Phi^+ \setminus D_0$.

If $r_{y_n} x \in \Phi^+ \setminus D_0$, let $z \in D(r_{y_n} x)$. We will show that there are at most $(n-1)$ possible values for z. Observe that this establishes

$$r_{y_n} x \in \biguplus_{m \leq n-1} D_m.$$

Since $r_{y_n} x$ dom z, Lemma 3.3.2 (ii) yields that $x \operatorname{dom}_W r_{y_n} z$. Hence either $r_{y_n} z = y_i$, for $1 \leq i \leq n-1$ or $r_{y_n} z \in \Phi^-$. If $r_{y_n} z \in \Phi^-$, then $(y_n, z) > 0$, yielding

$$\begin{aligned} 1 \leq (r_{y_n} x, z) &= (x - 2(x, y_n) y_n, z) \\ &= (x, z) - 2(x, y_n)(y_n, z). \end{aligned}$$

Now $(y_n, z) > 0$ and $(x, y_n) \geq 1$ (since $x \operatorname{dom}_W y_n$), therefore we must have $(x, z) \geq 1$. Similar to the proof of Theorem 3.3.8, we can conclude that $x \operatorname{dom}_W z$. Hence $z \in \{y_1, \cdots, y_n\}$. By Proposition3.3.4(iii) we know that $z \neq y_n$. Thus if $r_{y_n} z \in \Phi^-$, then $z \in \{y_1, \cdots, y_{n-1}\}$. Summing up, if $z \in D(r_{y_n} x)$, then

$$\begin{aligned} z \in \{\, r_{y_n}(y_i) \mid r_{y_n}(y_i) \in \Phi^+,\, i \in \{\,1, \ldots, n-1\,\}\,\} \\ \cup \{\, y_i \mid r_{y_n}(y_i) \in \Phi^-,\, i \in \{\,1, \ldots, n-1\,\}\,\} \end{aligned}$$

and this is clearly a disjoint union of size $n-1$. Thus $r_{y_n}(x) \in D_m$, for some $m \leq n-1$, and hence

$$D_n \subseteq \{\, r_a b \mid a \in D_0,\, b \in \biguplus_{m \leq n-1} D_m \,\}.$$

□

It turns out that we can obtain reasonably nice upper bonds for $|D_n|$, indeed we can deduce immediately:

Corollary 3.3.10. *Suppose that* $|S| < \infty$. *Then for* $n \in \mathbb{N}$, $|D_n| < \infty$. *Indeed*

$$|D_n| \leq |D_0|^{n+1} - |D_0|^n.$$

Proof. Induction on n. The case $n = 1$ has been shown in Theorem 3.3.8 and thus we may assume that $n > 1$. Since $D_i \cap D_j = \emptyset$ if $i \neq j$, it follows from Theorem 3.3.9 that

$$D_n \subseteq \{\, r_a b \mid a \in \mathscr{E},\, b \in \biguplus_{m \leq n-1} D_m \,\} \setminus (\biguplus_{m<n} D_m).$$

Hence in the case $|S| < \infty$,

(3.3.7)

$$\begin{aligned} |D_n| &\leq |D_0|(|D_0| + |D_1| + \cdots + |D_{n-1}|) - (|D_0| + |D_1| + \cdots + |D_{n-1}|) \\ &= (|D_0| - 1)(|D_0| + |D_1| + \cdots + |D_{n-1}|). \end{aligned}$$

Now the inductive hypothesis yields that

$$\begin{aligned} &|D_0| + |D_1| + |D_2| + \cdots + |D_{n-1}| \\ &\leq |D_0| + |D_0|^2 - |D_0| + |D_0|^3 - |D_0|^2 + \cdots \\ &\qquad + |D_0|^{n-1} - |D_0|^{n-2} + |D_0|^n - |D_0|^{n-1} \\ &= |D_0|^n. \end{aligned}$$

Thus (3.3.7) yields that $|D_n| \leq |D_0|^{n+1} - |D_0|^n$, as required.

$\square$

Remark 3.3.11. (i) It is seen in the proof of Theorem 3.3.9 that if $x,\ y \in \Phi^+$ with $x \operatorname{dom} y$ and $x \in D_n$, then $r_y x \in D_m$, for some $m < n$. It turns out that we can say a bit more on this and we shall do so in Proposition 3.3.16 and Proposition 3.3.24 below.

(ii) Having shown that for all $n \in \mathbb{N}$, $|D_n| < \infty$ if $|S| < \infty$, it is not immediately clear, at this stage, that for all $n \in \mathbb{N}$, $D_n \neq \emptyset$. Lemma 3.3.12 to Corollary 3.3.23 below will, amongst other things, establish that $D_n \neq \emptyset$ for all $n \in \mathbb{N}$ if W is a finite-rank infinite Coxeter group.

Lemma 3.3.12. *For $n \in \mathbb{N}$,*

$$\{\, wa \mid a \in D_0,\ w \in W \text{ with } l(w) < n \,\} \cap D_n = \emptyset.$$

Proof. Suppose not. Then there exists some $n \in \mathbb{N}$ and $x = wa \in D_n$, with $a \in D_0$, $w \in W$ and $l(w) < n$. Let $D(x) = \{\, y_1, \ldots, y_n \,\}$. Since dominance is W-invariant, it follows that

$$w^{-1}x = a \operatorname{dom} w^{-1}y_1, \ldots, w^{-1}y_n.$$

Note that $a \notin \{\, w^{-1}y_1, \cdots, w^{-1}y_n \,\}$, for otherwise $x \in \{\, y_1, y_2 \ldots, y_n \,\}$ which is absurd. Then $w^{-1}y_1, \cdots, w^{-1}y_n \in \Phi^-$ (since $a \in D_0$). Hence $y_1, \cdots, y_n \in N(w^{-1})$, but this gives a contradiction to the fact that $|N(w^{-1})| = l(w^{-1}) = l(w) < n$. $\square$

Lemma 3.3.13.

$$RD_0 \subseteq -D_0 \uplus D_0 \uplus D_1.$$

Proof. Let $r \in R$ and $x \in D_0$ be arbitrary. If $rx \in \Phi^+$, then Lemma 3.3.12 above yields that $rx \in D_0 \uplus D_1$. On the other hand, if $rx \in \Phi^-$, then $x \in \Pi$ and $r = r_x$. Thus $rx = -x \in -\Pi$. Since r and x were chosen arbitrarily, it follows that $RD_0 \subseteq -\Pi \uplus D_0 \uplus D_1$. $\square$

Generalising Lemma 3.3.13, we have:

Lemma 3.3.14. *For all $n \geq 1$*

$$RD_n \subseteq D_{n-1} \uplus D_n \uplus D_{n+1}.$$

Proof. Suppose that $n \geq 1$, and let $x \in D_n$, and $z \in \Pi$ be arbitrary. Since $x \neq z$ it follows that $r_z x \in \Phi^+$.

Suppose for a contradiction that $r_z x \in D_m$ for some $m \geq n+2$. Let $D(r_z x) = \{ y_1, \ldots, y_m \}$. Then $x \operatorname{dom}_W r_z y_1, \cdots, r_z y_m$, for dominance is W-invariant. Since $x \in D_n$, and $m \geq n+2$, it follows that there are $1 \leq i < j \leq m$ such that $r_z y_i \in \Phi^-$ and $r_z y_j \in \Phi^-$. But this is impossible since r_z could only make one positive root negative. Therefore we may conclude that $r_z x \notin D_m$ where $m \geq m+2$. A similar argument also shows that $r_z x \notin D_{m'}$ where $m' \leq n-2$, and we are done. □

Lemma 3.3.15. *Suppose that x, y are in Φ^+ with $y \preceq x$. Let $w \in W$ be such that $x = wy$ and $dp(x) = dp(y) + l(w)$. Then $y \in D_m$ implies that $x \in D_n$ for some $n \geq m$. Furthermore, $wD(y) \subseteq D(x)$.*

Proof. It is enough to show that the desired result holds in the case that $w = r_a$ for some $a \in \Pi$. The more general proof then follows from an induction on $l(w)$.

Since $x = r_a y$ and $y \prec x$ it follows from Lemma 2.3.3 (v) that $(a, y) < 0$. Let $D(y) = \{ z_1, z_2, \ldots, z_m \}$. Then Lemma 3.3.2 (v) yields that $a \notin D(y)$. Since $a \in \Pi$, this in turn implies that $r_a D(y) \subset \Phi^+$. Since dominance is W-invariant, it follows that $x \operatorname{dom}_W r_a z_i$ for all $i \in \{ 1, 2 \ldots, m \}$. Hence $\{ r_a z_1, r_a z_2, \ldots, r_a z_m \} \subseteq D(x)$, and thus $x \in D_n$ for some $n \geq m$. □

The next proposition, somewhat an analogy to Lemma 2.3.3 (v) above, has many applications, among which, we can deduce for arbitrary positive root x, the integer n for which $x \in D_n$. Furthermore, it enables us to compute $D(x)$ explicitly as well as to obtain an algorithm to compute all the D_n's systematically.

Proposition 3.3.16. *Suppose that $x \in D_n$, $n \geq 1$, and $a \in \Pi$. Then*

(i) *$r_a x \in D_{n-1}$ if and only if $(x, a) \geq 1$;*

(ii) *$r_a x \in D_{n+1}$ if and only if $(x, a) \leq -1$;*

(iii) *$r_a x \in D_n$ if and only if $(x, a) \in (-1, 1)$.*

Proof. (i) Suppose that $x \in D_n$, $a \in \Pi$ such that $r_a x \in D_{n-1}$. Let $D(x) = \{ z_1, z_2, \ldots, z_n \}$. Then $r_a x \operatorname{dom}_W r_a z_i$ for all $i \in \{ 1, 2, \ldots, n \}$ since dominance is W-invairant. Thus at least one of $r_a z_1, \ldots, r_a z_n$ must be negative. Without loss of generality, we may assume that $r_a z_1 \in \Phi^-$. Since $a \in \Pi$, it follows that $a = z_1$. Then $(z, a) \geq 1$ by Lemma 3.3.2 (v) since $x \operatorname{dom}_W a$.

