

Empirical Saddlepoint Approximations for Stratified Random Sampling

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Confidence intervals or tests for finite population mean from simple random sampling and stratified sampling require Studentized statistics.

Normal or Student-t approximations have been used for many years but are inadequate.

Booth, Butler and Hall (JASA, 1994) obtained bootstrap methods for both these situations.

For simple random samples, Edgeworth approximations were obtained by Sugden and Smith (JRSS B, 1998, 2000) and Dai and Robinson (SPL, 2001) obtained empirical saddlepoint approximations.

We extend the work on empirical approximations to the case of stratified random sampling and note that this gives a saddlepoint approximation to the bootstrap.

A population is divided into a finite number of disjoint subpopulations, which are called strata in sample surveys.

Assume that a finite population with N elements is divided into k strata, and the i th stratum possesses N_i elements a_{i1}, \dots, a_{iN_i} , $i = 1, \dots, k$, so $N = \sum_{i=1}^k N_i$. Denote the overall population mean by

$$\bar{a} = \frac{1}{N} \sum_{i=1}^k \sum_{j=1}^{N_i} a_{ij} = \sum_{i=1}^k \frac{N_i}{N} \bar{a}_i = \sum_{i=1}^k \omega_i \bar{a}_i,$$

where $\omega_i = N_i/N$, and $\bar{a}_i = \sum_{j=1}^{N_i} a_{ij}/N_i$ is the population mean for stratum i , $i = 1, \dots, k$.

Suppose that X_{i1}, \dots, X_{in_i} is a simple random sample without replacement selected from stratum i , with the sample mean

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij},$$

and the sample variance

$$S_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2, \quad i = 1, \dots, k.$$

The stratified sample mean is

$$\bar{X}_{st} = \frac{1}{N} \sum_{i=1}^k N_i \bar{X}_i,$$

and its estimated variance is

$$\hat{V}ar(\bar{X}_{st}) = \frac{1}{N^2} \sum_{i=1}^k N_i^2 \hat{V}ar(\bar{X}_i) = \sum_{i=1}^k \omega_i^2 (1 - f_i) S_i^2 / n_i,$$

where $f_i = n_i/N_i$, $i = 1, \dots, k$.

We want the distribution of

$$U_n = \frac{\bar{X}_{st} - \bar{a}}{\sqrt{\hat{V}ar(\bar{X}_{st})}},$$

where $n = \sum_{i=1}^k n_i$.

This distribution can be used to obtain tests and confidence intervals for the population mean. For example, a $(1 - \alpha)100\%$ confidence interval for \bar{a} is

$$\left(\bar{X}_{st} - F_{U_n}^{-1}(1 - \alpha/2) \sqrt{\hat{V}ar(\bar{X}_{st})}, \bar{X}_{st} - F_{U_n}^{-1}(\alpha/2) \sqrt{\hat{V}ar(\bar{X}_{st})} \right)$$

The bootstrap

F_{U_n} is unknown. Generate an empirical population by repeating the sample from the i th stratum $[N_i/n_i]$ times together with a random subsample of $N_i - [N_i/n_i]n_i$ elements of the sample.

Obtain a bootstrap sample by randomly sampling, without replacement, from this empirical population and calculate the bootstrap statistic. These two steps are repeated B times to get bootstrap replicates U_{ni}^* , $i = 1, \dots, B$. The empirical distribution of these, F_n^* , is the bootstrap approximation of the distribution of U_n .

So a bootstrap approximation to the confidence interval is

$$\left(\bar{X}_{st} - F_n^{*-1}(\alpha/2) \sqrt{\hat{V}ar(\bar{X}_{st})}, \bar{X}_{st} - F_n^{*-1}(1 - \alpha/2) \sqrt{\hat{V}ar(\bar{X}_{st})} \right)$$

We will obtain an empirical saddlepoint approximation, which is, in fact, when N/n is an integer, just the saddlepoint approximation to the distribution of the bootstrap statistic.

1. Saddlepoint approximation for non-Studentized statistic.
2. Saddlepoint and empirical saddlepoint approximation to F_{U_n} for a simple random sample.
3. Saddlepoint and empirical saddlepoint approximation to F_{U_n} for a stratified sample.
4. Numerical results.

