Tuesday, August 29, 2017 2:27 PM

The Jost functions satisfy

$$\varphi \sim e^{-i\zeta x} \quad x\to-\infty \qquad \Rightarrow \quad B=0\,,\ A=\frac{1}{-1+i\zeta}$$

$$\psi \sim e^{i\zeta x} \quad x\to+\infty \qquad \Rightarrow \quad A=0\,,\ B=\frac{1}{1-i\zeta}$$

$$\bar\psi \sim e^{-i\zeta x} \quad x\to+\infty \qquad \Rightarrow \quad B=0\,,\ A=\frac{1}{1+i\zeta}$$

We have $\quad \varphi = a(\zeta)\,\bar\psi + b(\zeta)\,\psi$

$$\Rightarrow \quad \frac{e^{-i\zeta x}}{-1+i\zeta}\left(\tanh x + i\zeta\right) = a(\zeta)\cdot\frac{e^{-i\zeta x}}{1+i\zeta}\left(\tanh x+i\zeta\right)+b(\zeta)\cdot\frac{e^{i\zeta x}}{1-i\zeta}\left(\tanh x - i\zeta\right)$$

$$\Rightarrow \begin{cases} b(\zeta)\equiv 0 \\ a(\zeta) = \dfrac{-(1+i\zeta)}{(1-i\zeta)} \end{cases} \qquad \Rightarrow \quad \text{simple zero at } \zeta_1 = i, \quad \dot a(\zeta_1) = -\frac{i}{2} \quad (\text{Check!})$$

$$\Rightarrow \quad R(\zeta)\equiv 0$$

We also need b_1: relate $\varphi(x,\zeta_1)$ to $\psi(x,\zeta_1)$

$$\varphi(x,\zeta_1) = \varphi_1 = \tfrac12\operatorname{sech} x, \qquad \psi(x,\zeta_1)=\psi_1 = \tfrac12 \operatorname{sech} x.$$

$$\varphi_1 = b_1\psi_1 \quad \Rightarrow \quad b_1 = 1.$$

$$\gamma_1 = \frac{b_1}{\dot a_1} = \frac{1}{-i/2} = 2i$$

Time evolution: $\quad b_1(t) = b_1(0)\, e^{8i\zeta_1^3 t} = e^{8t}$

$$\gamma_1(t) = 2i\,e^{8t}$$

$$\Rightarrow \quad B(z) = -i\gamma_1 e^{i\zeta_1 z} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} 0\cdot e^{i\xi z}\,d\xi = 2e^{8t}\cdot e^{-z}$$

$\Rightarrow$ The GLM eqn becomes (for time t)

$$K(x,\tau) + 2e^{-(x+\tau)+8t} + 2e^{8t}\int_x^\infty K(x,s)\, e^{-(s+\tau)}\,ds = 0, \quad \text{for } \tau > x.$$

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Separation of variables:

$$K(x,\tau) = L(x)\,e^{-\tau} \quad \Rightarrow \quad L(x)\cancel{e^{-\tau}} + 2e^{-x\cancel{-\tau}+8t} + 2e^{8t}\int_x^\infty L(x)\,e^{-s}\,e^{-s\cancel{-\tau}}\,ds = 0$$

$$\Rightarrow \quad L(x)\left(1 + 2e^{8t}\cdot\left[\frac{e^{-2s}}{-2}\right]_x^\infty\right) + 2e^{-x+8t} = 0$$

$$\Rightarrow \quad L(x) = -\frac{2e^{-x+8t}}{1+e^{-2x+8t}}$$

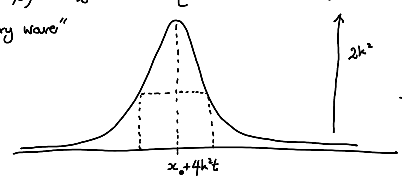
$$\Rightarrow \quad K(x,x) = -2\,\frac{e^{-2x+8t}}{1+e^{-2x+8t}} = \frac{-2}{e^{2x-8t}+1}$$

$$\Rightarrow \quad u(x,t) = 2\,\frac{d}{dx}\big(K(x,x)\big) = \cdots = 2\operatorname{sech}^2(x-4t)$$

Or using scaling invariance of the KdV and arbitrary shift in x by x_0

$$\Rightarrow \quad u(x,t) = 2k^2\operatorname{sech}^2\left[k(x-4k^2t-x_0)\right]$$

a "solitary wave"



travels to the right as t increases with speed $4k^2$.