Conversely, suppose that $x \in D_n$ and $a \in \Pi$ such that $(x, a) \geq 1$. Then by Lemma 3.3.2 (v) $x \operatorname{dom}_W a$; furthermore, Lemma 2.3.3 (v) yields that $r_a x \prec x$. Hence Lemma 3.3.15 yields that

$$r_a D(r_a x) \subseteq D(x). \tag{3.3.8}$$

Now suppose for a contradiction that $r_a x \notin D_{n-1}$. Then Lemma 3.3.14 yields that $r_a x \in D_n \uplus D_{n+1}$. From (3.3.8) it is clear that $r_a x \notin D_{n+1}$. But if $r_a x \in D_n$, then (3.3.8) yields that $r_a D(r_a x) = D(x)$. Observe that $a \in D(x)$ and $a \notin r_a D(r_a x)$, producing a contradiction as desired. This completes the proof of (i).

(ii) Replace x by $r_a x$ in (i) above then we may obtain the desired result.

(iii) Follows from (i), (ii) and Lemma 3.3.14.

□

Definition 3.3.17. For each $x \in \Phi^+$, define

$$S(x) = \{ w \in W \mid l(w) = dp(x) - 1 \text{ and } w^{-1}x \in \Pi \},$$

$$T(x) = \{ w \in W \mid l(w) = dp(x) \text{ and } w^{-1}x \in \Phi^- \}.$$

In other words, for $x \in \Phi^+$, $S(x)$ (respectively, $T(x)$) consists of all $w \in W$ of minimal length such that $w^{-1}x \in \Pi$ (repectively, $w^{-1}x \in \Phi^-$).

Proposition 3.3.18. *Suppose that $x \in \Phi^+$. Then $x \in D_n$ where $n = |\{ b \in N(w^{-1}) \mid x \operatorname{dom}_W b \}|$ for all $w \in S(x)$. In particular,*

$$D(x) = \{ b \in N(w^{-1}) \mid (x, b) \geq 1 \}$$

for all $w \in S(x)$.

Proof. Let $x \in \Phi^+$ and write $x = wa$ where $w \in S(x)$ and $a \in \Pi$. Let $w = r_{a_1} \cdots r_{a_l}$ with $l = l(w)$. For all $i \in \{2, \ldots, l\}$,

$$\begin{aligned} & w^{-1}(r_{a_1} r_{a_2} \cdots r_{a_{i-2}}) a_{i-1} \\ &= r_{a_l} \cdots r_{a_1} r_{a_1} \cdots r_{a_{i-2}} a_{i-1} \\ &= r_{a_l} \cdots r_{a_i} r_{a_{i-1}} a_{i-1} \\ &= -r_{a_l} \cdots r_{a_i} a_{i-1}. \end{aligned}$$

Since $l(r_l r_{l-1} \cdots r_i) = l(r_l \cdots r_i r_{i-1}) - 1$, it follows from Lemma 2.3.3 (iv) that $r_{a_l} \cdots r_{a_i} a_{i-1} \in \Phi^+$. Thus

$$w^{-1}(r_{a_1} r_{a_2} \cdots r_{a_{i-2}}) a_{i-1} = -r_{a_l} \cdots r_{a_i} a_{i-1} \in \Phi^-. \tag{3.3.9}$$

Now by Proposition 3.3.16, we can immediately deduce that $x \in D_n$ where

$$\begin{aligned} n &= |\{ i \colon (a_{i-1} \,,\, r_{a_i} r_{a_{i+1}} \cdots r_{a_l} a) \le -1 \}| \\ &= |\{ i \colon (r_{a_1} \cdots r_{a_{i-1}}(a_{i-1}) \,,\, r_{a_1} \cdots r_{a_l}(a)) \le -1 \}| \\ &= |\{ i \colon (r_{a_1} \cdots r_{a_{i-1}}(a_{i-1}) \,,\, x) \le -1 \}| \\ &= |\{ i \colon (-r_{a_1} \cdots r_{a_{i-2}}(a_{i-1}) \,,\, x) \ \le -1 \}| \\ &= |\{ b \in N(w^{-1}) \colon (-b, x) \le -1 \}| \\ &= |\{ b \in N(w^{-1}) \colon (-b, x) \ge 1 \}|. \end{aligned}$$

Lemma 3.3.2 (v) yields that either $x \operatorname{dom}_W b$ or $b \operatorname{dom}_W x$. Since all such b are in $N(w^{-1})$ where $w \in S(x)$, it follows that $w^{-1}x \in \Pi$ and $w^{-1}b \in \Phi^-$. Thus b cannot dominate x. So we may conclude that $x \in D_n$, where

$$n = |\{ b \in N(w^{-1}) \mid x \operatorname{dom}_W b \}| \tag{3.3.10}$$

for all $w \in S(x)$. But (3.3.10) says precisely that $D(x) \subseteq N(w^{-1})$ and

$$\begin{aligned} D(x) &= \{ b \in N(w^{-1}) \mid x \operatorname{dom}_W b \} \\ &= \{ b \in N(w^{-1}) \mid (x, b) \ge 1 \}. \end{aligned}$$

□

Immediately from Proposition 3.3.18, we have:

Corollary 3.3.19. *Let* $x \in \Phi^+$*. Then* $D(x) \subseteq \bigcap_{w \in S(x)} N(w^{-1})$*. And* $D(x) = \{ b \in N(w^{-1}) \mid (x, b) \geq 1 \}$ *is independent of the particular choice of* $w \in S(x)$*.*

It turns out that we can also say something about the roots in $\bigcap_{w \in S(x)} N(w^{-1}) \setminus D(x)$. Indeed in the next two lemmas we deduce that if $b \in \bigcap_{w \in S(x)} N(w^{-1})$, then $(x, b) > 0$.

Lemma 3.3.20. *Suppose that* $x \in \Phi^+$*,* $w \in T(x)$ *and* $b \in N(w^{-1})$*. Then* $(b, x) > 0$*.*

Proof. If $\mathrm{dp}(x) = 1$, then $x \in \Pi$. Hence $T(x) = \{ r_x \}$ and $x = b$, and so $(b, x) = 1$ as required.

Thus we may assume that $\mathrm{dp}(x) > 1$ and proceed by an induction on $\mathrm{dp}(x)$. Let $a \in \Pi \cap N(w^{-1})$. Then

$$l(r_a w) = l(w^{-1} r_a) = l(w^{-1}) - 1 = l(w) - 1.$$

Now since $(r_a w)^{-1}(r_a x) = w^{-1}x \in \Phi^-$, it follows that

$$\mathrm{dp}(r_a x) \leq l(r_a w) < l(w) = \mathrm{dp}(x).$$

Thus Lemma 2.3.3 (v) yields that $(a, x) > 0$. If $b = a$ then we are done. So we may assume that $b \neq a$ and let $w' = r_a w$. Observe that then $w' \in T(r_a x)$. Since $b \in N(w^{-1})$, it follows that $w'^{-1} r_a b = w^{-1} r_a r_a b = w^{-1} b \in \Phi^-$. Thus $r_a b \in N(w'^{-1})$. And so the inductive hypothesis yields that $(r_a b, r_a x) > 0$. Since $(\, , \,)$ is W-invariant, it follows that $(b, x) > 0$ as required. □

Lemma 3.3.21. *Suppose that* $x \in \Phi^+$*,* $w \in S(x)$ *and* $b \in N(w^{-1})$*. Then* $(b, x) > 0$*.*

Proof. Follows from Lemma 3.3.20 and the fact that for each $w \in S(x)$ there is a $w' \in T(x)$ such that $N(w^{-1}) \subset N(w'^{-1})$. □

Lemma 3.3.22. *For $n \in \mathbb{N}$, if $D_n = \emptyset$, then $D_m = \emptyset$ for all $m \in \mathbb{N}$ such that $m > n$.*

Proof. Suppose for a contradiction that there exists $n \in \mathbb{N}$ such that $D_n = \emptyset$ and yet $D_{n+1} \neq \emptyset$. Let $x \in D_{n+1}$. Then Lemma 3.3.14 yields that $r_a x \notin D_n$ for all $a \in \Pi$. Hence if $a \in \Pi$ such that $r_a x \prec x$ then $r_a x \in D_{n+1}$ still. Write $x = wb$, where $b \in \Pi$, and $w \in S(x)$. Let $w = r_{a_1} \cdots r_{a_l}$ ($a_i \in \Pi$, for $i = 1, \cdots, l$) be a reduced expression for w. Then for all $i \in \{1, \ldots, l\}$, we have that $r_{a_i} \cdots r_{a_2} r_{a_1} x \in D_{n+1}$. In particular, $b = r_{a_l} \cdots r_{a_1} x \in D_{n+1}$, contradicting the fact that $b \in \Pi \subset D_0$. □

Corollary 3.3.23. *Suppose that W is a finite rank infinite Coxeter group. Then for all nonnegative integers n, $D_n \neq \emptyset$.*

Proof. It is clear from the definition of the D_n's that $\Phi^+ = \biguplus_{n \geq 0} D_n$. Since W is an infinite Coxeter group, it follows that $|\Phi^+| = \infty$. On the other hand, since W is of finite rank, Theorem 3.3.9 yields that $|D_n| < \infty$. Thus Lemma 3.3.22 implies that $D_n \neq \emptyset$ for all nonnegative integers n. □

The following is a generalization of Proposition 3.3.16:

Proposition 3.3.24. *Let $x \in D_n$ with $n > 0$, and let $a \in \Phi^+$. Then*

(i) $|D(r_a x)| < n$ *if* $(x, a) \geq 1$;
(ii) $|D(r_a x)| > n$ *if* $(x, a) \leq -1$.