Saddlepoint for sample means:

From $n + m = N$ units, n are chosen at random. The observations are x_1, \dots, x_N . Let $Y_i = x_{R(i)}$, where $R(1), \dots, R(N)$ are random permutations of $1, \dots, N$. Define \bar{Y}_1 as the mean of the first n of Y_1, \dots, Y_N . Then consider

$$p = P(\bar{Y}_1 \geq x).$$

Again, under permutations $n\bar{Y}_1 + m\bar{Y}_2 = N\bar{x}$ and $\sum Y_i^2 = \sum x_i^2$ are constant so the test statistic is equivalent to the usual t-statistic.

Saddlepoint approximation

Here we use the fact that

$$P(\bar{Y}_1 \geq y) = P\left(\sum_{i=1}^N I_i x_i \geq y \mid \sum_{i=1}^N I_i = n\right)$$

where I_1, \dots, I_N are iid Bernoulli variables with $P(I_1 = N/n) = n/N = f$. Then we approximate the conditional distribution by the ratio of the joint distribution and the marginal distribution of the conditioning sum, using saddlepoint approximations for each of these.

Let $\bar{Z}_1 = \sum I_i x_i / N$ and $\bar{Z}_0 = \sum I_i / N$. Define

$$\kappa(t, s) = N^{-1} \log E e^{Nt\bar{Z}_1 + Ns\bar{Z}_0} = N^{-1} \sum \log(1 - f + f e^{(tx_i + s)/f})$$

If

$$(\tilde{t}, \tilde{s}) = \arg \max_{(t, s)} [tx + s - \kappa(t, s)].$$

Let

$$L(x) = \tilde{t}x + \tilde{s} - \kappa(\tilde{t}, \tilde{s})$$

then

$$P(\bar{Z}_1 \geq x | \bar{Z}_0 = 1) = \int_x^\infty f_{\bar{Z}_1 | \bar{Z}_0}(y, 1) dy (1 + O(N^{-1})),$$

where we use the formal saddlepoint density approximation

$$f_{\bar{Z}_1|\bar{Z}_0}(x, 1) = \frac{\Delta_{00}(1, 0)^{1/2} e^{-NL(x)}}{(2\pi/N)^{1/2} \Delta(1, x)^{1/2}}$$

for $\Delta(1, x) = \det \kappa''(\tilde{t}, \tilde{s})$ and $\Delta_{00}(1, 0) = f(1 - f)$ the first element of $\det \kappa''(0, 0)$.

Now a change of variable and integration by parts yields

$$P(\bar{Z}_1 \geq x | \bar{Z}_0 = 1) = [1 - \Phi(\sqrt{N}w^*)][1 + O(N^{-1})]$$

where

$$w^* = w - \frac{\log \psi}{nw}, \quad w = \sqrt{2L(x)}, \quad \psi = \frac{w\sqrt{pq}}{\tilde{t}\sqrt{\det(\kappa''(\tilde{t}, \tilde{s}))}}.$$

Finite Population Sampling

Consider approximations for the Studentized mean of a simple random sample without replacement from a finite population. Let a_1, \dots, a_N be the units where N is the population size. A simple random sample without replacement X_1, \dots, X_n ($n \leq N$) is drawn from the population with the sample mean $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ and sample variance $S^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$.

Consider the Studentized sample mean

$$U_n = \sqrt{\frac{n}{1-f}} \frac{\bar{X} - \bar{a}}{S},$$

where $f = n/N$ and $\bar{a} = N^{-1} \sum_{i=1}^N a_i$.

This looks similar to the two sample permutation case but here only the

sample is available. So a Studentized statistic is needed.

An Edgeworth approximation was surprisingly only given in 1997 (Sugden and Smith).

Assume $\sum_{i=1}^N a_i = 0$ and $\sum_{i=1}^N a_i^2/N = 1$. Let I_1, \dots, I_N be i.i.d. with $P(I_i = N/n) = n/N = f$, $P(I_i = 0) = 1 - f$, $i = 1, \dots, N$. Let

$$\bar{Z}_0 = \frac{1}{N} \sum_{i=1}^N I_i,$$

$$\bar{Z}_1 = \frac{1}{N} \sum_{i=1}^N I_i a_i,$$

$$\bar{Z}_2 = \frac{1}{N} \sum_{i=1}^N I_i a_i^2.$$

Then \bar{X} and $\sum_{i=1}^n X_i^2/n$ are equal in distribution to the conditional distribution of \bar{Z}_1 and \bar{Z}_2 , given $\bar{Z}_0 = 1$, and U_n/\sqrt{n} is equal in distribution to the conditional distribution of

