soliton

Recall reflectionless solutions
$$u(x,\cdot) = 2\frac{d^2}{dx^2}\Big(\log\big(\det(I+C)\big)\Big) \quad , I+C \text{ being an } N\times N \text{ matrix}.$$

We rewrite this as (from intermediate steps in proving the above)
$$u(x) = -2\frac{d}{dx}\sum_{k=1}^{N}\underbrace{c_k\tilde{\psi}_k e^{-\eta_k x}}_{=:P_k}$$
i.e. $u(x) = -2\frac{d}{dx}\sum_{k=1}^{N} P_k(x)$.

where
$$(6.5)\qquad \frac{e^{2\eta_m x}}{c_m^2}P_m(x) + \sum_{j=1}^{N}\frac{P_j(x)}{\eta_m+\eta_j} = 1 \qquad \left(\text{from } (I+C)\Psi = E\right)$$

Differentiating in x,
$$(6.6)\qquad \frac{e^{2\eta_m x}}{c_m^2}P_m'(x) + \frac{2\eta_m}{c_m^2}e^{2\eta_m x}P_m(x) + \sum_{j=1}^{N}\frac{P_j'(x)}{\eta_m+\eta_j} = 0$$

Consider the coordinate frame
$$\chi = x - 4\eta_k^2 t \qquad (\text{think of } \chi \text{ as fixed, while } t\to\pm\infty)$$

Using $c_m(t) = c_m(0)\,e^{4\eta_m^2 t}$, we get
$$(c_m(t))^{-2}e^{2\eta_m x} = \underbrace{(c_m(0))^{-2}e^{2\eta_m\chi}}_{=:\tilde{c}_m(\chi)}\,e^{-8\eta_m(\eta_m^2-\eta_k^2)t}$$

In this coord frame, (6.5) & (6.6) become
$$\begin{cases}\tilde{c}_m(\chi)\,e^{-8\eta_m(\eta_m^2-\eta_k^2)t}P_m + \sum_{j=1}^{N}\frac{P_j}{\eta_m+\eta_j} = 1 & (6.5')\\ \tilde{c}_m(\chi)\,e^{-8\eta_m(\eta_m^2-\eta_k^2)t}P_m' + \sum_{j=1}^{N}\frac{P_j'}{\eta_m+\eta_j} = -2\eta_m\tilde{c}_m(\chi)\,e^{-8\eta_m(\eta_m^2-\eta_k^2)t}P_m & (6.6')\end{cases}$$

Order the eigenvalues
$$\eta_1 > \eta_2 > \cdots > \eta_N > 0$$

Case $t\to+\infty$

The largest terms in (6.5') & (6.6') depend on whether $m<k$, $m=k$ or $m>k$.

(6.5') as $t\to+\infty$
$$\begin{cases}\sum_{j=1}^{N}\frac{P_j}{\eta_m+\eta_j} = 1 & \text{if } m=1,2,\dots,k-1.\\ \tilde{c}_k P_k + \sum_{j=1}^{N}\frac{P_j}{\eta_m+\eta_j} = 1 & \text{if } m=k\\ P_m = 0 & \text{if } m=k+1,\dots,N.\end{cases}$$

Or, equivalently,
$$(6.7)\qquad \sum_{j=1}^{k}\frac{P_j}{\eta_m+\eta_j} = 1 - \delta_{km}\tilde{c}_k P_k, \quad m=1,\dots,k \qquad \left(\delta_{km} = \begin{cases}0 & \text{if } k\neq m\\ 1 & \text{if } k=m\end{cases}\right)$$

Similarly, from (6.6')
$$(6.8)\qquad \sum_{j=1}^{k}\frac{P_j'}{\eta_m+\eta_j} = -\delta_{km}\tilde{c}_k\left(2\eta_k P_k + P_k'\right), \quad m=1,\dots,k$$

Note: $P_m' = 0 = 2\eta_m P_m$, for $m=k+1,\dots,N$.

Note:
$$0 < \det C = \det\left[\frac{c_j c_k e^{-(\eta_j+\eta_k)x}}{\eta_j+\eta_k}\right]$$
$$= \left(\prod_{j=1}^{N}c_j\right)^2\exp\left[-\sum_{i=1}^{N}2\eta_i x\right]\det(K_N)$$
where $(K_k)_{jm} = \frac{1}{\eta_j+\eta_m}$, $j,m=1,\dots,k$.