Proof. (i) If $\mathrm{dp}(a) = 1$ then this is just Propostion 3.3.16. Suppose now that $\mathrm{dp}(a) > 1$, and proceed by induction.

Write $a = r_b c$, where $b \in \Pi$, $c \in \Phi^+$ such that

$$\mathrm{dp}(a) = \mathrm{dp}(c) + 1 \tag{3.3.11}$$

Now since $(x, a) = (x, r_b c) = (r_b x, c) \geq 1$, it follows from the inductive hypothesis that

$$|D(r_c(r_b x))| < |D(r_b x)|. \tag{3.3.12}$$

Now we have three possibilities:

1) $(b, x) \geq 1$;
2) $(b, x) \leq -1$;
3) $(b, x) \in (-1, 1)$.

If 1) is the case, then Proposition 3.3.16 yields that $r_b x \in D_{n-1}$. And thus we have in this case

$$\begin{aligned} |D(r_a x)| &= |D(r_b(r_c r_b x))| \\ &\leq |D(r_c(r_b x))| + 1 && \text{(follows from Proposition 3.3.16)} \\ &\leq |D(r_b x)| && \text{(follows from (3.3.12))} \\ &= n - 1 \end{aligned}$$

as required.

If 2) is the case, then Proposition 3.3.16 yields that $r_b x \in D_{n+1}$, and $(b, r_c(r_b x)) = (b, r_b x - 2(r_b x, c\,)c\,) = \underbrace{(b, r_b x)}_{\geq 1} - 2(x, a)(b, c)$. By Lemma 2.3.3 (v), equation(3.3.11) above yields that $(b, c) < 0$. Hence

$$(b, r_c(r_b x)) > (b, r_b x) \geq 1. \tag{3.3.13}$$

Then Proposition 3.3.16 yields that

$$\begin{aligned} |D(r_a x)| &= |D(r_b(r_c r_b x)| \\ &= |D(r_c r_b x)| - 1 && \text{(by (3.3.13) above)} \\ &\leq |D(r_b x)| - 2 && \text{(by (3.3.12))} \\ &\leq n - 1 && \text{(since } r_b x \in D_{n+1} \text{ in case 2))} \end{aligned}$$

as required.

If 3) is the case, then we are done unless $|D(r_c(r_b x))| = n-1$, and $(b, r_c r_b x) \leq -1$. But this is impossible, since

$$\begin{aligned}
&(b, r_c r_b x)\\
&= (b, r_b x) - 2(r_b x, c)(b, c)\\
&= (b, r_b x) - 2(a, x)(b, c)\\
&> (b, r_b x) \qquad (\text{ since } (a, x) \geq 1, \text{ and } (b, c) < 0 \text{ })\\
&> -1.
\end{aligned}$$

Thus $|D(r_a x)| = |D(r_b r_c r_b x)| < n$ in this case too. This completes the proof of (i).

(ii) Replace x by $r_a x$, then apply (i) above.

□

Next we give an algorithm to systematically compute all the D_n's for an arbitrary finite-rank Coxeter group W.

Lemma 3.3.25. *Suppose that $x \in D_n$ with $n \geq 1$. Then there exists $y \in D_{n-1}$ such that $y \prec x$.*

Proof. Suppose that the contrary is true. Let $x \in D_n$ be such that there is no root in D_{n-1} preceding x. Write $x = wa$, where $a \in \Pi$, and $w \in S(x)$. Let $w = r_{a_1} \cdots r_{a_l}$ $(a_i \in \Pi, i = 1, \ldots, l)$ be a reduced expression for w. Then $a = r_{a_l} \cdots r_{a_1} x$. Observe that in such case, for all $i \in \{1, \ldots, l-1\}$, $r_{a_i} \cdots r_{a_1} x \prec r_{a_{i-1}} \cdots r_{a_1} x$. Thus Lemma 2.3.3 (v) yields that $(r_{a_{i-1}} \cdots r_{a_1} x, a_i) > 0$ for $i \in \{2, \cdots, l\}$. So by Proposition 3.3.16, our assumption that x is not preceded by any root in D_{n-1} is equivalent to the condition that $(r_{a_{i-1}} \cdots r_{a_1} x, a_i) \in (0, 1)$ for all $i \in \{2, \cdots, l\}$. Proposition 3.3.16 (iii) then yields that

$$r_{a_i} r_{a_{i-1}} \cdots r_{a_1} x \in D_n$$

for all $i \in \{1, \cdots, l\}$. In particular, $a = r_{a_l} \cdots r_{a_1} x \in D_n$ where $n \geq 1$, contradicting that $a \in \Pi \subset D_0$. □

Proposition 3.3.26. *Suppose that W is a finite-rank Coxeter group. For $n \geq 1$, there is an algorithm to compute D_n provided that D_{n-1} is known.*

Proof. We outline such an algorithm:

1) Set $D = \emptyset$.
2) Enumerate all the elements of D_{n-1} in some order, that is, write $D_{n-1} = \{x_1, \cdots, x_m\}$, where $m = |D_{n-1}|$.
3) Starting with x_1, apply all simple reflections r_a $(a \in \Pi)$, to x_1, one at a time. If $(a, x_1) \leq -1$, then add $r_a x_1$ to D.
4) Repeat 3) to $x_2, \cdots, x_m$.
5) Enumerate all the elements of the modified set D in some order, that is, write $D = \{x'_1, x'_2, \cdots, x'_{|D|}\}$.
6) Starting with x'_1, apply all simple reflections r_a $(a \in \Pi)$ to x'_1, one at a time. If $(a, x'_1) \in (-1, 0)$ and $r_a x'_1 \notin D$, then add $r_a x'_1$ to D.
7) Repeat 6) to $x'_2, \cdots, x'_{|D|}$.
8) Repeat steps 5) to 7) above.
9) Repeat 8) until no new elements can be added to D.
10) Set $D_n = D$.

Next we show that the above algorithm will be able to produce all elements of D_n within a finite number of iterations.

Let $x \in D_n$ $(n \geq 1)$ be arbitrary. Lemma 3.3.25 yields that there exists a $y \in D_{n-1}$ with $y \prec x$. Write $x = wy$ for some $w \in W$ with $l(w) = \mathrm{dp}(x) - \mathrm{dp}(y)$. Let $w = r_{a_1} r_{a_2} \cdots r_{a_l}$ $(a_1, \ldots, a_l \in \Pi)$ be a reduced expression for w. Then

$$y \prec r_{a_l} y \prec r_{a_{l-1}} r_{a_l} y \prec \cdots \prec r_{a_1} r_{a_2} \cdots r_{a_l} y = x.$$

Since $x \in D_n$ and $y \in D_{n-1}$, it follows from Lemma 3.3.15 that

$$r_{a_l} y, r_{a_{l-1}} r_{a_l} y, \ldots, r_{a_2} r_{a_3} \cdots r_{a_l} y \in D_{n-1} \uplus D_n$$

furthermore, there exists $i \in \{1, 2, \ldots, l\}$ such that

$$\begin{aligned} y &\in D_{n-1} \\ r_{a_l} y &\in D_{n-1} \\ &\vdots \\ r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y &\in D_{n-1} \end{aligned}$$

and

$$\begin{aligned} r_{a_i}(r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y) &\in D_n \\ r_{a_{i-1}} r_{a_i}(r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y) &\in D_n \\ &\vdots \\ r_{a_1} r_{a_2} \cdots r_{a_l} y = x &\in D_n. \end{aligned}$$

Since $r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y \in D_{n-1}$, it follows that $r_{a_i} r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y$ is an element of D_n obtainable by going through steps 3) and 4) above. This in turn implies that $r_{a_{i-1}} r_{a_i} \cdots r_{a_l} y$ is an element obtainable by going through steps 5) to 7). It then follows that $r_{a_{i-2}} r_{a_{i-1}} r_{a_i} \cdots r_{a_l} y$ and so on are all obtainable by (repeated) application of step 8). In particular $x = r_{a_1} \cdots r_{a_l} y$ can be obtained after $(i-2)$ iterations of step 8). Thus x can be obtained by going through setps 1) to 8), with step 8) repeated finitely many times. Since $x \in D_n$ was arbitrary, it follows that every element of D_n can be obtained from the above algorithm in this manner with step 8) repeated finitely many times.

Finally W is of finite rank, so $|D_n| < \infty$. Therefore step 9) will only be repeated a finite number of times and hence the algorithm will terminate completing the proof. □

Corollary 3.3.27. *If $|S| < \infty$, then we may compute D_n, for all $n \in \mathbb{N}$.*

Proof. [**5**] gives a complete description of D_0 when $|S| < \infty$. Now combine [**5**] and Proposition 3.3.26, the result follows immediately. □

Now a question worth asking is whether, given an arbitrary (finite rank) Coxeter group W, the successive D_n's expand or contract in size? At this stage, unfortunately, a full answer has not yet been found. However we do know that for a family of special subsets $D_{n,1}$'s of the D_n's to be defined below, the successive $D_{n,1}$'s tend to increase in size.

For $I \subseteq \Pi$, the Coxeter group generated by $\{ r_a \mid a \in I \}$ is denoted by W_I, and the root system of W_I on the subspace of V spanned by I is denoted by Φ_I. As usual, write Φ_I^+ for $\Phi_I \cap \mathrm{PLC}(I)$ and write Φ_I^- for $-\Phi_I^+$. The Coxeter graph $\Gamma(I)$ of W_I has vertex set I, and two vertices a_r and a_s are adjoined by an edge of weight m_{rs}, where $r, s \in S$ and $m_{rs} \neq 2$.

Each root x can be written uniquely as $\sum_{a\in\Pi} \lambda_a a$, and we say λ_a is *the coefficient* (witten $\mathrm{coeff}_a(x)$) of a in x. The *support* (written $\mathrm{supp}(x)$) of x is the set of all $a \in \Pi$ with $\mathrm{coeff}_a(x) \neq 0$, and $\Gamma(x)$ defined by $\Gamma(\mathrm{supp}(x))$ is the corresponding graph. It is readily seen that $\Gamma(\mathrm{supp}(x))$ is finite and connected.