$$v(\bar{Z}_1, \bar{Z}_2) = c_n \frac{\bar{Z}_1}{\sqrt{\bar{Z}_2 - \bar{Z}_1^2}},$$

where

$$c_n = \sqrt{\frac{n-1}{1-f}}$$

The formal saddlepoint density approximation

$$f_{12|0}(x) = \frac{\Delta_{00}(1, 0, 1)^{1/2} e^{-N\Lambda(x)}}{(2\pi/N)\Delta(x)^{1/2}},$$

where $\Lambda(x) = t(x).x - K(t(x))$,

$$K(t) = \frac{1}{N} \sum_{j=1}^N \log \left[(1 - f) + f e^{(t_0 + t_1 a_j + t_2 a_j^2)/f} \right],$$

$$t(x) = \arg \max_t [t.x - K(t)],$$

$$\Delta_{00}(1, 0, 1) = f(1 - f) \text{ and } \Delta(x) = \det(K''(t(x))).$$

We transform from (\bar{Z}_1, \bar{Z}_2) to $(v(\bar{Z}_1, \bar{Z}_2), \bar{Z}_2)$, with $v = c_n x_1 / \sqrt{x_2 - x_1^2}$ and inverse $x_1(v, x_2) = v \sqrt{x_2} / \sqrt{c_n^2 + v^2}$. Then if

$$L(v, x_2) = \Lambda(1, x_1(v, x_2), x_2)$$

we can use the Laplace approximation to the integral over x_2 by solving

$$\frac{\partial L(v, x_2)}{\partial x_2} = \frac{v}{2\sqrt{x_2}\sqrt{c_n^2 + v^2}} t_1(0, x_1(v, x_2), x_2) + t_2(0, x_1(v, x_2), x_2) = 0,$$

with solution $x_2(v)$. Also set $x_1(v) = x_1(v, x_2(v))$. Then

$$P(U_n \geq u\sqrt{n}) = \int_u^\infty G(v) e^{-NH(v)} dv (1 + O(N^{-1})),$$

where

$$H(v) = \inf_{x_2} L(v, x_1(v, x_2)) = L(v, x_1(v, x_2(v))),$$

$$G(v) = \frac{\Delta_{00}(1, 0, 1)^{1/2} \left[\frac{\partial^2 L(v, x_2)}{\partial x_2^2} \right]^{-1/2}}{(2\pi/n)^{1/2} \left| \frac{\partial v(x_1, x_2)}{\partial x_1} \right|_{(x_1(v), x_2(v))} \Delta(0, x_1(v), x_2(v))^{1/2}}.$$

We can obtain the Barndorff-Nielsen form

$$P(U_n \geq u\sqrt{n}) = [1 - \Phi(\sqrt{N}w^\dagger)](1 + O(N^{-1})), \quad (1)$$

where

$$w^\dagger = w - \frac{\log(\psi(w))}{Nw},$$

for $w = \sqrt{2H(u)}$ and $\psi(w) = wG(u)/H'(u)$.

Empirical Saddlepoint Approximation

An empirical saddlepoint approximation can be obtained as a saddlepoint approximation to the bootstrap.

An empirical cgf is

$$\hat{K}(t) = -f(t_0 + t_2) + \frac{1}{n} \sum_{j=1}^n \log \left[(1 - f) + f e^{(t_0 + t_1 A_j + t_2 A_j^2)} \right],$$

where $A_j = (X_j - \bar{X}) / \sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 / n}$. Then we can obtain $\hat{t}(x)$ the solution of $\hat{K}'(t) = x$, $\hat{\Lambda}(x)$, $\hat{h}(u)$, $\hat{G}(u)$ and $\hat{\psi}(\hat{w})$, from \hat{K} as in the general case considered earlier. Then we can approximate $P(U_n \geq u\sqrt{n})$