$\Rightarrow \det(K_k) > 0$ for all k.

Let cofactors of K_k be $\{R_{im}\}$.

Consider $i\neq j$. Then
$$\sum_{m=1}^{k}\frac{R_{im}}{\eta_j+\eta_m} = 0$$
because the LHS is the det of a matrix obtained from K_k by replacing i^{th} row by j^{th} row.

But $i=j \Rightarrow$ LHS $= \det(K_k)$.

Multiply (6.7) by R_{im} and sum over m:
$$\sum_{m=1}^{k}\sum_{j=1}^{k}\frac{P_j R_{im}}{\eta_j+\eta_m} = \sum_{m=1}^{k}R_{im} - \tilde{c}_k R_{ik}P_k.$$

$$\Rightarrow \quad (6.9) \qquad P_i \det(K_k) = \sum_{m=1}^{k} R_{im} - \tilde{c}_k R_{ik} P_k$$

Similarly from (6.8), we get

$$(6.10) \qquad P_i' \det(K_k) = -\tilde{c}_k R_{ik}\left(2\eta_k P_k + P_k'\right)$$

Let L_k be the matrix obtained from K_k by replacing last column by a column of 1's.

$$\Rightarrow \quad \det(L_k) = \sum_{m=1}^{k} R_{km}.$$

Taking $i=k$, in (6.9) $\Rightarrow$ $$P_k = \frac{\det(L_k)}{\det(K_k) + \tilde{c}_k \det(K_{k-1})} \qquad (6.11)$$

$$(6.10) \Rightarrow \quad P_k' = \frac{-2\eta_k \tilde{c}_k P_k \det(K_{k-1})}{\det(K_k) + \tilde{c}_k \det(K_{k-1})} \qquad (6.12)$$

Using these, we get

$$\sum_{i=1}^{N} P_i'(x) = \frac{-2\eta_k \tilde{c}_k}{\left[\dfrac{\det(K_k)}{\det(L_k)} + \tilde{c}_k \dfrac{\det(K_{k-1})}{\det(L_k)}\right]^2} \quad ; \text{ we used (6.12) in (6.10)}$$

Exercise:

(i) $$\det K_k = \frac{\prod_{i=1}^{k-1}(\eta_k - \eta_i)}{\prod_{i=1}^{k}(\eta_k + \eta_i)} \cdot \det L_k$$

(ii) $$\det L_k = \frac{\prod_{i=1}^{k-1}(\eta_k - \eta_i)}{\prod_{i=1}^{k-1}(\eta_k + \eta_i)} \cdot \det K_{k-1}$$

$$\Rightarrow \lim_{\substack{t\to+\infty \\ x \text{ fixed}}} u(x,t) = -2\lim_{\substack{t\to\infty \\ x\text{ fixed}}} \sum_{k=1}^{k} P_i'(x)$$

$$= \frac{4\eta_k \tilde{c}_k}{\left[\dfrac{\prod_{i=1}^{k-1}(\eta_k-\eta_i)}{\prod_{i=1}^{k}(\eta_k+\eta_i)} + \tilde{c}_k \dfrac{\prod_{i=1}^{k-1}(\eta_k+\eta_i)}{\prod_{i=1}^{k-1}(\eta_k-\eta_i)}\right]^2}$$

$$= \frac{4\eta_k (c_k(0))^{-2} e^{2\eta_k x} \cdot 4\eta_k^2 \prod_{i=1}^{k-1}\left(\dfrac{\eta_k+\eta_i}{\eta_k-\eta_i}\right)^2}{\left[1 + 2\eta_k (c_k(0))^{-2} e^{+2\eta_k x} \prod_{i=1}^{k-1}\left(\dfrac{\eta_k+\eta_i}{\eta_k-\eta_i}\right)^2\right]^2}$$

$$= \frac{8\eta_k^2 e^{2\eta_k(x-x_k)}}{\left(1 + e^{2\eta_k(x-x_k)}\right)^2}, \quad \text{where } e^{-2\eta_k x_k} = 2\eta_k (c_k(0))^{-2} \prod_{i=1}^{k-1}\left(\frac{\eta_k+\eta_i}{\eta_k-\eta_i}\right)^2$$

$$= 2\eta_k^2 \operatorname{sech}^2\left[\eta_k\left(x - 4\eta_k^2 t - x_k\right)\right] \qquad !$$