Definition 3.3.28. For $n \in \mathbb{N}$, set

$$D_{n,1} = D_n \cap \{ \beta \in \Phi^+ \mid \mathrm{coeff}_x(\beta) = 1 \text{ , for some } x \in \Pi \}.$$

In Proposition 3.3.30 below we shall prove that, amongest other things, $|D_{n,1}| \geq |D_{m,1}|$, if $n > m$.

Lemma 3.3.29. ((4.4) Lemma of [**5**]) *Let β be a positive root and $x \in \Pi$ with $\mathrm{coeff}_x(\beta) = 1$. Then $x \preceq \beta$; that is, there exists a $w \in W_{\mathrm{supp}(\beta)\setminus\{x\}}$ such that $\beta = wx$ and $dp(\beta) = l(w) + 1$.* □

Proposition 3.3.30. *Let $I \subseteq \Pi$, $x \in I$ and I_1, $I_2 \subseteq I$ be such that $\Gamma(I_1 \setminus \{x\})$ and $\Gamma(I_2 \setminus \{x\})$ are unions of connected components of $\Gamma(I \setminus \{x\})$ with $I = I_1 \cup I_2$, and $I_1 \cap I_2 = \{x\}$. Then*

(i)

$$\begin{aligned}\phi\colon \{\, (\beta_1,\beta_2) \in \Phi^+_{I_1} \times \Phi^+_{I_2} \mid \operatorname{coeff}_x(\beta_1) = \operatorname{coeff}_x(\beta_2) = 1 \,\}\\ \to \{\, \alpha \in \Phi^+_I \mid \operatorname{coeff}_x(\alpha) = 1 \,\}\\ (\beta_1,\beta_2) \mapsto \beta_1 + \beta_2 - x\end{aligned}$$

defines a bijection. Moreover, $\operatorname{dp}(\phi(\beta_1,\beta_2)) = \operatorname{dp}(\beta_1) + \operatorname{dp}(\beta_2) - 1$ *and* β_1, $\beta_2 \preceq \phi(\beta_1,\beta_2)$ *for* $\beta_i \in \Phi^+_{I_1}$, *with* $\operatorname{coeff}_x(\beta_i) = 1$ $(i = 1, 2)$.
(ii) ϕ *restricts to a bijection*

$$\begin{aligned}\{\, (\beta_1,\beta_2) \in (\Phi^+_{I_1} \cap D_i) \times (\Phi^+_{I_2} \cap D_j) \mid \operatorname{coeff}_x(\beta_1) = \operatorname{coeff}_x(\beta_2) = 1 \,\}\\ \leftrightarrow \{\, \beta \in \Phi^+_I \cap D_{i+j} \mid \operatorname{coeff}_x(\beta) = 1 \,\}.\end{aligned}$$

Proof. (i) Lemma 4.2 of [**5**].

(ii) Let $\beta \in \Phi^+_I$ with $\operatorname{coeff}(\beta) = 1$. Then Lemma 3.3.29 above yields that $\beta = wx$, for some $w \in W_{I\setminus\{x\}}$. We may write $w = w_1w_2$ where $w_1 \in W_{I_1\setminus\{x\}}$ and $w_2 \in W_{I_2\setminus\{x\}}$ with $l(w) = l(w_1) + l(w_2)$. Observe that under this construction, $W_{I_1\setminus\{x\}}$ commutes with $W_{I_2\setminus\{x\}}$ and $W_{I_1\setminus\{x\}}$ fixes $\Phi^+_{I_2\setminus\{x\}}$ pointwise and vice versa. Set $\beta_1 = w_1x$ and $\beta_2 = w_2x$.

We first show that $\beta_1 \in D_i$, $\beta_2 \in D_j$ implies that $\beta \in D_{i+j}$.

Suppose for a contradiction that $\beta \in D_n$ for some $n \in \mathbb{N}$ and $n \neq i + j$. Let $\{\, z_1, \dots, z_i \,\} = D(\beta_1)$ and $\{\, z_{i+1}, \dots, z_{i+j} \,\} = D(\beta_2)$. Since

$$\beta_1 = w_1x \operatorname{dom}_W z_1, \dots, z_i,$$

it follows from the W-invariance of dominance that

$$x \operatorname{dom}_W w_1^{-1}z_1, \dots w_1^{-1}z_i.$$

Since $x \in \Pi$ it follows that $w_1^{-1}z_1, \dots, w_1^{-1}z_i \in \Phi^-$. Consequently $z_1, \dots, z_i \in N(w_1^{-1}) \subseteq \Phi^+_{I_1\setminus\{x\}}$. Similar reasoning also yields that $z_{i+1}, \dots, z_{i+j} \in N(w_2^{-1}) \subseteq \Phi^+_{I_2\setminus\{x\}}$. Observe that in particular, we have

$D(\beta_1) \cap D(\beta_2) = \emptyset$. By Lemma 3.3.15,

$$\begin{aligned} w_2 D(\beta_1) &= w_2 D(w_1 x) \\ &\subseteq D(w_2 w_1 x) \qquad \text{(since } w_1 x \prec w_2 w_1 x\text{)} \\ &= D(wx) \\ &= D(\beta). \end{aligned}$$

Similarly $w_1 D(\beta_2) \subseteq D(\beta)$. Since w_2 fixes $z_1, \ldots, z_i$ pointwise and w_1 fixes $z_{i+1}, \ldots, z_{j+j}$ pointwise, it follows that

$$\{ z_1, \ldots, z_i, z_{i+1}, \cdots, z_{i+j} \} \subseteq D(\beta). \tag{3.3.14}$$

Observe that (3.3.14) implies that, in particular, $\beta \in D_n$ for some $n > i + j$. Thus there exists $z \in \Phi^+$ such that $z \in D(\beta)$ and

$$z \neq z_1, \ldots, z_i, z_{i+1}, \ldots, z_{i+j}. \tag{3.3.15}$$

$\beta = wx \operatorname{dom}_W z$ implies that $x \operatorname{dom}_W w^{-1}z$. Because $x \in \Pi$, this forces that $w^{-1}z \in \Phi^-$. Thus $z \in N(w^{-1}) \subseteq \Phi^+_{I \setminus \{x\}}$. Now the connectedness of $\Gamma(z)$ implies that either $z \in \Phi^+_{I_1 \setminus \{x\}}$ or $z \in \Phi^+_{I_2 \setminus \{x\}}$. Suppose that $z \in \Phi^+_{I_1 \setminus \{x\}}$. Then $wx \operatorname{dom}_W z$ implies that

$$w_1 x = w_2^{-1}(wx) \operatorname{dom}_W w_2^{-1} z = z.$$

Then $z \in \{ z_1, \ldots, z_i \}$, contradicting (3.3.15). Next suppose that $z \in \Phi^+_{I_2 \setminus \{x\}}$. Then $\beta = wx \operatorname{dom}_W z$ implies that

$$w_2 x = w_1^{-1}(wx) \operatorname{dom}_W w_1^{-1} z = z.$$

Then $z \in \{ z_{i+1}, \ldots, z_{i+j} \}$, again contradicting (3.3.15). Therefore if $\beta_1 \in D_i$ and $\beta_2 \in D_j$ then $\beta \in D_{i+j}$, as required.

The converse can be shown by a similar argument. □

3.4. The Imaginary Cone and Standard Dominance

The section is devoted to a preliminary study of the so called *imaginary cone* (introduced by Dyer in [**3**] and [**4**]) of a Coxeter group and a stronger form of dominance. In particular, we will show that if x and

y are roots with $x \operatorname{dom}_W y$, then $x - y$ is in this imaginary cone. We stress that throughout this section S is assumed to be finite.

Definition 3.4.1. Let V^* be the dual space of V (recall that W acts on V^* via the folowing: for all $w \in W$, $x \in V$, and $f \in V^*$, $(wf)(x) := f(w^{-1}x)$). For any convex cone C in V, define the dual of C to be

$$C^* = \{\, \phi \in V^* \mid \phi(v) \geq 0, \text{ for all } v \in C \,\},$$

and similarly for a convex cone F in V^*, we define its dual to be

$$F^* = \{\, v \in V \mid f(v) \geq 0, \text{ for all } f \in F \,\}.$$

Let $P = \mathrm{PLC}(\Pi) \cup \{0\}$, and we set $U = \bigcup_{w \in W} wP^*$.

Remark 3.4.2. Given the finite dimensinality of V (and hence V^*), it is well known that if C (respectively, F) is a convex cone in V (respectively, V^*), then then $(C^*)^*$ (respectively, $(F^*)^*$) is the topological closure of C (respectively, F) in V with respect to the standard topology on $V \cong \mathbb{R}^{|S|}$ (respectively, V^* with respect to the standard topology). Furthermore C^* (repectively, F^*) is always a convex cone in V (respectively, V^*), even if C (respectively, F) is neither convex nor a cone in V (respectively, V^*).

