$$\Rightarrow \quad (6.9) \qquad P_i \det(K_k) = \sum_{m=1}^{k} R_{im} - \tilde{c}_k R_{ik} P_k$$

Similarly from (6.8), we get

$$(6.10) \qquad P_i' \det(K_k) = -\tilde{c}_k R_{ik} \left(2\eta_k P_k + P_k'\right)$$

Let L_k be the matrix obtained from K_k by replacing last column by a column of 1's.

$$\Rightarrow \quad \det(L_k) = \sum_{m=1}^{k} R_{km}.$$

Taking $i=k$, in (6.9) $\Rightarrow$

$$P_k = \frac{\det(L_k)}{\det(K_k) + \tilde{c}_k \det(K_{k-1})} \qquad (6.11)$$

$$(6.10) \Rightarrow \qquad P_k' = \frac{-2\eta_k \tilde{c}_k P_k \det(K_{k-1})}{\det(K_k) + \tilde{c}_k \det(K_{k-1})} \qquad (6.12)$$

Using these, we get

$$\sum_{i=1}^{N} P_i'(x) = \frac{-2\eta_k \tilde{c}_k}{\left[\dfrac{\det(K_k)}{\det(L_k)} + \tilde{c}_k \dfrac{\det(K_{k-1})}{\det(L_k)}\right]^2} \quad ; \text{ we used (6.12) in (6.10)}$$

Exercise:

(i) $$\det K_k = \frac{\prod_{i=1}^{k-1} (\eta_k - \eta_i)}{\prod_{i=1}^{k} (\eta_k + \eta_i)} \cdot \det L_k$$

(ii) $$\det L_k = \frac{\prod_{i=1}^{k-1} (\eta_k - \eta_i)}{\prod_{i=1}^{k-1} (\eta_k + \eta_i)} \cdot \det K_{k-1}$$

$t \to \infty$

$$\lim_{\text{fixed } x} u(x,t) = -2 \lim_{\substack{t\to\infty \\ x \text{ fixed}}} \sum_{i=1}^{k} P_i'(x)$$

$$= \frac{4\eta_k \tilde{c}_k}{\left[\dfrac{\prod_{i=1}^{k-1}(\eta_k-\eta_i)}{\prod_{i=1}^{k}(\eta_k+\eta_i)} + \tilde{c}_k \dfrac{\prod_{i=1}^{k-1}(\eta_k+\eta_i)}{\prod_{i=1}^{k-1}(\eta_k-\eta_i)}\right]^2}$$

$$= \frac{4\eta_k (c_k(0))^{-2} e^{2\eta_k x} \, 4\eta_k^2 \prod_{i=1}^{k-1} \left(\eta_k - \eta_i\right)}{\left[1 + 2\eta_k (c_k(0))^{-2} e^{+2\eta_k x} \prod_{i=1}^{k-1}\left(\dfrac{\eta_k+\eta_i}{\eta_k-\eta_i}\right)^2\right]^2}$$

$$= \frac{8\eta_k^2 e^{2\eta_k(x-x_k)}}{\left(1 + e^{2\eta_k(x-x_k)}\right)^2}, \quad \text{where } e^{-2\eta_k x_k} = 2\eta_k (c_k(0))^{-2} \prod_{i=1}^{k-1}\left(\frac{\eta_k+\eta_i}{\eta_k-\eta_i}\right)^2$$

$$= 2\eta_k^2 \operatorname{sech}^2\left[\eta_k\left(x - 4\eta_k^2 t - x_k\right)\right] \qquad !$$

Case $t \to -\infty$:

Equations (6.5') & (6.6') now give non-zero values of P_m, P_m' in the range $m = k, \dots, N$, while they are zero for $m = 1, \dots, k-1$.

Resulting equations have same structure and asymptotic behaviour, except that the phase is now given by

$$e^{-2\eta_k \bar{x}_k} = 2\eta_k (c_k(0))^{-2} \prod_{i=k+1}^{N} \left(\frac{\eta_k+\eta_i}{\eta_k-\eta_i}\right)^2$$

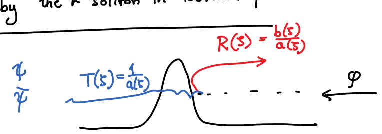
So we get the k-th soliton again, but with a phase shift.

Remark:

(i) the number of eigenvalues corresponds to the number of solitons present in a reflectionless potential.

