existence of transformations of solutions.

7.1 Darboux transformations

Consider 2 copies of the (stationary) Schrödinger eqn

$$y'' + (u(x) - \zeta^2) y = 0 \qquad (7.1)$$

$$z'' + (v(x) - \zeta^2) z = 0 \qquad (7.2)$$

where solutions are related by

$$z(x,\zeta) = A(x,\zeta) y + y_x \qquad (7.3)$$

Then $\exists\ \tilde{y}(x)$ satisfying

$$\tilde{y}'' + (u(x) + \tilde{\lambda}) \tilde{y} = 0$$

s.t. (i) $A = -\dfrac{\tilde{y}'}{\tilde{y}}$

(ii) $v(x) = u(x) + 2(\ln \tilde{y})'' \qquad (7.4)$

Proof: Substitute (7.3) into (7.2) and collect coeffts of y, y' separately.

$$z = Ay + y' \Rightarrow z' = A'y + Ay' + y'' = [A' - (u - \zeta^2)]y + Ay'$$

$$z'' = \ldots$$

$$\Rightarrow \quad y' > \quad u - v = 2A'$$

$$y > \quad A'' - 2AA' - u_x = 0$$

$$\Rightarrow \boxed{A' - A^2 - u = \text{const} = \tilde{\lambda}}, \text{ say} \qquad (7.5)$$

Riccati eqn

Linearizable: $A = -\dfrac{\tilde{y}'}{\tilde{y}} \Rightarrow A' = -\dfrac{\tilde{y}''}{\tilde{y}} + \dfrac{\tilde{y}'^2}{\tilde{y}^2}$

$$\Rightarrow -\frac{\tilde{y}''}{\tilde{y}} - u = \tilde{\lambda} \Rightarrow \tilde{y}'' + (u + \tilde{\lambda})\tilde{y} = 0$$

$$\Rightarrow v = u - 2\left(-\frac{\tilde{y}'}{\tilde{y}}\right)' = u + 2(\ln \tilde{y})''.$$

as desired □.

In other words, from (7.3),

$$z = y_x - y\left(\frac{\tilde{y}'}{\tilde{y}}\right) \quad \text{— called a Darboux transformation.}$$

Remark: $2(\ln \tilde{y})'' = \dfrac{\tilde{y}''}{\tilde{y}} - \tilde{y}\left(\dfrac{1}{\tilde{y}}\right)''$ [Check!]

Then it follows that

$$v(x) = u(x) + \frac{\tilde{y}''}{\tilde{y}} - \tilde{y}\left(\frac{1}{\tilde{y}}\right)''$$

$$= \cancel{u(x)} - (\cancel{u(x)} + \tilde{\lambda}) - \tilde{y}\left(\frac{1}{\tilde{y}}\right)''$$

So (7.2) becomes

$$z'' + \left(-\zeta^2 + \tilde{\zeta}^2 - \tilde{y}\left(\frac{1}{\tilde{y}}\right)''\right) z = 0 \qquad \text{where } \tilde{\lambda} = -\tilde{\zeta}^2.$$

So when $\tilde{\zeta} = \zeta$ one solution is $z = \dfrac{1}{\tilde{y}}$.

7.2 Miura transformations

The transformation (7.5) now rewritten for $A = -v$, $\tilde{\lambda} = \lambda$ is

$$u = -v' - v^2 + \lambda \qquad (7.6)$$

$$\Rightarrow \quad u_t + 6uu_x + u_{xxx} = -\left(2v + \frac{\partial}{\partial x}\right)\left(v_t - 6v^2 v_x + v_{xxx} - 6\lambda v_x\right)$$

[Check!]

So if v satisfies

$$v_t - 6v^2 v_x + v_{xxx} - 6\lambda v_x = 0 \qquad (7.7)$$

then u satisfies the KdV eqn.

(7.7) is called a generalized modified KdV eqn. (The MKdV is case $\lambda = 0$.)

(7.6) is a Miura transformation.

Next lecture to be held Thursday 14th Sept @ 2pm in Rm 829.

Note: $u = \lambda - v^2 - v_x$

where $v_t - 6v^2 v_x + v_{xxx} - 6\lambda v_x = 0$ is invariant under $v \mapsto -v$.