Lemma 3.4.3. *Let $w \in W$ be arbitrary and suppose that $f \in wP^*$. Then $f(a) \geq 0$ for all but finitely many positive roots a.*

Proof.

$$\begin{aligned} wP^* &= \{\, w\phi \in V^* \mid \phi(a) \geq 0 \text{ for all } a \in \Phi^+ \,\} \\ &= \{\, f \in V^* \mid (w^{-1}f)(a) \geq 0 \text{ for all } a \in \Phi^+ \,\} \\ &= \{\, f \in V^* \mid f(wa) \geq 0 \text{ for all } a \in \Phi^+ \,\} \\ &= \{\, f \in V^* \mid f(b) \geq 0 \text{ for all } b \in w\Phi^+ \,\} \\ &= \{\, f \in V^* \mid \text{ if } c \in \Phi^+ \text{ then } f(c) < 0 \text{ only if } c \in N(w^{-1}) \,\} \end{aligned}$$

Since $N(w^{-1})$ is a finite set, the desired result follows. □

Lemma 3.4.4. *U consists of all $\phi \in V^*$ such that $\phi(a) \geq 0$ for all but finitely many positive roots a.*

Proof. Let

$$X := \{ \psi \in V^* \mid \psi(a) \geq 0 \text{ for all but finitely many } b \in \Phi^+ \}.$$

The previous lemma yields that $U \subseteq X$. Now let $\psi \in X$ be arbitrary. We shall show that $\psi \in U$ by an induction on the size of the set $\mathrm{Neg}(\psi) := \{ b \in \Phi^+ \mid \psi(b) < 0 \}$. If $|\,\mathrm{Neg}(\psi)\,| = 0$, then $\psi \in P^* \subset U$, so we may assume that $\mathrm{Neg}(\psi) \neq \emptyset$. Then there must exist $b \in \Pi$ such that $\psi(b) < 0$. It is readily observed that the size of $\mathrm{Neg}(r_b\psi)$ is one less than the size of $\mathrm{Neg}(\psi)$. Indeed $\mathrm{Neg}(r_b\psi) = r_b(\mathrm{Neg}(\psi) \setminus \{b\})$. Hence the inductive hypothesis yields that $r_b\psi \in U$. Since U is W-invariant, if follows that $\psi \in U$ since U. Since $\psi \in X$ was arbitrary, it follows that $X \subseteq U$. □

Remark 3.4.5. By the lemma above, we see that U is a convex cone, and we call it the *Tits Cone*.

Lemma 3.4.6.

$$U^* = \bigcap_{w \in W} wP.$$

Proof.

$$\begin{aligned}
U^* &= \{ v \in V \mid f(v) \geq 0, \text{ for all } f \in U \} \\
&= \{ v \in V \mid (w\phi)(v) \geq 0 \text{ , for all } \phi \in P^*, \text{ and for all } w \in W \} \\
&= \{ v \in V \mid \phi(w^{-1}v) \geq 0 \text{ , for all } \phi \in P^*, \text{ and for all } w \in W \} \\
&= \bigcap_{w \in W} \{ v \in V \mid \phi(w^{-1}v) \geq 0 \text{ , for all } \phi \in P^* \} \\
&= \bigcap_{w \in W} \{ wv \in V \mid \phi(v) \geq 0, \text{ , for all } \phi \in P^* \} \\
&= \bigcap_{w \in W} \{ wv \in V \mid v \in (P^*)^* \} \\
&= \bigcap_{w \in W} wP \qquad ((P^*)^* = P, \text{ since } P \text{ is topologically closed}).
\end{aligned}$$

□

Adpoting the concept introduced in [**3**] and [**4**], we make the following definition.

Definition 3.4.7. We define the *imaginary cone* Q of W to be

$$Q = \{\, v \in U^* \mid (v, a) \le 0 \text{ for all but finitely many } a \in \Phi^+ \,\}.$$

Observe that Q is indeed a convex cone.

Lemma 3.4.8. *Suppose that $v \in V$ has the property that $(a, v) \le 0$ for all $a \in \Pi$. Then $wv - v \in P$ for all $w \in W$. Moreover, if $v \in P$, then $v \in U^*$.*

Proof. Use induction on $l(w)$. Noting that if $l(w) = 0$ then there is nothing to prove. If $l(w) \ge 1$, then we may write $w = w' r_a$ with $w' \in W$, $a \in \Pi$ and $l(w) = l(w') + 1$. Then

$$\begin{aligned} wv - v = w' r_a(v) - v &= w'(v - 2(v, a)a) - v \\ &= (w'v - v) + 2|(a, v)| w'a. \end{aligned}$$

Observe that $w'a \in P$, and $w'v - v \in P$ by the inductive hypothesis. Since P is a cone it follows that $wv - v \in P$ as required.

If $v \in P$, then $wv = (wv - v) + v \in P$ for all $w \in W$, and so $v \in \bigcap_{w \in W} w^{-1} P = U^*$.

□

Now we are ready to give an alternative characterization of Q:

Proposition 3.4.9.

$$Q = \{\, wv \mid w \in W, v \in P \textit{ such that } (v, a) \le 0 \textit{ for all } a \in \Phi^+ \,\}.$$

Proof. First set

$$X = \{\, wv \mid w \in W, v \in \mathrm{PLC}(\Pi) \text{ such that } (v, a) \le 0 \text{ for all } a \in \Phi^+ \,\}.$$

Suppose that $u \in Q$. Lemma 3.4.6 yields that u can indeed be expressed as wv for some $w \in W$ and $v \in P$. In particular, $u \in P$. For each $b \in P$ set $\operatorname{Pos}(b) := \{ c \in \Phi^+ \mid (b,c) > 0 \}$. If $\operatorname{Pos}(u) = \emptyset$, then trivially $u \in Q$ (taking $w = 1$). Thus we may assume that $\operatorname{Pos}(u) \neq \emptyset$ and proceed by an induction on $|\operatorname{Pos}(v)|$. Choose $a \in \Pi$ such that $(u, a) > 0$. Now it is readily checked that $\operatorname{Pos}(r_a u) = r_a(\operatorname{Pos}(u) \setminus \{a\})$. Thus the inductive hypothesis yields that $r_a u \in X$. Thus $u \in X$ since X is clearly W-invariant. Since $u \in Q$ was arbitrary, it follows that $Q \subseteq X$.

Conversely, if $x \in X$, then $x = wv$ for some $w \in W$ and $v \in P$ such that $(v, a) \leq 0$ for all $a \in \Pi$. By the previous lemma, $x \in U^*$. Now let $y \in \Phi^+$. Then $(x, y) = (wv, y) = (v, w^{-1}y)$ and $(x, y) > 0$ only for those roots $b \in N(w^{-1})$. Clearly there are finitely (indeed, at most $l(w)$) many such roots. Thus $x \in Q$. Since $x \in X$ was arbitrary, it follows that $X \subseteq Q$. □

Proposition 3.4.10. *Suppose that $x, y \in \Phi$ are distinct with $x \operatorname{dom}_W y$. Then for all $w \in W$, $w(x - y) \in \operatorname{PLC}(\Pi)$, that is, $x - y \in U^*$.*

Proof. Let W' be the (infinite) dihedral subgroup of W generated by r_x and r_y. Let $S(W') = \{s, t\}$ and $\triangle(W') = \{\alpha_s, \alpha_t\}$. Proposition 3.3.4 (i) and Proposition 4.5.4 (i) of [**13**] combined yield that $\alpha_s,\ \alpha_t \in \Phi^+$ with $(\alpha_s, \alpha_t) = -(x, y) \leq -1$. Since x and $y \in \Phi(W') = W'\Delta(W')$, Lemma 3.2.1 yields that there are integers m and n such that

$$x = \frac{\sinh(n \pm 1)\theta}{\sinh(\theta)}\alpha_s + \frac{\sinh(n\theta)}{\sinh(\theta)}\alpha_t$$

and

$$y = \frac{\sinh(m \pm 1)\theta}{\sinh(\theta)}\alpha_s + \frac{\sinh(m\theta)}{\sinh(\theta)}\alpha_t$$

where $\theta = \ln(-(\alpha_s, \alpha_t) + \sqrt{(\alpha_s, \alpha_t)^2 - 1}\ ) = \cosh^{-1}(-(\alpha_s, \alpha_t))$.

Keeping all notation as in Section 3.2, we write c_i for $\frac{\sinh(i\theta)}{\sinh(\theta)}$, for all $i \in \mathbb{N}$. Now let us consider the possible values of (x, y):

(a) If $x = c_{n+1}\alpha_s + c_n\alpha_t$ and $y = c_{m+1}\alpha_s + c_m\alpha_t$, then

$$(x, y) = \cosh((n-m)\theta) \geq 1.$$

(b) If $x = c_{n+1}\alpha_s + c_n\alpha_t$ and $y = c_{m-1}\alpha_s + c_m\alpha_t$, then

$$(x, y) = -\cosh((n+m)\theta) \leq -1.$$

(c) If $x = c_{n-1}\alpha_s + c_n\alpha_t$ and $y = c_{m+1}\alpha_s + c_m\alpha_t$, then

$$(x, y) = -\cosh((n+m)\theta) \leq -1.$$

(d) If $x = c_{n-1}\alpha_s + c_n\alpha_t$ and $y = c_{m-1}\alpha_s + c_m\alpha_t$, then

$$(x, y) = \cosh((n-m)\theta) \geq 1.$$

Since $x \operatorname{dom}_W y$, Lemma 3.3.2 (v) yields that $(x, y) \geq 1$. Therefore we can rule out cases (b) and (c) above and conclude that either

$$x = \frac{\sinh(n+1)\theta}{\sinh(\theta)}\alpha_s + \frac{\sinh(n\theta)}{\sinh(\theta)}\alpha_t,\ y = \frac{\sinh(m+1)\theta}{\sinh(\theta)}\alpha_s + \frac{\sinh(m\theta)}{\sinh(\theta)}\alpha_t;$$

or

$$x = \frac{\sinh(n-1)\theta}{\sinh(\theta)}\alpha_s + \frac{\sinh(n\theta)}{\sinh(\theta)}\alpha_t,\ y = \frac{\sinh(m-1)\theta}{\sinh(\theta)}\alpha_t + \frac{\sinh(m\theta)}{\sinh(\theta)}\alpha_t.$$

Next we shall show that $n > m$. Suppose for a contradiction that $m \geq n$. Then either $x = y$ (when $n = m$), or there will be a $w \in W'$ such that $wx \in \Phi(W') \cap \Phi^-$ and yet $wy \in \Phi(W') \cap \Phi^+$ (when $n < m$), all contradicting the fact that $x \operatorname{dom}_W y$. Since $c_n > c_m$ if $n > m$, it follows that $x - y \in \mathrm{PLC}(\Pi)$. Finally because dominance is W-invariant, so for any $w \in W$, repeat the above argument with x replaced by wx and y replaced by wy, we may conclude that $w(x-y) \in \mathrm{PLC}(\Pi)$. □

Theorem 3.4.11. *Suppose that x, $y \in \Phi$ are distinct with $x \operatorname{dom}_W y$. Then there exists $w \in W$ such that $(w(x-y), z) \leq 0$, for all $z \in \Phi^+$, that is, $x - y \in Q$.*

Proof. By the previous proposition, we know that $x - y \in U^*$. Thus if we can prove that there exists $w \in W$ with $(w(x-y), z) \leq 0$ for all $z \in \Phi^+$, then $x - y \in Q$ by Proposition 3.4.9 (since U^* is $W-$invariant).