Remember this is a reflectionless potential.

Recall

$$\psi(x,\zeta) = e^{i\zeta x} + \frac{1}{\zeta}\int_x^\infty u(y)\,\psi(y,\zeta)\,\sin[\zeta(x-y)]\,dy$$

which gives

$$\psi(x,\zeta)\,e^{-i\zeta x} = \sum_{j=0}^\infty \frac{g_j(x,\zeta)}{\zeta^j}, \qquad \Im \zeta \geq 0$$

$$\Rightarrow \quad g_0 = 1$$

$$g_1 = \frac{1}{2i}\int_x^\infty u(y)\,dy$$

$$\vdots$$

$$\Rightarrow \quad \psi(x,\zeta)\,e^{-i\zeta x} = 1 + \frac{1}{2i\zeta}\int_x^\infty u(y)\,dy + O\!\left(\frac{1}{\zeta^2}\right)$$

$$\Rightarrow \quad \boxed{u(x) = -\lim_{\substack{|\zeta|\to\infty \\ \Im\zeta\geq 0}}\left(2i\zeta\,\frac{\partial}{\partial x}\left[\psi(x,\zeta)\,e^{-i\zeta x}-1\right]\right)} \qquad (\ast)$$

We will also need (for $R(\zeta)\equiv 0$)

$$\psi(x,\zeta)\,e^{-i\zeta x} = 1 - \sum_{k=1}^N \frac{\gamma_k\,\psi_k\,e^{i\zeta_k x}}{\zeta+\zeta_k} \qquad (\ast\ast)$$

Remembering that ζ_k are pure imaginary, we write $\zeta_k = i\eta_k$, $\eta_k\in\mathbb{R}^+$.

So

$$\psi_j\,e^{\eta_j x} = 1 + i\sum_{k=1}^N \frac{\gamma_k\,\psi_k\,e^{-\eta_k x}}{\eta_j+\eta_k}, \qquad j = 1,\dots,N.$$

These are linear equations, actually a system of linear eqns for ψ_j.

Rewrite $\quad \tilde\psi_k = c_k\psi_k, \quad c_k^2 = \left(\int_{-\infty}^\infty (\psi_k(x))^2\,dx\right)^{-1}$

$$\Rightarrow \quad \|\tilde\psi_k\| = 1.$$

$$\Rightarrow \quad \frac{\tilde\psi_j}{c_j}\,e^{\eta_j x} = 1 + i\sum_{k=1}^N \frac{\overbrace{\gamma_k}^{i c_k^2}\cdot\dfrac{\tilde\psi_k}{c_k}\,e^{-\eta_k x}}{\eta_j+\eta_k}$$

$$\Rightarrow \quad \tilde\psi_j(x) + \sum_{k=1}^N \frac{c_j c_k\,\tilde\psi_k\,e^{-(\eta_j+\eta_k)x}}{\eta_j+\eta_k} = c_j\,e^{-\eta_j x}$$

Or, in matrix form

$$\boxed{(I + C)\,\Psi = E}$$

where $\quad I$ is the $N\times N$ identity matrix,

$$\Psi = \begin{pmatrix}\tilde\psi_1\\ \vdots \\ \tilde\psi_N\end{pmatrix}, \qquad E = \begin{pmatrix} c_1 e^{-\eta_1 x} \\ \vdots \\ c_N e^{-\eta_N x}\end{pmatrix}$$

and C is an $N\times N$ matrix with entries

$$C_{jk} = \frac{c_j c_k\, e^{-(\eta_j+\eta_k)x}}{\eta_j+\eta_k}$$

Lemma: $\quad C$ is a positive-definite matrix.

Proof: Take an N-vector $v = (v_1, \dots, v_N)$.

$$v\,C\,v^t = \sum_{j=1}^{N}\sum_{k=1}^{N} v_j v_k \frac{c_j c_k e^{-(\eta_j+\eta_k)x}}{\eta_j+\eta_k}$$

$$= \int_x^\infty \left(\sum_{j=1}^{N} v_j c_j e^{-\eta_j z}\right)^2 dz \qquad \text{(Check!)}$$

The RHS is positive (for $\underset{\sim}{v} \neq \underset{\sim}{0}$)

$\Rightarrow$ C is positive definite. $\square$

$\therefore$ $I + C$ is also positive definite.

