From (7.8) $\tilde{w}_x = 2\lambda - w_x - \frac{1}{2}(\tilde{w}-w)^2 \Rightarrow$ in (7.10) we get

$$\hat{\tilde{w}}_x - w_x = 2\mu - 2\lambda + \frac{1}{2}\left(\cancel{\tilde{w}^2} - 2\tilde{w}w + w^2\right) - \frac{1}{2}\left(\hat{\tilde{w}}^2 - 2\hat{\tilde{w}}\tilde{w} + \cancel{\tilde{w}^2}\right)$$

$$= 2(\mu-\lambda) - \tilde{w}w + \hat{\tilde{w}}\tilde{w} - \frac{1}{2}(\hat{\tilde{w}}-w)(\hat{\tilde{w}}+w)$$

$$\Leftrightarrow \quad \tilde{w} = \frac{1}{2}(\hat{\tilde{w}}+w) + \frac{(\hat{\tilde{w}}-w)_x + 2(\lambda-\mu)}{(\hat{\tilde{w}}-w)} \qquad \text{(Check!)}$$

Use this and its x-derivative in (7.10) to show

$$\lambda+\mu = (\hat{\tilde{w}}+w)_x + \partial_x^2\left[\log(\hat{\tilde{w}}-w)\right] + \frac{1}{2}\left[\partial_x \log(\hat{\tilde{w}}-w)\right]^2 + \frac{1}{8}(\hat{\tilde{w}}-w)^2 + 2\frac{(\lambda-\mu)^2}{(\hat{\tilde{w}}-w)^2}$$

This is invariant under $\lambda \leftrightarrow \mu$ and $\hat{\tilde{\ }} \leftrightarrow \tilde{\hat{\ }}$. The same invariance holds if we had started with the t-part of the BT i.e. (7.9).

I.e. given w, λ, μ any solution $\hat{\tilde{w}}$ also solves the equation for $\tilde{\hat{w}}$.

$\therefore$ We identify $\hat{\tilde{w}} = \tilde{\hat{w}}$ (w.l.o.g.)

So we get the Bianchi permutability theorem

$$\begin{tikzcd} & \tilde{w} \arrow[dr, "BT_\mu"] & \\ w \arrow[ur, "BT_\lambda"] \arrow[dr, "BT_\mu"'] & & \hat{\tilde{w}} \\ & \hat{w} \arrow[ur, "BT_\lambda"'] & \end{tikzcd}$$

Recall that we have Bäcklund transformations

$BT_\lambda : w \mapsto \tilde{w}$ $\qquad BT_\mu \circ BT_\lambda : \tilde{w} \to \hat{\tilde{w}}$

$BT_\mu : w \mapsto \hat{w}$ $\qquad BT_\lambda \circ BT_\mu : \hat{w} \to \tilde{\hat{w}}$

where

$$(\tilde{w}+w)_x = 2\lambda - \frac{(\tilde{w}-w)^2}{2} \qquad (7.8)$$

$$(\hat{w}+w)_x = 2\mu - \frac{(\hat{w}-w)^2}{2} \qquad (7.9)$$

$$(\hat{\tilde{w}}+\tilde{w})_x = 2\mu - \frac{(\hat{\tilde{w}}-\tilde{w})^2}{2} \qquad (7.10)$$

$$(\tilde{\hat{w}}+\hat{w})_x = 2\lambda - \frac{(\tilde{\hat{w}}-\hat{w})^2}{2} \qquad (7.11)$$

Consider (7.8) $-$ (7.9) :

$$(\tilde{w}-\hat{w})_x = 2(\lambda-\mu) - \left(\frac{\tilde{w}^2 - 2\tilde{w}w - \hat{w}^2 + 2\hat{w}w}{2}\right)$$

Similarly (7.10) $-$ (7.11)

$$(\tilde{w}-\hat{w})_x = 2(\mu-\lambda) - \left(\frac{-2\hat{\tilde{w}}\tilde{w} + \tilde{w}^2 + 2\tilde{\hat{w}}\hat{w} - \hat{w}^2}{2}\right)$$

Subtracting, we get

$$0 = 4(\lambda-\mu) + \tilde{w}w - \hat{w}w - \hat{\tilde{w}}\tilde{w} + \tilde{\hat{w}}\hat{w}$$

$$\Leftrightarrow \quad \boxed{(\hat{\tilde{w}}-w)(\hat{w}-\tilde{w}) = 4(\mu-\lambda)}$$

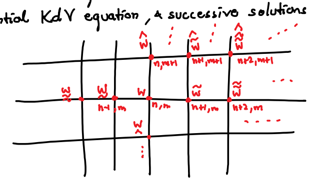
an algebraic equation relating the images of BTs to the original solution w.