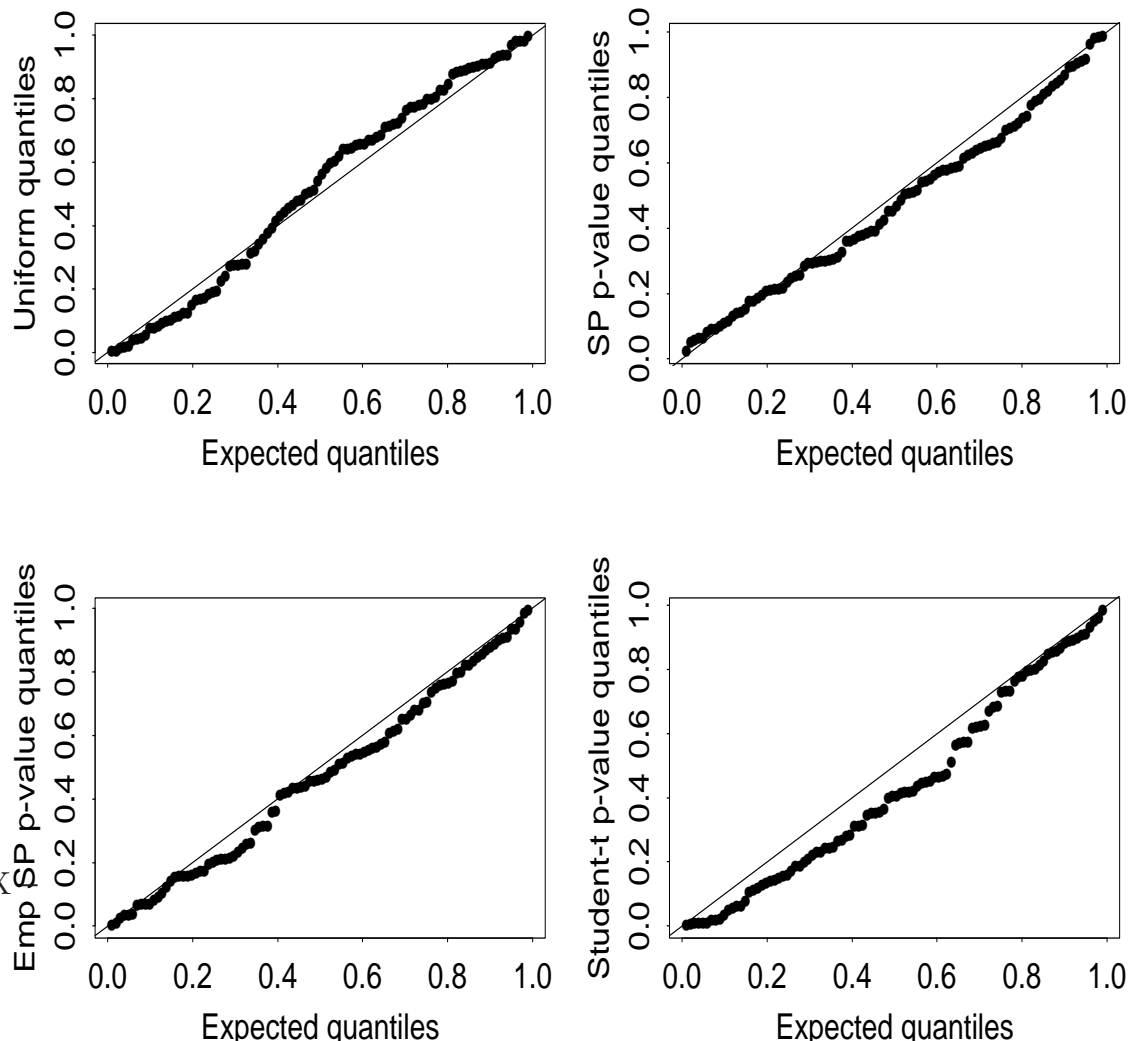
by

$$1 - \Phi(\sqrt{N}\hat{w}^\dagger(u)) \quad (2)$$

We can obtain information on errors and relative errors of the bootstrap from these results.

u	Simulation	Student-t	Edgeworth	Empirical SP	Saddlepoint
-0.6	0.0133	0.0031	0.0091	0.0134	0.0134
-0.5	0.0269	0.0098	0.0236	0.0278	0.0268
-0.4	0.0534	0.0285	0.0520	0.0551	0.0531
-0.3	0.1029	0.0733	0.1032	0.1055	0.1028
-0.2	0.1908	0.1636	0.1912	0.1930	0.1904
-0.1	0.329	0.3108	0.3292	0.3300	0.3283
0.1	0.2929	0.3108	0.2937	0.2936	0.2937
0.2	0.139	0.1636	0.1393	0.1387	0.1392
0.3	0.0508	0.0733	0.0508	0.0493	0.0502
0.4	0.0139	0.0285	0.0158	0.0130	0.0137
0.5	0.0031	0.0098	0.0058	0.0026	0.0029
0.6	0.0004	0.0031	0.0028	0.0004	0.0005