$\Rightarrow \tilde{u} = \lambda - v^2 + v_x$ is again a solution of the KdV equation.

$$\Rightarrow \begin{cases} \tilde{u} + u = 2\lambda - 2v^2 \\ \tilde{u} - u = +2v_x \end{cases}$$

Taking $u = w_x$, we obtain (using $\tilde w - w = 2v$)

$$(\tilde w + w)_x = 2\lambda - \tfrac{1}{2}(\tilde w - w)^2 \qquad (7.8)$$

Notice that the KdV eqn becomes

$$w_{xt} + 6 w_x w_{xx} + w_{xxxx} = 0$$

$$\Rightarrow \quad w_t + 3w_x^2 + w_{xxx} = 0 \qquad (\text{wlog})$$

$\nwarrow$ the potential KdV equation

Also $\tilde w_t + 3\tilde w_x^2 + \tilde w_{xxx} = 0$

$$\Rightarrow \quad (\tilde w + w)_t + 3(\tilde w_x^2 + w_x^2) + (\tilde w + w)_{xxx} = 0$$

But from (7.8), we get

$$(\tilde w + w)_{xx} = -(\tilde w - w)(\tilde w_x - w_x)$$

$$(\tilde w + w)_{xxx} = -(\tilde w_x - w_x)^2 - (\tilde w - w)(\tilde w - w)_{xx}$$

$$\Rightarrow \quad (\tilde w + w)_t = -2\tilde w_x^2 - 2w_x^2 - 2\tilde w_x w_x + (\tilde w - w)(\tilde w - w)_{xx} \qquad (7.9)$$

Theorem: If $w(x,t)$ solves the potential KdV equation, then the function $\tilde w(x,t)$ satisfying (for any λ)

$$\begin{cases} (\tilde w + w)_x = 2\lambda - \tfrac{1}{2}(\tilde w - w)^2 & (7.8) \\ (\tilde w + w)_t = (\tilde w - w)(\tilde w - w)_{xx} - 2(\tilde w_x^2 + w_x\tilde w_x + w_x^2) & (7.9) \end{cases}$$

with $\tilde w \neq w$, also satisfies the PKdV, i.e.

$$\tilde w_t + 3\tilde w_x^2 + \tilde w_{xxx} = 0$$

Proof: $\partial_t (7.8) \Rightarrow$

$$(\tilde w + w)_{xt} = -(\tilde w - w)(\tilde w - w)_t \qquad (*)$$

$\partial_x (7.9) \Rightarrow$

$$(\tilde w + w)_{tx} = (\tilde w_x - w_x)(\tilde w_{xx} - w_{xx}) + (\tilde w - w)(\tilde w_{xxx} - w_{xxx})$$
$$- 2(2\tilde w_x \tilde w_{xx} + w_{xx}\tilde w_x + w_x \tilde w_{xx} + 2w_x w_{xx})$$

$$= (\tilde w - w)(\tilde w_{xxx} - w_{xxx}) - 3(\tilde w_x + w_x)(\tilde w_{xx} + w_{xx}) \qquad (\Delta)$$

& Using $\partial_x (7.8)$: $\tilde w_{xx} + w_{xx} = -(\tilde w - w)(\tilde w_x - w_x)$

Subtracting (Δ) from $(*)$

$$0 = (\tilde w - w)\left[\left(w_t + 3w_x^2 + w_{xxx}\right) - \left(\tilde w_t + 3\tilde w_x^2 + \tilde w_{xxx}\right)\right] \qquad (\text{Check!})$$

$\Rightarrow$ either $\tilde w = w$ or $\tilde w$ satisfies the PKdV eqn. $\square$

The system (7.8), (7.9) forms a system of transformations called <u>Bäcklund transformations</u>

Example: Suppose $w \equiv 0$.

Then $(7.8) \Rightarrow \tilde w_x = 2\lambda - \tfrac{1}{2}\tilde w^2 \quad (*) \longleftarrow$ Riccati eqn for $\tilde w$, can be linearized.

$(7.9) \Rightarrow \tilde w_t = -2\tilde w_x^2 + \tilde w \tilde w_{xx}$

$(*)$ has the solution: $\tilde w = 2k \tanh(kx + c(t))$, $\lambda = k^2$

$\Rightarrow$ 1-soliton solution of KdV. Applying the BT again gives a 2-soliton solution, ..., N-soliton solution after N applications (N pos integer).

