Integrable Systems:
Lecture 1

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Contents

In this lecture, we introduced the Korteweg–de Vries equation and derived its solitary wave solution.

We followed the exposition from Chapter 1: *The Korteweg–de Vries equation* and Chapter 2: *Elementary solutions of the KdV equation* from [DJ1989].

References

[DJ1989] P. G. Drazin and R. S. Johnson, *Solitons: an introduction*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1989.

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Korteweg-de Vries equation (KdV)

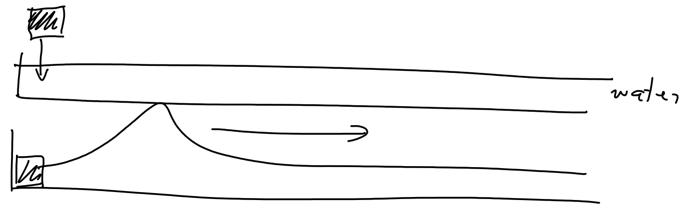
- describes the wave propagation

$$u_t - 6uu_x + u_{xxx} = 0$$

$u = u(x,t)$ $\quad$ t - time

(1895)

<u>1834</u> J. Scott Russell observed <u>a solitary wave</u> in the Edinburgh-Glasgow channel.



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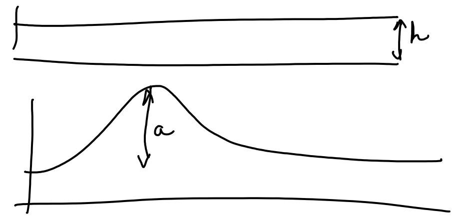
empirically : the volume of water in the wave = the volume of water displaced

- $c^2 = g(h+a)$

c - the speed of the wave
g - the acceleration of gravity

h - depth of water
a - amplitude



higher waves travel faster

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Boussinesq (1871)
Lord Rayleigh (1876)

- assumed that the length of the wave is much bigger than the depth of the channel

They deduced, using the equations of motion for inviscid incompressible fluid, the Russell's formula for c and found the equation for the wave profile is:

$$\zeta(x,t) = a \operatorname{sech}^2(\beta(x-ct))$$

$$\beta = \sqrt{\frac{3a}{4h^2(h+a)}}$$

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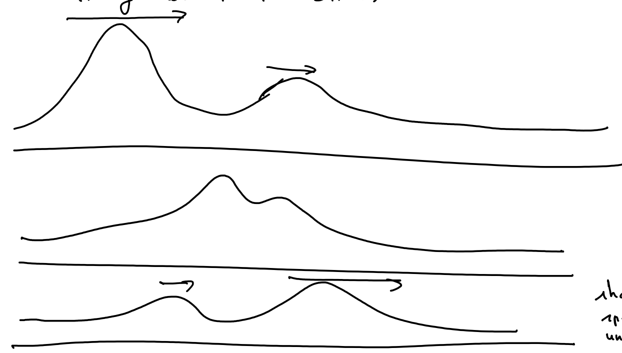
1894 Korteweg - de Vries

$$u_t - 6 u u_x + u_{xxx} = 0$$

Solitary waves interactions



shapes and speeds are unchanged

dissipation

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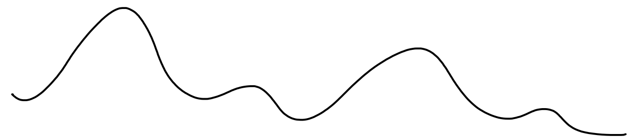
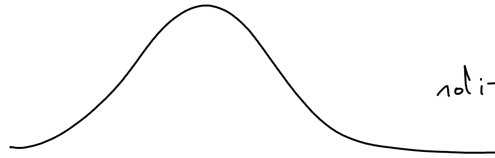
(i) represent a wave of a permanent form

(ii) localised (approaches a constant at $\pm\infty$)

(iii) strongly interact with other such solutions, keeping own identity

"soliton" - the name which was coined by Zabusky and Kruskal in 1960's





solitary wave

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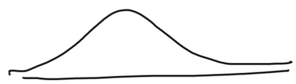
$u_t - 6uu_x + u_{xxx} = 0$

travelling wave solution:

$u(x,t) = f(\xi) \qquad \xi = x - ct$

$-cf' - 6ff' + f''' = 0$

integrate



$-cf - 3f^2 + f'' = A \quad / \cdot f'$

$-cff' - 3f^2 f' + f'f'' = Af' \quad / \int$

$-\frac{c}{2}f^2 - f^3 + \frac{1}{2}(f')^2 = Af + B$

$f'^2 = f^2(c + 2f) \qquad \int \frac{df}{f\sqrt{c+2f}} = \int d\xi$

the wave is localized

$\Downarrow$

$f, f', f'' \to 0$ as $x \to \pm\infty$

$\Downarrow$

$A = B = 0$

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$$f(x-ct) = -\frac{1}{2} c \cdot \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c}\,(x-ct-x_0)\right)$$