Figure : Q-Q plots of random uniform, saddlepoint p-values, empirical saddlepoint p-values and Student-t p-values for 100 samples of 25 from lung cancer survival times



Saddlepoint for stratified samples

Recall

$$U_n = \frac{\bar{X}_{st} - \bar{a}}{\sqrt{\widehat{Var}(\bar{X}_{st})}},$$

Assume that I_{ij} ($i = 1, \dots, k, j = 1, \dots, N_i$) are independent with

$$P(I_{ij} = 1/f_i) = f_i, \quad P(I_{ij} = 0) = 1 - f_i,$$

and that

$$\bar{Z}_{0i} = \frac{1}{N_i} \sum_{j=1}^{N_i} I_{ij}, \quad \bar{Z}_{1i} = \frac{1}{N_i} \sum_{j=1}^{N_i} I_{ij} a_{ij}, \quad \bar{Z}_{2i} = \frac{1}{N_i} \sum_{j=1}^{N_i} I_{ij} a_{ij}^2.$$

The distribution of U_n is the same as the conditional distribution of

$$V_n = \frac{\sum_{i=1}^k \omega_i (\bar{Z}_{1i} - \bar{a}_i)}{\sqrt{\sum_{i=1}^k \omega_i^2 c_i (\bar{Z}_{2i} - \bar{Z}_{1i}^2)}} := g(\bar{Z}),$$

given that $\bar{Z}_{0i} = 1$, $i = 1, \dots, k$, where $\bar{Z} = (\bar{Z}_{11}, \bar{Z}_{21}, \dots, \bar{Z}_{1k}, \bar{Z}_{2k})$ and $c_i = (1 - f_i)/(n_i - 1)$.

For $i = 1, \dots, k$, let

$$\kappa_i(t_i) = \frac{1}{N_i} \sum_{j=1}^{N_i} \log (1 - f_i + f_i e^{(t_{0i} + t_{1i} a_{ij} + t_{2i} a_{ij}^2) / f_i}),$$

$$t_i(x_i) = \operatorname{argsup}_{t_i} [t_i \cdot x_i - \kappa_i(t_i)], \text{ for } x_i = (x_{0i}, x_{1i}, x_{2i}),$$

$$L_i(x_i) = t_i(x_i) \cdot x_i - \kappa_i(t_i(x_i)) = \sup_{t_i} [t_i \cdot x_i - \kappa_i(t_i)],$$

and let

$$L(x) = \sum_{i=1}^k \omega_i L_i(1, x_{1i}, x_{2i}), \text{ for } x = (x_{11}, x_{21} \cdots, x_{1k}, x_{2k}).$$

Write

$$\mu_i = E(\bar{Z}_{0i}, \bar{Z}_{1i}, \bar{Z}_{2i}) = \left(1, \frac{1}{N_i} \sum_{j=1}^{N_i} a_{ij}, \frac{1}{N_i} \sum_{j=1}^{N_i} a_{ij}^2\right), \quad i = 1, \dots, k.$$

Then $t_i(\mu_i) = 0$ and $L_i(\mu_i) = \inf_{x_{1i}, x_{2i}} L_i(1, x_{1i}, x_{2i}) = 0$.

So

$$P(\bar{Z}_{1i} \leq z_{1i}, \bar{Z}_{2i} \leq z_{2i} | \bar{Z}_{0i} = 1) = \int_{-\infty}^{z_{1i}} \int_{-\infty}^{z_{2i}} f_{\bar{Z}_i | \bar{Z}_{0i}}(x_{1i}, x_{2i}) dx_{1i} dx_{2i} (1 + O(N_i^{-1})),$$

with

$$f_{\bar{Z}_i|\bar{Z}_{0i}}(x_{1i}, x_{2i}) = \frac{N_i \Delta_i^{1/2} \det(\kappa_i''(0))^{1/2}}{2\pi \det(\kappa_i''(t_i(x_i(1))))^{1/2}} e^{-N_i L_i(x_i(1))},$$