Note that the BT above depends on a parameter λ.

Suppose we take

$$BT_\lambda: \quad w \overset{\lambda}{\longmapsto} \tilde w \qquad (\tilde w + w)_x = 2\lambda - \tfrac{1}{2}(\tilde w - w)^2$$

$$BT_\mu: \quad w \overset{\mu}{\longmapsto} \hat w \qquad (\hat w + w)_x = 2\mu - \tfrac{1}{2}(\hat w - w)^2$$

And apply BT_μ to $\tilde w$ and BT_λ to $\hat w$:

$$BT_\mu \circ BT_\lambda: \quad \tilde w \overset{\mu}{\longmapsto} \hat{\tilde w} \qquad (\hat{\tilde w} + \tilde w)_x = 2\mu - \tfrac{1}{2}(\hat{\tilde w} - \tilde w)^2 \qquad (7.10)$$

$$BT_\lambda \circ BT_\mu: \quad \hat w \overset{\lambda}{\longmapsto} \tilde{\hat w} \qquad (\tilde{\hat w} + \hat w)_x = 2\lambda - \tfrac{1}{2}(\tilde{\hat w} - \hat w)^2 \qquad (7.11)$$

Schematically:

$$\begin{tikzcd} & \tilde w \arrow[dr, "BT_\mu"] & \\ w \arrow[ur, "BT_\lambda"] \arrow[dr, "BT_\mu"'] & & \hat{\tilde w} \overset{?}{=} \tilde{\hat w} \\ & \hat w \arrow[ur, "BT_\lambda"'] & \end{tikzcd}$$

$\Big\}$ Are these equal?

First we remove $\tilde w$ from (7.10).

From (7.8) $\quad \tilde{w}_x = 2\lambda - w_x - \frac{1}{2}(\tilde{w} - w)^2 \Rightarrow$ in (7.10) we get

$$\begin{aligned}\hat{\tilde{w}}_x - w_x &= 2\mu - 2\lambda + \tfrac{1}{2}\left(\cancel{\tilde{w}^2} - 2\tilde{w}w + w^2\right) - \tfrac{1}{2}\left(\hat{\tilde{w}}^2 - 2\hat{\tilde{w}}\tilde{w} + \cancel{\tilde{w}^2}\right) \\ &= 2(\mu - \lambda) - \tilde{w}w + \hat{\tilde{w}}\tilde{w} - \tfrac{1}{2}\left(\hat{\tilde{w}} - w\right)\left(\hat{\tilde{w}} + w\right)\end{aligned}$$

$$\Leftrightarrow \qquad \tilde{w} = \frac{1}{2}\left(\hat{\tilde{w}} + w\right) + \frac{\left(\hat{\tilde{w}} - w\right)_x + 2(\lambda - \mu)}{\left(\hat{\tilde{w}} - w\right)} \qquad (\text{Check!})$$

Use this and its x-derivative in (7.10) to show

$$\lambda + \mu = \left(\hat{\tilde{w}} + w\right)_x + \partial_x^2\left[\log\left(\hat{\tilde{w}} - w\right)\right] + \frac{1}{2}\left[\partial_x \log\left(\hat{\tilde{w}} - w\right)\right]^2 + \frac{1}{8}\left(\hat{\tilde{w}} - w\right)^2 + 2\frac{(\lambda - \mu)^2}{\left(\hat{\tilde{w}} - w\right)^2}$$

This is invariant under $\lambda \leftrightarrow \mu$ and $\hat{\tilde{\ }} \leftrightarrow \tilde{\hat{\ }}$. The same invariance holds if we had started with the t-part of the BT i.e. (7.9).

I.e. given w, λ, μ any solution $\hat{\tilde{w}}$ also solves the equation for $\tilde{\hat{w}}$.

$\therefore$ We identify $\hat{\tilde{w}} = \tilde{\hat{w}}$ (w.l.o.g.)

So we get the Bianchi permutability theorem

$$\begin{tikzcd} & \tilde{w} \arrow[dr, "BT_\mu"] & \\ w \arrow[ur, "BT_\lambda"] \arrow[dr, "BT_\mu"'] & & \hat{\tilde{w}} \\ & \hat{w} \arrow[ur, "BT_\lambda"'] & \end{tikzcd}$$