Clearly it is enough to prove that $(w(x-y), z) \leq 0$ for all $z \in \Pi$ and we give an algorithm to find such a w below.

Let W' be the (infinite) dihedral subgroup of W generated by r_x and r_y, and let $\triangle(W') = \{a_0, b_0\}$. By Propostion 3.3.4 (i), a_0, $b_0 \in \Phi^+$, and $(a_0, b_0) = -(x, y) \leq -1$, and there is some $u \in \langle r_x, r_y \rangle$ such that either

(3.4.1)

$$u(x) = a_0 \quad \text{and} \quad u(y) = -b_0 \quad \text{or} \quad u(x) = b_0 \quad \text{and} \quad u(y) = -a_0.$$

At any rate, $u(x-y) = a_0 + b_0$. Obviously $(a_0 + b_0, a_0) \leq 0$ and $(a_0+b_0, b_0) \leq 0$. However there may be $c_1 \in \Pi$ such that $(a_0+b_0, c_1) > 0$. If this is the case, set

$$a_1 = r_{c_1} a_0$$

and

$$b_1 = r_{c_1} b_0.$$

Observe that $(d, c_1) \leq 0$ for all $d \in \Pi \setminus \{c_1\}$, so

$$c_1 \in \text{supp}(a_0) \cup \text{supp}(b_0) \tag{3.4.2}$$

Note that $a_0 \neq c_1$ and $b_0 \neq c_1$, since $(a_0 + b_0, c_1) > 0$ whereas $(a_0 + b_0, a_0) \leq 0$ and $(a_0 + b_0, b_0) \leq 0$. Thus $a_1, b_1 \in \Phi^+$, and $(a_1, b_1) = (a_0, b_0) \leq -1$. Consequently Proposition 3.1.3 (ii) yields that a_1, b_1 are the canonical roots for the root subsystem $\Phi(\langle r_{a_1}, r_{b_1} \rangle)$. Observe that applying r_{c_1} to a_0 and b_0 will only change the coefficient of c_1, therefore (3.4.2) yields that

$$\text{supp}(a_1) \cup \text{supp}(b_1) \subseteq \text{supp}(a_0) \cup \text{supp}(b_0).$$

Furthermore,

$$\sum_{a \in \Pi} \text{coeff}_a(a_1) + \sum_{a \in \Pi} \text{coeff}_a(b_1) < \sum_{a \in \Pi} \text{coeff}_a(a_0) + \sum_{a \in \Pi} \text{coeff}_a(b_0)$$

since the coefficient for c_1 has been stricly decreased as $(a_0+b_0, c_1) > 0$. Moreover, since $(a_0 + b_0, c_1) > 0$, it follows that at least one of (a_0, c_1)

or (b_0, c_1) must be strictly positive. Hence Lemma 2.3.3 (v) yields that

$$dp(a_1) + dp(b_1) \le dp(a_0) + dp(b_0).$$

Repeat this process and we can obtain new pairs of positive roots $\{ a_2, b_2 \}, \dots, \{ a_{m-1}, b_{m-1} \}, \{ a_m, b_m \}$ with

$$\begin{aligned}&\mathrm{supp}(a_m) \cup \mathrm{supp}(b_m) \subseteq \mathrm{supp}(a_{m-1}) \cup \mathrm{supp}(b_{m-1}) \subseteq \cdots \\ &\subseteq \mathrm{supp}(a_0) \cup \mathrm{supp}(b_0)\end{aligned}$$

so long as we can find a $c_m \in \Pi$ such that $(a_{m-1} + b_{m-1}, c_m) > 0$. And this process only terminates at $\{ a_n, b_n \}$ for some n, if for all $z \in \Pi$, $(a_n + b_n, z) \le 0$. Thus if we could show that this process terminates at $\{a_n, b_n\}$ for some finite n, then we have in fact found a $w = r_{c_n} r_{c_{n-1}} \cdots r_{c_1} u \in W$ such that

$$(w(x-y), z) = (r_{c_n} \cdots r_{c_1}(a_0 + b_0), z) \le 0$$

for all $z \in \Pi$, which in turn will establish that $x - y \in Q$.

Since the set of positive roots having depth less than a specific bound and support in a fixed finite subset of Π is finite, it then follows that the possible pairs of positive roots $\{a_i, b_i\}$ obtained in this process must be finite too. Finally since

$$\sum_{a\in\Pi} \mathrm{coeff}_a(a_j) + \sum_{a\in\Pi} \mathrm{coeff}_a(b_j) < \sum_{a\in\Pi} \mathrm{coeff}_a(a_i) + \sum_{a\in\Pi} \mathrm{coeff}_a(b_i)$$

for all $j > i$, therefore the sequence $\{a_0, b_0\}, \{a_1, b_1\}, \cdots$ must terminate at $\{a_n, b_n\}$.

Incidentally, observe that for all $i \in \{1, \cdots, n\}$, where n is as above, $r_{c_i} a_{i-1} \in \Phi^+$ and $r_{c_i} b_{i-1} \in \Phi^+$. So we can easily deduce that $r_{c_i} \cdots r_{c_1}(a_0) \in \Phi^+$, and $r_{c_i} \cdots r_{c_1}(b_0) \in \Phi^+$, for all $i \in \{1, \cdots, n\}$. Hence, let $w \in W$ be as above, (3.4.2) yields that either

$$wx = r_{c_n} \cdots r_{c_1} u a_0 \in \Phi^+ \qquad \text{and} \qquad wy = r_{c_n} \cdots r_{c_1} u(-b_0) \in \Phi^-$$

or

$$wx = r_{c_n} \cdots r_{c_1} u b_0 \in \Phi^+ \qquad \text{and} \qquad wy = r_{c_n} \cdots r_{c_1} u(-a_0) \Phi^-.$$

□

The observation made at the end of the proof of Theorem 3.4.11 yields that:

Corollary 3.4.12. *Let $x, y \in \Phi$ be distinct with $x \operatorname{dom}_W y$. Then there exists $w \in W$ such that $wx \in \Phi^+$, $wy \in \Phi^-$, and $(w(x-y), z) \leq 0$ for all $z \in \Phi^+$.* □

Corollary 3.4.13. *Suppose that x, $y \in \Phi$ are distinct with $x \operatorname{dom}_W y$. Then the following are equivalent:*

(i) *whenever $x \operatorname{dom}_W z \operatorname{dom}_W y$ for some $z \in \Phi$, then either $z = x$ or $z = y$;*
(ii) *there exists a $w \in W$ such that $w(x) \in D_0$ and $w(y) \in -D_0$.*

Proof. (i) implies (ii): Let w be as in the previous corollary. First we show that then $wx \in D_0$. Suppose for a contradiction that $wx \notin D_0$. Let $z \in D(wx)$. Then the last corollary yields $(wy, z) \geq 1$ too, for $(wy, z) \geq (wx, z) \geq 1$. Since the last Corollary yields that $wy \in \Phi^-$, it follows that $z \operatorname{dom}_W wy$. But this gives us $x \operatorname{dom}_W w^{-1}z \operatorname{dom}_W y$ with $x \neq w^{-1}z \neq y$, contradicting (i). Therefore $wx \in D_0$, as required. Similarly we may also deduce that $wy \in -D_0$.

(ii) implies (i): Clear.

□

Definition 3.4.14. Suppose that $x, y \in \Phi$, $x \operatorname{dom}_W y$ satisfy both (i) and (ii) of Corollary 3.4.13, then we say that the dominance between x and y is *minimal*.

Proposition 3.4.15. *Suppose that $x, y \in \Phi$ are distinct, and $x \operatorname{dom}_W y$. Then the dominance between x and y with respect to the subgroup $\langle r_x, r_y \rangle$ is minimal.*

Proof. Let a and b be the canonical roots for $\Phi(\langle r_x, r_y \rangle)$. Again as in the proof of Proposition 3.3.4 (i) we see that every root in $\Phi(\langle r_x, r_y \rangle)$

must be in exactly one of the following two dominace chains:

$$\cdots \operatorname{dom}_W\ r_a r_b r_a(b)\ \operatorname{dom}_W\ r_a r_b(a)\ \operatorname{dom}_W r_a(b)\ \operatorname{dom}_W\ a \\ \operatorname{dom}_W\ (-b)\ \operatorname{dom}_W\ r_b(-a)\ \operatorname{dom}_W\ r_b r_a(-b)\ \operatorname{dom}_W\ \cdots \tag{3.4.3}$$

and

$$\cdots\ \operatorname{dom}_W\ r_b r_a r_b(a)\ \operatorname{dom}_W\ r_b r_a(b)\ \operatorname{dom}_W\ r_b(a)\ \operatorname{dom}_W\ b \\ \operatorname{dom}_W(-a)\ \operatorname{dom}_W\ r_a(-b)\ \operatorname{dom}_W\ r_a r_b(-a)\ \operatorname{dom}_W\ \cdots . \tag{3.4.4}$$

Upon inspecting (3.4.3) and (3.4.4), we can readily see that the only elementary roots in this root subsystem with respect to $\operatorname{dom}_{\langle r_x, r_y\rangle}$ are a and b. By Proposition 3.3.4 (i), we know that there is some $w \in \langle r_x, r_y\rangle$ such that either

$$wx = a \text{ and } wy = -b$$

or

$$wx = b \text{ and } wy = -a,$$

therefore Corollary 3.4.13 yields that the dominance between x and y with respect to $\langle r_x, r_y\rangle$ is minimal. □

3.5. Dominance in Φ_1

In this section we generalize the results obtained in Section 3.3 into the non-orthogonal geometric realization studied in Chapter 1 and Chapter 2. In particular, we prove an analogue of Theorem 3.3.9 and Corollary 3.3.10 adapted to the non-orthogonal setting, namely, for a suitable definition of *roots*, the set of roots dominating precisely n other positive roots is finite in size, for any positive integer n. The conclusion drawn towards the end of this section is that the dominance concept, especially those results analogous to Theorem 3.3.9 and Corollary 3.3.10 are in fact dependent only on the underlying Coxeter group and not on the particular geometric realization used.