(ii) each discrete eigenvalue η_k gives soliton's amplitude and wavespeed.

(iii) the total phase shift $x_k - \bar{x}_k$ is the sum of the shifts suffered by the k^{th} soliton in isolated pairwise interaction with every other soliton.



7. Transformations

Integrability $\leftrightarrow$ solvability through an associated pair of linear systems.

$\downarrow$

existence of transformations of solutions.

7.1 Darboux transformations

Consider 2 copies of the (stationary) Schrödinger eqn

$$y'' + \left(u(x) - \xi^2\right) y = 0 \qquad (7.1)$$

$$z'' + \left(v(x) - \xi^2\right) z = 0 \qquad (7.2)$$

where solutions are related by

$$z(x, \xi) = A(x, \xi) y + y_x \qquad (7.3)$$

Then $\exists\ \tilde{y}(x)$ satisfying

$$\tilde{y}'' + \left(u(x) + \tilde{\lambda}\right) \tilde{y} = 0$$

s.t. (i) $A = -\dfrac{\tilde{y}'}{\tilde{y}}$

(ii) $v(x) = u(x) + 2\left(\ln \tilde{y}\right)'' \qquad (7.4)$

Proof: Substitute (7.3) into (7.2) and collect coeffts of y, y' separately.

$z = Ay + y' \Rightarrow z' = A'y + Ay' + y'' = \left[A' - (u - \xi^2)\right] y + A y'$

$z'' = \ldots$

$\Rightarrow\ y' > \quad u - v = 2A'$

$y > \quad A'' - 2AA' - u_x = 0$

$$\Rightarrow \boxed{A' - A^2 - u = \text{const} = \tilde{\lambda}}, \text{ say} \qquad (7.5)$$

Riccati eqn

Linearizable: $A = -\dfrac{\tilde{y}'}{\tilde{y}} \Rightarrow A' = -\dfrac{\tilde{y}''}{\tilde{y}} + \dfrac{\tilde{y}'^2}{\tilde{y}^2}$

$$\Rightarrow -\frac{\tilde{y}''}{\tilde{y}} - u = \tilde{\lambda} \Rightarrow \tilde{y}'' + (u + \tilde{\lambda})\tilde{y} = 0$$

$$\Rightarrow v = u - 2\left(-\frac{\tilde{y}'}{\tilde{y}}\right)' = u + 2\left(\ln \tilde{y}\right)''.$$

as desired $\square$.

In other words, from (7.3),

$$z = y_x - y\left(\frac{\tilde{y}'}{\tilde{y}}\right) \quad \text{— called a Darboux transformation.}$$

Remark: $2\left(\ln \tilde{y}\right)'' = \dfrac{\tilde{y}''}{\tilde{y}} - \tilde{y}\left(\dfrac{1}{\tilde{y}}\right)''$ [Check!]

Then it follows that

$$v(x) = u(x) + \frac{\tilde{y}''}{\tilde{y}} - \tilde{y}\left(\frac{1}{\tilde{y}}\right)''$$

$$= \cancel{u(x)} - \left(\cancel{u(x)} + \tilde{\lambda}\right) - \tilde{y}\left(\frac{1}{\tilde{y}}\right)''$$

So (7.2) becomes

$$z'' + \left(-\xi^2 + \tilde{\xi}^2 - \tilde{y}\left(\frac{1}{\tilde{y}}\right)''\right) z = 0 \qquad \text{where } \tilde{\lambda} = -\tilde{\xi}^2.$$

So when $\tilde{\xi} = \xi$ one solution is $z = \dfrac{1}{\tilde{y}}$.

7.2 Miura transformations

The transformation (7.5) now rewritten for $A = -v$, $\tilde{\lambda} = \lambda$ is

$$u = -v' - v^2 + \lambda \qquad (7.6)$$

$$\Rightarrow \quad u_t + 6uu_x + u_{xxx} = -\left(2v + \frac{\partial}{\partial x}\right)\left(v_t - 6v^2 v_x + v_{xxx} - 6\lambda v_x\right)$$

[Check!]

So if v satisfies

$$v_t - 6v^2 v_x + v_{xxx} - 6\lambda v_x = 0 \qquad (7.7)$$

then u satisfies the KdV eqn.

(7.7) is called a generalized modified KdV eqn. (The MKdV is case $\lambda = 0$.)

(7.6) is a Miura transformation.