We now transition to a discrete system by identifying

$w = w_{n,m}$ $\qquad \hat{w} = w_{n,m+1}$

$\tilde{w} = w_{n+1,m}$ $\qquad \hat{\tilde{w}} = w_{n+1,m+1}$

That is, we embed the equation on a lattice, where each vertex has a solution of the potential KdV equation, & successive solutions are related through a BT.



2D lattice composed of squares (or quadrilaterals)

We now have a partial difference eqn

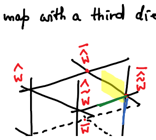
$$(w_{n+1,m+1} - w_{n,m})(w_{n,m+1} - w_{n+1,m}) = 4(\mu-\lambda)$$

Recall that the BT could have been applied with any number of parameters.

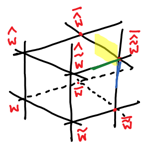
Take another parameter ν, say, and define

$BT_\nu : w \mapsto \bar{w}$

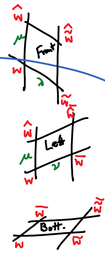
and identify this map with a third direction on a 3D lattice with pts (n,m,k).



3D lattice composed of cubes (or any space-filling solid with



3D lattice
composed of cubes
(or any space-filling solid with
6 faces and 8 vertices)



Is this a self-consistent lattice in 3D ?

front: $(\hat{\tilde{w}} - w)(\hat{w} - \tilde{w}) = 4(\mu-\lambda)$

left: $(\hat{\bar{w}} - w)(\hat{w} - \bar{w}) = 4(\mu-\nu)$

bottom: $(\tilde{\bar{w}} - w)(\bar{w} - \tilde{w}) = 4(\nu-\lambda)$

top: $(\hat{\tilde{\bar{w}}} - \hat{w})(\hat{\bar{w}} - \hat{\tilde{w}}) = 4(\nu-\lambda)$

right: $(\hat{\tilde{\bar{w}}} - \tilde{w})(\hat{\tilde{w}} - \tilde{\bar{w}}) = 4(\mu-\nu)$

rear: $(\hat{\tilde{\bar{w}}} - \bar{w})(\hat{\bar{w}} - \tilde{\bar{w}}) = 4(\mu-\lambda)$

$Q(w, \tilde{w}, \hat{w}, \hat{\tilde{w}}) = 0$
is polynomial
$Q(x, u, v, y)$
$= (y-x)(\cancel{v}-u) - 4(\mu-\lambda)$
$=0$

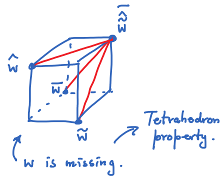
We solve the first 3 eqns for $\hat{\tilde{w}}, \hat{\bar{w}}, \tilde{\bar{w}}$ and substitute into the remaining equations to get $\hat{\tilde{\bar{w}}}$. E.g.

$$\bar{\hat{\tilde{w}}} = \hat{w} + 4\frac{(\nu-\lambda)}{(\hat{\bar{w}} - \hat{\tilde{w}})}$$ from the "top" eqn

$$= \hat{w} + \frac{4(\nu-\lambda)}{\left(w + \frac{4(\mu-\nu)}{\hat{w}-\bar{w}}\right) - \left(w + \frac{4(\mu-\lambda)}{\hat{w}-\tilde{w}}\right)}$$

....

$$= \frac{\hat{w}\,\tilde{w}\,(\lambda-\mu) + \hat{w}\,\bar{w}\,(\mu-\nu) + \bar{w}\,\tilde{w}\,(\nu-\lambda)}{\tilde{w}(\nu-\mu) + \hat{w}(\lambda-\nu) + \bar{w}(\mu-\lambda)}$$



This is invariant under permutation of $(\hat{w}, \tilde{w}, \bar{w})$ along with the same permutation of (μ, λ, ν).
$\therefore$ $\bar{\hat{\tilde{w}}} = \tilde{\hat{\bar{w}}} = \hat{\tilde{\bar{w}}}$. Called "consistency-around-a-cube" (CAC). The same argument extends to N-dim'l cubes.

Exercise: Show that the equation
$$\tilde{u} + \hat{u} + \bar{u} + u = 0$$
placed on the faces of a 3D cube is also CAC.