$\Rightarrow$ $I+C$ has positive eigenvalues

$\Rightarrow$ $I+C$ is invertible.

Let $$\Delta = \det(I+C) = \sum_{j=1}^{N}\left(\delta_{jk} + \frac{c_j c_k e^{-(\eta_j+\eta_k)x}}{\eta_j+\eta_k}\right) Q_{jk}$$

where Q_{jk} are cofactors of $I+C$.

Cramer's rule gives

$$\boxed{\tilde{\psi}_k(x) = \Delta^{-1} \sum_{j=1}^{N} c_j e^{-\eta_j x} Q_{jk}}$$

To get $u(x)$ (using $\star$), we need to know how to differentiate a determinant.

Consider a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = ad - bc$$, with a,b,c,d depending on x.

$$\Rightarrow \frac{d}{dx}(\det A) = a'd - b'c + ad' - bc'$$
$$= \sum_{i,j=1}^{2} \frac{d}{dx}(a_{ij})\, Q_{ij}, \quad \text{for } A = (a_{ij})$$

By induction (on the size of the matrix) we can prove a general formula for the derivative of a determinant of $A = (a_{ij})$

$$\frac{d}{dx}(\det A) = \sum_{i,j=1}^{N} \frac{d}{dx}(a_{ij}) \cdot Q_{ij}$$

where Q_{ij} is the (i,j)-th cofactor.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad (-1)^{i+j} \qquad Q_{11} = \begin{vmatrix} e & f \\ h & i \end{vmatrix} \cdot (-1)^2$$

We get

$$\frac{d}{dx}(\Delta) = -\sum_{k,j=1}^{N} c_j c_k e^{-(\eta_j+\eta_k)x} Q_{jk} \qquad \left(\text{Remember } \Delta = \det(I+C)\right)$$

Recalling $\tilde{\psi}_k(x) = \Delta^{-1} \sum_{j=1}^{N} c_j e^{-\eta_j x} Q_{jk}$

$$\Rightarrow \sum_{k=1}^{N} c_k \tilde{\psi}_k(x) e^{-\eta_k x} = \Delta^{-1} \sum_{k=1}^{N}\sum_{j=1}^{N} c_k c_j e^{-(\eta_j+\eta_k)x} Q_{jk}$$

$$= -\Delta^{-1} \frac{d}{dx}\Delta. \qquad (\dagger)$$

We combine the above results to get

$$u(x) = \lim_{\zeta\to\infty}\left(-2i\zeta \frac{\partial}{\partial x}\left[\psi(x,\zeta)e^{-i\zeta x} - 1\right]\right)$$

$$= \lim_{\zeta\to\infty}\left(-2i\zeta \frac{\partial}{\partial x}\left[-\sum_{k=1}^{N} \gamma_k \frac{\psi_k e^{-\eta_k x}}{(\zeta + i\eta_k)}\right]\right) \quad \text{from } (\star\star)$$

$$= -2\cancel{i} \frac{\partial}{\partial x}\left[\sum_{k=1}^{N} \cancel{i} c_k \tilde{\psi}_k e^{-\eta_k x}\right] \quad \text{after taking the limit}$$

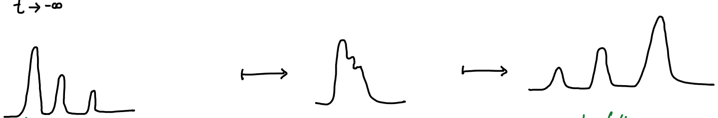
$$= 2\frac{d}{dx}\left(\Delta^{-1}\frac{d}{dx}\Delta\right)$$

$$= 2\frac{d^2}{dx^2}\left(\log\left(\det(I+C)\right)\right)$$

These are all reflectionless potentials. The results are all rational in each exponential $\{e^{-\eta_k x}\}$. Check that the case $N=1$ in this result agrees with the solution we considered yesterday: $u(x,t) = 2\,\mathrm{sech}^2(x-4t)$.

We will see that for any N integer > 0, we get a train of solitons

$t \to -\infty$ t finite $t \to +\infty$



each solitary wave interacts with others

* come out of the interaction with no change in height, speed or shape except for a phase shift.

"Particle-like" interaction

soliton