First, we find the scattering data for the **continuous spectrum:** $\lambda > 0$**.** We have:

$$\varphi(x) = \begin{cases} e^{-ikx} & \text{for } x < -1; \\ Ae^{ik_1 x} + Be^{-ik_1 x} & \text{for } x \in (-1, 1); \\ Ce^{ikx} + De^{-ikx} & \text{for } x > 1, \end{cases}$$

with $k = \sqrt{\lambda}$, $k_1 = \sqrt{\lambda - \alpha}$, with the assumption $\alpha \neq \lambda$. The continuity conditions at point $x = -1$ are:

$$\varphi(-1_-) = \varphi(-1_+), \quad \varphi'(-1_-) = \varphi'(-1_+)$$

$$\Longleftrightarrow$$

$$Ae^{-ik_1} + Be^{ik_1} = e^{ik}, \quad Ae^{-ik_1} - Be^{ik_1} = -\frac{k}{k_1}e^{ik}.$$

From the system of linear equations we get:

$$A = \frac{k_1 - k}{2k_1}e^{i(k+k_1)}, \quad B = \frac{k + k_1}{2k_1}e^{i(k-k_1)}.$$

Now, the continuity conditions at $x = 1$:

$$\varphi(-1_-) = \varphi(-1_+), \quad \varphi'(-1_-) = \varphi'(-1_+)$$

lead to the following system:

$$\begin{aligned} Ce^{ik} + De^{-ik} &= \frac{k_1 - k}{2k_1}e^{i(k+2k_1)} + \frac{k + k_1}{2k_1}e^{i(k-2k_1)}, \\ Ce^{ik} - De^{-ik} &= \frac{k_1 - k}{2k}e^{i(k+2k_1)} + \frac{k + k_1}{2k}e^{i(k-2k_1)}. \end{aligned}$$

The solution is:

$$\begin{aligned} C &= \frac{k_1^2 - k^2}{4kk_1}e^{2ik_1} + \frac{(k + k_1)^2}{4kk_1}e^{-2ik_1}, \\ D &= -\frac{(k - k_1)^2}{4kk_1}e^{2i(k+k_1)} + \frac{k^2 - k_1^2}{4kk_1}e^{2i(k-k_1)}. \end{aligned}$$

Since $\psi(x) = e^{ikx}$ on $(1, +\infty)$ and $\varphi = a\bar{\psi} + b\psi$, we have:

$$a(k) = D, \quad b(k) = C.$$

Thus the reflection coefficient is:

$$r(k) = \frac{b(k)}{a(k)} = \frac{C}{D} = -e^{-2ik}\frac{(k_1^2 - k^2)e^{2ik_1} + (k + k_1)^2 e^{-2ik_1}}{(k - k_1)^2 e^{2ik_1} - (k^2 - k_1^2)e^{-2ik_1}}.$$

Now, assume $\alpha = \lambda > 0$. The corresponding eigenfuction is:

$$\varphi(x) = \begin{cases} e^{-ikx} & \text{for } x < -1; \\ Ax + B & \text{for } x \in (-1, 1); \\ Ce^{ikx} + De^{-ikx} & \text{for } x > 1. \end{cases}$$

The continuity conditions at point $x = -1$ are:

$$-A + B = e^{ik}, \quad A = -ike^{ik},$$

thus $B = (1 - ik)e^{ik}$.

The continuity conditions at point $x = 1$ are:

$$Ce^{ik} + De^{-ik} = (1 - 2ik)e^{ik}, \quad Ce^{ik} - De^{-ik} = -e^{ik},$$

thus

$$C = -ik, \quad D = (1 - ik)e^{2ik}.$$

The reflection coefficient is:

$$r(k) = \frac{C}{D} = -e^{-2ik}\frac{ik}{1 - ik}.$$

Now, suppose that $\lambda_s < 0$ is in the **discrete spectrum** and denote $\sqrt{\lambda_s} = i\kappa_s$, $\kappa_s > 0$.

First case: $\lambda_s - \alpha > 0$.

The corresponding eigenfunction is of the following form:

$$\varphi_s(x) = \begin{cases} e^{\kappa_s x} & \text{for } x < -1; \\ E\sin(k_2 x) + F\cos(k_2 x) & \text{for } x \in (-1, 1); \\ b_s e^{-\kappa_s x} & \text{for } x > 1, \end{cases}$$

with $k_2 = \sqrt{\lambda_s - \alpha}$. The continuity conditions at $x = -1$ give the following:

$$-E\sin k_2 + F\cos k_2 = e^{-\kappa_s}, \quad E\cos k_2 + F\sin k_2 = \frac{\kappa_s}{k_2}e^{-\kappa_s},$$

which is equivalent to:

$$E = e^{-\kappa_s}\left(\frac{\kappa_s}{k_2}\cos k_2 - \sin k_2\right), \quad F = e^{-\kappa_s}\left(\frac{\kappa_s}{k_2}\sin k_2 + \cos k_2\right). \qquad (1)$$