where here and below $x_i(1) = (1, x_{1i}, x_{2i})$ and

$$\Delta_i = \det \left(\frac{\partial^2 L_i(x_i(1))}{\partial \tilde{x}_i^2} \right) \Big|_{x_i(1)=\mu_i} = \det \begin{pmatrix} ([\kappa_i''(0)]^{-1})_{22} & ([\kappa_i''(0)]^{-1})_{23} \\ ([\kappa_i''(0)]^{-1})_{32} & ([\kappa_i''(0)]^{-1})_{33} \end{pmatrix},$$

where $\tilde{x}_i = (x_{1i}, x_{2i})$ and $([\kappa_i''(0)]^{-1})_{jl}$ denotes the $[j, l]$ element of the matrix $[\kappa_i''(0)]^{-1}$ for $j, l = 2, 3$.

Moreover,

$$P(\bar{Z} \leq z | \bar{Z}_{0i} = 1, i = 1, \dots, k) = \int_{x \leq z} f_{\bar{Z} | \bar{Z}_0}(x) dx (1 + O((\min N_i)^{-1})),$$

where $z = (z_{11}, z_{21}, \dots, z_{1k}, z_{2k})$, and

$$f_{\bar{Z} | \bar{Z}_0}(x) = e^{-NL(x)} \prod_{i=1}^k \frac{N_i \Delta_i^{1/2} \det(\kappa_i''(0))^{1/2}}{2\pi \det(\kappa_i''(t_i(x_i(1))))^{1/2}}.$$

Let us transform \bar{Z} to

$$(V_n = g(\bar{Z}), \bar{Z}'_{21} = \bar{Z}_{21} - \bar{Z}_{11}^2, \bar{Z}_{12}, \bar{Z}_{22}, \dots, \bar{Z}_{1k}, \bar{Z}_{2k}).$$

For the transform

$$v = g(x) = \frac{\sum_{i=1}^k \omega_i (x_{1i} - \bar{a}_i)}{\sqrt{\sum_{i=1}^k \omega_i^2 c_i (x_{2i} - x_{1i}^2)}}, \quad x'_{21} = x_{21} - x_{11}^2,$$

we get the inverse $x_{21}(v, x'_{-1}) = x'_{21} + x_{11}^2(v, x'_{-1})$ and

$$x_{11}(v, x'_{-1}) = \bar{a}_1 + \frac{v \sqrt{\omega_1^2 c_1 x'_{21} + \sum_{i=2}^k \omega_i^2 c_i (x_{2i} - x_{1i}^2)} - \sum_{i=2}^k \omega_i (x_{1i} - \bar{a}_i)}{\omega_1},$$

where $x'_{-1} = (x'_{21}, x_{12}, x_{22} \cdots, x_{1k}, x_{2k})$.

Then if

$$\Lambda(v, x'_{-1}) = L(x_{11}(v, x'_{-1}), x_{21}(v, x'_{-1}), x_{12}, x_{22} \cdots, x_{1k}, x_{2k}),$$

we have

$$P(U_n \geq u) = \int_u^\infty \left[\int_{\mathfrak{R}^{2k-1}} \frac{e^{-n\Lambda(v, x'_{-1})}}{|J|} \prod_{i=1}^k \frac{N_i \Delta_i^{1/2} \det(\kappa_i''(0))^{1/2}}{2\pi \det(\kappa_i''(t_i(x_i(1))))^{1/2}} dx'_{-1} \right] dv \times \\ \times (1 + O((\min N_i)^{-1})), \quad (3)$$

where

$$J = \det \begin{pmatrix} \partial g(x)/\partial x_{11} & \partial x'_{21}/\partial x_{11} \\ \partial g(x)/\partial x_{21} & \partial x'_{21}/\partial x_{21} \end{pmatrix}.$$

We can use the Laplace approximation to the inner integral of (3) by solving

$$\frac{\partial \Lambda(v, x'_{-1})}{\partial x'_{-1}} = 0; \quad (4)$$

Denote the solution of (4) by $x'_{-1}(v) = (x'_{21}(v), x_{12}(v), x_{22}(v) \cdots, x_{2k}(v))$, and set $x_{i1}(v) = x_{i1}(v, x'_{-1}(v))$, $i = 1, 2$ and $x(v) = (x_{11}(v), x_{21}(v), \cdots, x_{1k}(v), x_{2k}(v))$.