Keeping all notation as in Chapter 1, in this section we adapt the notion of dominance to the roots in Φ_1. We stress that all the results obtained throughout this section equally apply to the roots in Φ_2. We begin by define what is meant by saying that a root in Φ_1 dominates another root in Φ_1.

Definition 3.5.1. For $\alpha_1, \alpha_2 \in \Phi_1$, we say that α_1 *dominates* α_2 with respect to W (written $\alpha_1 \operatorname{dom}_W \alpha_2$) if

$$\{\, w \in W \mid w\alpha_1 \in \Phi_1^- \,\} \subseteq \{\, w \in W \;\; w\alpha_2 \in \Phi_1^- \,\}.$$

Lemma 3.5.2. *Suppose that $\alpha_1, \alpha_2 \in \Phi_1$. Then $\alpha_1 \operatorname{dom}_W \alpha_2$ if and only if $f_1(\alpha_1) \operatorname{dom}_W f_1(\alpha_2)$.*

Proof. By definition, $\alpha_1 \operatorname{dom}_W \alpha_2$ if and only if whenever $w \in W$ and $w\alpha_1 \in \Phi_1^-$ then $w\alpha_2 \in \Phi_1^-$. Since f_1 is W-equivariant, if follows from Lemma 2.3.6 that

$$w\alpha_1 \in \Phi_1^- \quad \text{if and only if} \quad f_1(w\alpha_1) = wf_1(\alpha_1) \in \Phi^-$$

and

$$w\alpha_2 \in \Phi_1^- \quad \text{if and only if} \quad f_1(w\alpha_2) = wf_1(\alpha_2) \in \Phi^-.$$

Thus $\alpha_1 \operatorname{dom}_W \alpha_2$ if and only if $f_1(\alpha_1) \operatorname{dom}_W f_1(\alpha_2)$, as required.

□

Lemma 3.5.2 combined with results obtained in Section 3.3 enables us to deduce the following facts concerning dominance in Φ_1.

Lemma 3.5.3. *Let $\alpha_1, \alpha_2 \in \Phi_1^+$ such that α_1 is not a scalar multiple of α_2 and $\alpha_1 \operatorname{dom}_W \alpha_2$. Then*

(i) $\langle \alpha_1, \phi(\alpha_2) \rangle > 0$;
(ii) $(w\alpha_1) \operatorname{dom}_W (w\alpha_2)$ *for all $w \in W$, that is, dominance is W-invariant;*
(iii) $\operatorname{dp}_1(\alpha_1) > \operatorname{dp}_1(\alpha_2)$;
(iv) $-\alpha_2 \operatorname{dom}_W -\alpha_1$.

Proof. (i) Lemma 3.3.2 (i), Corollary 2.3.9 and Lemma 3.5.2 combined yield the desired result.
(ii) Follows from Lemma 3.3.2 (ii) and Lemma 3.5.2.

(iii) Follows from Lemma 3.3.2 (i), Lemma 2.3.7 and Lemma 3.5.2.

(iv) Follows from Lemma 3.3.2 (iii) and Lemma 3.5.2. □

Lemma 3.5.4. *Suppose that W is a finite Coxeter group. Then the only domincance in Φ_1 is of the form $\alpha \operatorname{dom} \lambda\alpha$ for some $\lambda > 0$.*

Proof. If W is finite, then there is a unique $w_0 \in W$ of maximal length. So $l(wr_s) < l(w)$ for all $s \in S$, and this implies that $w\alpha_s \in \Phi_1^-$ for all $s \in S$. Thus $w_0(\Phi_1^+) = \Phi_1^-$. Hence $N_1(w_0) = \widehat{\Phi_1}$ and consequently $w \in W$, $l(w_0 w) = l(w_0) - l(w)$. Furthermore, $w_0 = w_0^{-1}$. Now suppose that $\alpha \in \Phi_1^+$. Then $-w_0\alpha \in \Phi_1^+$, and

$$\begin{aligned}
&\operatorname{dp}_1(-w_0\alpha) = \tfrac{1}{2}(l(r_{-(w_0\alpha)}) + 1) \quad \text{(by Lemma 1.3.19 of Chapter 1)}\\
&= \tfrac{1}{2}(l(w_0 r_\alpha w_0) + 1)\\
&= \tfrac{1}{2}(l(w_0) - l(r_\alpha w_0) + 1)\\
&= \tfrac{1}{2}(l(w_0) - l(w_0 r_\alpha) + 1)\\
&= \tfrac{1}{2}(l(w_0) - l(w_0) + l(r_\alpha) + 1)\\
&= \tfrac{1}{2}(l(r_\alpha) + 1)\\
&= \operatorname{dp}_1(\alpha) \qquad \text{(agaim by Lemma 1.3.19 of Chapter 1).}
\end{aligned}$$

Therefore the map sending each positive root α to $-w_0\alpha$ is a depth preserving permutation.

Now suppose for a contradiction that $\alpha_1, \alpha_2 \in \Phi_1$ which are not scalar multiples of each other and $\alpha_1 \operatorname{dom}_W \alpha_2$. Without loss of generality we may assume that $\alpha_1, \alpha_2 \in \Phi_1^+$. Then Lemma 3.5.3 (v) yields that $-w_0\alpha_2 \operatorname{dom}_W -w_0\alpha_1$, and by Lemma 3.5.3 (iv) this in turn implies that

$$\operatorname{dp}_1(\alpha_2) = \operatorname{dp}_1(-w_0\alpha_2) > \operatorname{dp}_1(-w_0\alpha_1) = \operatorname{dp}_1(\alpha_1),$$

contradicting Lemma 3.5.3 (iv). □

Now we give a geometric characterization of dominance in Φ_1:

Lemma 3.5.5. *Let α_1 and α_2 be arbitrary roots in Φ_1^+ which are not scalar multiples of each other. Then $\alpha_1 \operatorname{dom}_W \alpha_2$ if and only if the* *following three conditions are satisfied:*

$$\langle \alpha_2, \phi(\alpha_1)\rangle > 0, \quad \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq 1 \quad \textit{and} \quad \mathrm{dp}_1(\alpha_1) > \mathrm{dp}_1(\alpha_2).$$

Proof. Suppose that $\alpha_1 \operatorname{dom}_W \alpha_2$. By Lemma 3.5.3 (i) and (iv) we only need to prove that $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq 1$.

If $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle < 1$, then by Corollary 2.3.17, it follows that D the subgroup of W generated by r_{α_1} and r_{α_2} is finite. By Lemma 3.5.4 there is no non-trivial dominance in D, hence there exists $w \in D$ such that $w\alpha_1 \in \Phi_1^-$ and $w\alpha_2 \in \Phi_1^+$. Since $D \subseteq W$, this contradicts our hypothesis that $\alpha_1 \operatorname{dom}_W \alpha_2$. Thus $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq 1$ as required.

Conversely, assume that $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq 1$, $\langle \alpha_1, \phi(\alpha_2)\rangle > 0$ and $\mathrm{dp}_1(\alpha_1) > \mathrm{dp}(\alpha_2)$. First consider the case that $\alpha_2 = \alpha_r$ for some $r \in S$. Clearly $r_r\alpha_1 \in \Phi_1^+$, for α_1 is not a positive scalar multiple of α_2 (because $\mathrm{dp}_1(\alpha_1) > \mathrm{dp}_1(\alpha_2)$), and

$$\begin{aligned}
&\langle \alpha_1, \phi(r_r\alpha_1)\rangle\langle r_r\alpha_1, \phi(\alpha_1)\rangle \\
&= \langle \alpha_1, \phi(\alpha_1) - 2\langle \alpha_r, \phi(\alpha_1)\rangle\beta_r\rangle\langle \alpha_1 - 2\langle \alpha_1, \beta_r\rangle\alpha_r, \phi(\alpha_1)\rangle \\
&= (\langle \alpha_1, \phi(\alpha_1)\rangle - 2\langle \alpha_1, \beta_r\rangle\langle \alpha_r, \phi(\alpha_1)\rangle)^2 \\
&\geq 1.
\end{aligned}$$

Now direct calculations similar to those in Lemma 1.1.8 of Chapter 1 yield that there are infinitely many elements in $\widehat{\Phi_1}$ of the form $\widehat{\lambda\alpha_1 + \mu r_r\alpha_1}$, where $\lambda, \mu > 0$. Suppose for a contradiction that α_1 does not dominate α_r, and choose $w \in W$ such that $w\alpha_1 \in \Phi_1^-$ and $w\alpha_r \in \Phi_1^+$. Then

$$w(r_r\alpha_1) = w(\alpha_1 - 2\langle \alpha_1, \beta_r\rangle\alpha_r) = w\alpha_1 + 2\langle \alpha_1, \beta_r\rangle(-w\alpha_r).$$

By Corollary 2.3.13, $\langle \alpha_1, \beta_r\rangle > 0$ (since $\langle \alpha_r, \phi(\alpha_1)\rangle > 0$), so we see that $w(r_r\alpha_1)$ is negative. So $N_1(w)$ contains both α_1 and $r_r\alpha_1$, and hence all the roots of the form $\widehat{\lambda\alpha_1 + \mu r_r\alpha_1}$ where $\lambda, \mu > 0$, contradicting the finiteness of $N_1(w)$.

