THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Integrable Systems: Assessment 4

AMH2: Applied Mathematics Honours — Semester 2, 2017

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1. Find the Jost function solutions ϕ, ψ and $\overline{\psi}$ of the stationary Schrödinger equation

$$\psi_{xx} + (6\,\mathrm{sech}^2(x) + \zeta^2)\,\psi = 0$$

and hence find $a(\zeta)$ and $b(\zeta)$ explicitly. Where is $a(\zeta)$ analytic in the complex ζ-plane?

[*Hint:* You may find it useful to transform variables to $z = \tanh x$. Compare the resulting ODE with the one for Associated Legendre functions in any book on special functions. A good online resource is the Digital Library of Mathematical Functions: `http://dlmf.nist.gov/` .]

Solution: The change of variables to $z = \tanh x$ transforms the Schrödinger equation to

$$\left((1-z^2)\,\psi_z\right)_z + \left(\frac{\zeta^2}{1-z^2} + 6\right)\psi = 0$$

which is the *Associated Legendre equation*, whose general solutions are

$$\psi = k_1(\zeta)\;P_n^{i\zeta}(z) + k_2(\zeta)\,Q_n^{i\zeta}(z),$$

where $k_j(\zeta)$ are arbitrary, $n = 2$, and $P_n^m(z)$ and $Q_n^m(z)$ are associated Legendre functions of the first and second kind respectively. These functions have well known asymptotic behaviours as $|z| \to 1$: note as $x \to +\infty$, we get $z \to +1$ from below, while $x \to -\infty$ gives $z \to -1$ from above. By connecting these (for the potential being $c_0\,\mathrm{sech}^2(\mathrm{x})$), we obtain

$$\begin{aligned}\phi(x,\zeta) \underset{x\to+\infty}{\sim} & \frac{\Gamma(1-i\,\zeta)\,\Gamma(-i\,\zeta)}{\Gamma(1/2 - i\,\zeta + \sqrt{c_0+1/4})\,\Gamma(1/2 - i\,\zeta - \sqrt{c_0+1/4})}\,e^{-i\,\zeta\,x} \\ & + \frac{\Gamma(1-i\,\zeta)\,\Gamma(i\,\zeta)}{\Gamma(1/2 - \sqrt{c_0+1/4})\,\Gamma(1/2 + \sqrt{c_0+1/4})}\,e^{i\,\zeta\,x}\end{aligned}$$

The coefficient of $e^{-i\,\zeta\,x}$ is $a(\zeta)$, while that of $e^{i\,\zeta\,x}$ is $b(\zeta)$. But, the denominator of $b(\zeta)$ is $\Gamma(1/2 - \sqrt{c_0+1/4})\,\Gamma(1/2 + \sqrt{c_0+1/4}) = \pi\,/\cos(\pi\sqrt{c_0+1/4})$. So we have $b(\zeta) = 0$ for $c_0 = 6$. Clearly this b is analytic everywhere.

On the other hand, $\Gamma(s)$ is a meromorphic function in the s-plane with simple poles at $s = -N = -1, -2, -3, \ldots$ and no zeroes. These provide singularities for $a(\zeta)$ at $\zeta = -i(1+N)$, $\zeta = -i\,N$ which are in the lower half-plane. So $a(\zeta)$ is analytic in the whole ζ-plane except at $\zeta = -\,i, -2\,i, -3\,i, \ldots$.