Then

$$P(U_n \geq u) = \sqrt{\frac{N}{2\pi}} \int_u^\infty G(v) e^{-NH(v)} dv (1 + O((\min N_i)^{-1})), \quad (5)$$

where

$$H(v) = \inf_{x'_{-1}} L(x_{11}(v, x'_{-1}), x_{21}(v, x'_{-1}), x_{12}, x_{22} \cdots, x_{1k}, x_{2k}) = L(x(v)),$$

$$G(v) = \frac{\det(\partial^2 \Lambda(v, x'_{-1}) / \partial (x'_{-1})^2 |_{x'_{-1}(v)})^{-1/2}}{N^k |J|_{x(v)}} \prod_{i=1}^k \frac{N_i \Delta_i^{1/2} \det(\kappa_i''(0))^{1/2}}{\det(\kappa_i''(t_i(1, x_{1i}(v), x_{2i}(v))))^{1/2}}.$$

By (4), we have

$$H'(v) = \omega_1 [t_{11}(x'_1(1)) + 2t_{21}(x'_1(1))x_{11}(v, x'_{-1}(v))] \frac{\partial x_{11}(v, x'_{-1}(v))}{\partial v}.$$

Finally, approximating (5) via integration by parts yields the Barndorff-Nielsen form

$$P(U_n \geq u) = [1 - \Phi(\sqrt{N}w^\dagger)](1 + O((\min N_i)^{-1})), \quad (6)$$

where, for $u > 0$, $w = \sqrt{2H(u)}$, $w^\dagger = w - \log(\psi(w))/(Nw)$, and $\psi(w) = wG(u)/H'(u)$.

The empirical saddlepoint approximation

In general the stratum elements a_{i1}, \dots, a_{iN_i} , $i = 1, \dots, k$, are unknown, so that we are unable to obtain $\kappa_i(t)$ for evaluating the saddlepoint approximation of U_n . As an alternative, we adopt an estimate of $\kappa_i(t)$,

$$\hat{\kappa}_i(t) = \frac{1}{n_i} \sum_{j=1}^{n_i} \log \left[(1 - f_i) + f_i e^{(t_0 + t_1 X_{ij} + t_2 X_{ij}^2)/f_i} \right], \quad i = 1, \dots, k, \quad (7)$$

and proceed as in Section 2 to obtain $\hat{t}_i(x_i)$, $\hat{L}_i(x_i)$, $\hat{L}(x)$, $\hat{H}(u)$, \hat{w} , $\hat{G}(u)$ and $\hat{\psi}(\hat{w}(u))$, simply by replacing κ_i by $\hat{\kappa}_i$ and \bar{a}_i by \bar{X}_i . This results in an empirical saddlepoint approximation for $P(U_n \geq u)$,

$$1 - \Phi(\sqrt{N} \hat{w}^\dagger(u)) \quad (8)$$

where

$$\hat{w}^\dagger(u) = \hat{w}(u) - \frac{\log(\hat{\psi}(\hat{w}(u)))}{N\hat{w}(u)}. \quad (9)$$

We can consider the relative error of the approximation using the same methods as in Dai and Robinson (2001) to show that

$$\frac{\Phi(\sqrt{N}\hat{w}^\dagger(u)) - \Phi(\sqrt{N}w^\dagger(u))}{1 - \Phi(\sqrt{N}\hat{w}^\dagger(u))} = O_P(u^3\sqrt{N}). \quad (10)$$

The empirical saddlepoint is a saddlepoint approximation to the bootstrap approximation of Booth, Butler and Hall (1994), so this also gives the relative error of that bootstrap approximation to the true distribution of the statistics U_n .

Some numerical comparisons

We generated the population in each of 4 strata from exponential distributions with given parameters.

Randomly selected population with 4 strata:

i	N_i	n_i	distribution
1	100	10	exp(1)
2	150	12	exp(1.5)
3	200	15	exp(0.5)
4	350	40	exp(2)

u	0.5	1.0	1.5	2.0	2.5	3.0
SP	0.28891	0.13311	0.04541	0.01147	0.00223	0.00036
MC	0.29082	0.13357	0.04564	0.0114	0.00231	0.00039
ESP	0.29772	0.14507	0.05573	0.01689	0.00412	0.00084
BS	0.29778	0.14298	0.05482	0.01675	0.00414	0.00086
t	0.30929	0.16031	0.06896	0.02461	0.00733	0.00185