Proceeding by induction on $\mathrm{dp}_1(\alpha_2)$, suppose that now $\mathrm{dp}_1(\alpha_2) > 1$, and choose $s \in S$ such that $r_s\alpha_2 \prec_1 \alpha_2$. Since $\mathrm{dp}_1(\alpha_1) > \mathrm{dp}_1(\alpha_2) > 1$, it follows that $r_s\alpha_2 \in \Phi_1^+$. Now

$$\langle r_s\alpha_1, r_s\phi(\alpha_2)\rangle\langle r_s\alpha_2, r_s\phi(\alpha_1)\rangle = \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq 1$$

and $\langle r_s\alpha_2, r_s\phi(\alpha_1)\rangle = \langle \alpha_2, \phi(\alpha_1)\rangle > 0$, and furthermore,

$$\mathrm{dp}(r_s\alpha_1) \geq \mathrm{dp}_1(\alpha_1) - 1 > \mathrm{dp}_1(\alpha_2) - 1 = \mathrm{dp}_1(r_s\alpha_2).$$

Hence the inductive hypothesis yields that $r_s\alpha_1 \operatorname{dom}_W r_s\alpha_2$, and by Lemma 3.5.3 (ii), this implies that $\alpha_1 \operatorname{dom}_W \alpha_2$. □

Using similar arguments as those used in the proofs of Lemma 3.3.2 part (iv) and part (v), we may extend the last lemma to the following:

Proposition 3.5.6. *Let $\alpha_1, \alpha_2 \in \Phi_1$. Then there is dominance between α_1 and α_2 if and only if*

$$\langle \alpha_1, \phi(\alpha_2)\rangle > 0 \qquad \textit{and} \qquad \langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq 1.$$

□

Now we are able to prove the converse of Corollary 2.3.17:

Lemma 3.5.7. *Suppose $\alpha_1, \alpha_2 \in \Phi_1$ with $r_{\alpha_1} \neq r_{\alpha_2}$. If the subgroup of W generated by r_{α_1} and r_{α_2} is finite then $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle < 1$.*

Proof. Suppose for a contradiction that $\langle r_{\alpha_1}, r_{\alpha_2}\rangle$ is finite and yet $\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq 1$. Since $r_{\alpha_1} = r_{-\alpha_1}$, we may replace α_1 by $-\alpha_1$ if needed and assume that $\langle \alpha_1, \phi(\alpha_2)\rangle \geq 0$. Then Proposition 3.5.6 yields that there is dominance with respect to W between α_1 and α_2. Proposition 3.3.4 (ii) and Lemma 3.5.2 combined yield that there is dominance with respect to $\langle r_{\alpha_1}, r_{\alpha_2}\rangle$ between α_1 and α_2, contradicting Lemma 3.5.4. □

Recall the equivalence relation and the set of equivalence classes $\widetilde{\Phi_1}$ defined in the proof of Lemma 1.2.13.

Proposition 3.5.8. *There is a W-equivariant bijection $\widetilde{f_1}\colon \widetilde{\Phi_1} \to \Phi$ satisfying*

$$\widetilde{f_1}(\widetilde{\alpha}) = f(\alpha),$$

for all $\alpha \in \Phi_1$.

Proof. Since f is W-equivariant, it follows that if such a map $\widetilde{f_1}$ exists, then it must be W-equivariant. Furthermore, it can be readily checked that if such $\widetilde{f_1}$ exists, then it must be bijective. Thus all that remains is to prove that such $\widehat{f_1}$ is well-defined, that is, we need to show that if α and $\lambda\alpha$ are in Φ_1 with $\lambda > 0$ and $\lambda \neq 1$, then $f_1(\alpha) = f_1(\lambda\alpha)$. Our proof will be based on an induction on the depth of α (which is equal to the depth of $\lambda\alpha$).

If $\mathrm{dp}_1(\alpha) = 1$, then $\alpha = \mu\alpha_s$ for some $s \in S$ and $\mu > 0$. Then Corollary 1.2.19 yields that $\phi(\alpha) = \frac{1}{\mu}\beta_s$. Now Corollary 2.3.13 yields that $\mathrm{coeff}_{\gamma_t}(f_1(\alpha)) = 0$ for all $t \in S\setminus\{s\}$. Since $\mathrm{coeff}_{\alpha_s}(\alpha)\,\mathrm{coeff}_{\beta_s}(\phi(\alpha)) = 1$, it follows from Proposition 2.3.14 that $f_1(\alpha) = \gamma_s$. Similarly if $\lambda\alpha \in \Phi_1^+$, then $f_1(\lambda\alpha) = \gamma_s$. Hence $f_1(\alpha) = f_1(\lambda\alpha)$ when $\mathrm{dp}(\alpha) = 1$.

Next we may assume that $\mathrm{dp}_1(\alpha) > 1$, and choose $t \in S$ such that $r_t\alpha \prec_1 \alpha$. By Lemma 1.3.10 this means $\langle \alpha, \beta_t\rangle > 0$. Then $\langle \lambda\alpha, \beta_t\rangle > 0$ too, and so $r_t(\lambda\alpha) \prec_1 (\lambda\alpha)$ as well. Now by the inductive hypothesis

$$f_1(r_t\alpha) = f_1(r_t(\lambda\alpha)) = \gamma$$

for some $\gamma \in \Phi$. Then

$$f_1(\alpha) = r_t f_1(r_t(\alpha)) = r_t\gamma = r_t(f_1(r_t(\lambda\alpha))) = f_1(\lambda\alpha)$$

as required. □

Definition 3.5.9. Let $\widetilde{\alpha_1}, \widetilde{\alpha_2} \in \widetilde{\Phi_1}$, we say that $\widetilde{\alpha_1}$ *dominates* $\widetilde{\alpha_2}$ with respect to W (written $\widetilde{\alpha_1}\,\widetilde{\mathrm{dom}_W}\,\widetilde{\alpha_2}$), if $\alpha_1 \;\mathrm{dom}_W\; \alpha_2$. For each nonnegative integer n, define

$$\widetilde{D_{1,n}} = \{\, \widetilde{\alpha} \in \widetilde{\Phi_1^+} \mid \widetilde{\alpha} \text{ dominates exactly } n \text{ elements in } \widetilde{\Phi_1^+} \setminus \{\widetilde{\alpha}\}\,\}.$$

Proposition 3.5.10. *For each nonnegative integer n, $\widetilde{f_1}(\widetilde{D_{1,n}}) = D_n$.*

Proof. Follows readily from Proposition 3.5.8 and Lemma 3.5.2. □

Now we are ready to prove the following main result of this section:

Theorem 3.5.11. *Suppose that S is finite. Then $\widehat{D_{1,0}}$ is finite in size. Furthermore, for each $n \in \mathbb{N}$, $|\widehat{D_{1,n}}| < \infty$ and $|\widehat{D_{1,n}}| \leq |D_0|^{n+1} - |D_0|^n$.*

Proof. Follows from Theorem 3.3.8, Theorem 3.3.9, Proposition 3.5.10 and Corollary 3.3.10. □

References

[1] M. Dyer, *Hecke algebras and reflections in Coxeter groups*, PhD thesis, University of Sydney, 1987.

[2] M. Dyer, *Reflection subgroups of Coxeter systems*, J. Algebra 135 (1990), no. 1, 57–73

[3] M. Dyer, *Imaginary cone and dominance order in Coxeter groups*, in preparation.

[4] M. Dyer, *Imaginary cone and reflection subgroups of Coxeter groups*, in preparation.

[5] B. Brink, *The set of dominance-minimal roots*, J. Algebra **206** (1998), 371–412.

[6] B. Brink and R. B. Howlett, *A finiteness property and an automatic structure of Coxeter groups*, Math. Ann. **296** (1993), 179–190.

[7] R. B. Howlett, *Introduction to Coxeter groups*, Lectures given at ANU, 1996.

[8] R. B. Howlett, *Normalizers of parabolic subgroups of reflection groups*, J. London Math. Soc. (2) 21 (1980), no. 1, 62–80.

[9] H. Hiller, *Geometry of Coxeter Groups*. Research Notes in Mathematics 54, Pitman (Advanced Publishing Program), Boston-London, 1981.

[10] N. Bourbaki, *Groupes et algebras de Lie, Chapitres 4, 5 et 6* , Hermann, Paris, 1968

[11] D. Krammer, *The conjugacy problem for Coxeter groups*, PhD thesis, Universiteit Utrecht, 1994.

[12] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge studies in advanced mathematics 29, 1990.

[13] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics, GTM 231, Springer, 2005

[14] F. Bergeron, N. Bergeron, R.B. Howlett and D.E. Taylor *A Decomposition of the Descent Algebra of a Finite Coxeter Group*, Journal of Algebraic Combinatorics, **1** (1992), 23–44

[15] C. Hohlweg and M. Schocker, *On a parabolic symmetry of finite Coxeter groups*, Bull. London Math. Soc. 36 (2004), no. 3, 289–293.

[16] D. Blessenohl, C. Hohlweg and M. Schocker, *A symmetry of the descent algebra of a finite Coxeter group*, Adv. Math. 193 (2005), no. 2, 416–437.

[17] A. Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, University Lecture Series, 15. American Mathematical Society, Providence, RI, 1999.

[18] V.G. Kac, *Infinite-Dimensional Lie Algebras, Theta Functions and Modular Forms*, Advances in Mathematics, **53**, 125–264, (1984).