On the other hand, from the continuity conditions at $x = 1$:

$$E\sin k_2 + F\cos k_2 = b_s e^{-\kappa_s}, \quad E\cos k_2 - F\sin k_2 = -\frac{b_s\kappa_s}{k_2}e^{-\kappa_s},$$

we get:

$$E = b_s e^{-\kappa_s}\left(\sin k_2 - \frac{\kappa_s}{k_2}\cos k_2\right), \quad F = b_s e^{-\kappa_s}\left(\frac{\kappa_s}{k_2}\sin k_2 + \cos k_2\right). \qquad (2)$$

Now, expression (1) and (2) hold simultaneously if and only if:

$$b_s = 1,\ \kappa_s = k_2\tan k_2 \quad \text{or} \quad b_s = -1,\ \kappa_s = -k_2\,\text{cotan}\,k_2.$$

Finally, the scattering data are:

- each solution $\kappa \in (0, \sqrt{-\alpha})$ of the equation $\kappa = \sqrt{-\kappa^2 - \alpha} \tan \sqrt{-\kappa^2 - \alpha}$ and $b = 1$;
- each solution $\kappa \in (0, \sqrt{-\alpha})$ of the equation $\kappa = -\sqrt{-\kappa^2 - \alpha} \operatorname{cotan} \sqrt{-\kappa^2 - \alpha}$ and $b = -1$.

Second case: $\lambda_s = \alpha < 0$.
The corresponding eigenfunction is of the following form:

$$\varphi_s(x) = \begin{cases} e^{\kappa_s x} & \text{for } x < -1; \\ Ex + F & \text{for } x \in (-1, 1); \\ b_s e^{-\kappa_s x} & \text{for } x > 1. \end{cases}$$

The continuity conditions at $x = -1$ are:

$$-E + F = e^{-\kappa_s}, \quad E = \kappa_s e^{-\kappa_s},$$

thus $F = (1 + \kappa_s) e^{-\kappa_s}$.
At $x = 1$, we get:

$$E + F = b_s e^{-\kappa_s}, \quad E = -b_s \kappa_s e^{-\kappa_s},$$

thus $F = b_s(1 + \kappa_s) e^{-\kappa_s}$.
Since $\kappa_s > 0$, the two expressions for E imply $b_s = -1$, and the two expressions for F give $b_s = 1$, which is impossible.

Third case: $\lambda_s - \alpha < 0$.
The corresponding eigenfunction is of the following form:

$$\varphi_s(x) = \begin{cases} e^{\kappa_s x} & \text{for } x < -1; \\ E e^{k_3 x} + F e^{-k_3 x} & \text{for } x \in (-1, 1); \\ b_s e^{-\kappa_s x} & \text{for } x > 1, \end{cases}$$

with $k_3 = \sqrt{\alpha - \lambda_s}$. The continuity conditions at $x = -1$ are:

$$E e^{-k_3} + F e^{k_3} = e^{-\kappa_s}, \quad E e^{-k_3} - F e^{k_3} = \frac{\kappa_s}{k_3} e^{-\kappa_s},$$

which gives:

$$E = \frac{1}{2} \left(1 + \frac{\kappa_s}{k_3} \right) e^{k_3 - \kappa_s}, \quad F = \frac{1}{2} \left(1 - \frac{\kappa_s}{k_3} \right) e^{-k_3 - \kappa_s}. \tag{3}$$

The continuity conditions at $x = 1$ are:

$$E e^{k_3} + F e^{-k_3} = b_s e^{-\kappa_s}, \quad E e^{k_3} - F e^{-k_3} = -\frac{b_s \kappa_s}{k_3} e^{-\kappa_s},$$

thus

$$E = \frac{b_s}{2} \left(1 - \frac{\kappa_s}{k_3} \right) e^{-k_3 - \kappa_s}, \quad F = \frac{b_s}{2} \left(1 + \frac{\kappa_s}{k_3} \right) e^{k_3 - \kappa_s}. \tag{4}$$

From (3) and (4) we get $E = b_s F$ and $F = b_s E$, thus $b_s \in \{-1, 1\}$. Finally, we get:

$$\left(1 + \frac{\kappa_s}{\sqrt{\alpha + \kappa_s^2}}\right) e^{2\sqrt{\alpha+\kappa_s^2}} = b_s \left(1 - \frac{\kappa_s}{\sqrt{\alpha + \kappa_s^2}}\right),$$

which has no solutions since the lefthandside is bigger the righthandside.