Integrable Systems:
Lecture 2

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Contents

In this lecture, we introduced the Lax representation for the KdV equation and proved the corresponding isospectral property, see Section 1.1: *Integrability of Korteweg - de Vries equation* from [Dub2009].

References

[Dub2009] B. A. Dubrovin, *Integrable systems and Riemann surfaces*, 2009. lecture notes, available at the author's web-page.

KdV: $u_t - 6uu_x + u_{xxx} = 0$

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we want to solve the initial value problem $u(x,0)$ - given

find $u(x,t)$

KdV and Schrödinger equation

Schrödinger operator: $L = -\partial_x^2 + u(x)$

$$L\psi = -\frac{\partial^2 \psi}{\partial x^2} + u(x)\psi$$

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(T) The KdV equation is equivalent to the following operator equation:

$$L_t = [L, A], \text{ with}$$

$$A = 4\partial_x^3 - 6u\partial_x - 3u_x .$$

$[L, A] = LA - AL$
commutator

Proof. $L = -\partial_x^2 + u \qquad L_t = u_t$

$$[L, A] = [-\partial_x^2 + u,\ 4\partial_x^3 - 6u\partial_x - 3u_x]$$
$$= 6[\partial_x^2, u\partial_x] + 3[\partial_x^2, u_x] + 4[u, \partial_x^3] - 6[u, u\partial_x]$$

$$[\partial_x^2, u\partial_x]\Psi = \partial_x^2(u\partial_x\Psi) - u\partial_x(\partial_x^2\Psi)$$
$$= \partial_x^2(u\Psi_x) - u\Psi_{xxx}$$
$$= u_{xx}\Psi_x + 2u_x\Psi_{xx} + \cancel{u\Psi_{xxx}} - \cancel{u\Psi_{xxx}}$$

3 $[\partial_x^2, u_x]\Psi = \partial_x^2(u_x\Psi) - u_x(\partial_x^2\Psi) =$

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$= u_{xxx}\Psi + 2u_{xx}\Psi_x + \cancel{u_x\Psi_{xx}} - \cancel{u_x\Psi_{xx}}$

$= (\boxed{u_{xxx}} + 2u_{xx}\partial_x)\Psi$

4 $[u, \partial_x^3]\Psi = u\partial_x^3\Psi - \partial_x^3(u\Psi)$

$= \cancel{u\Psi_{xxx}} - (\boxed{u_{xxx}\Psi} + 3u_{xx}\Psi_x + 3u_x\Psi_{xx} + \cancel{u\Psi_{xxx}})$

$[u, u\partial_x]\Psi = u(u\partial_x\Psi) - u\partial_x(u\Psi)$ $\quad -6$

$= \cancel{u^2\Psi_x} - u(u_x\Psi + \cancel{u\Psi_x})$

$= -uu_x\Psi$

$[\partial_x^2, u\partial_x] = u_{xx}\partial_x + 2u_x\partial_{xx}$ $\quad 6$

$[L, A] = 6uu_x - u_{xxx}$

$L_t = u_t$

$[L, A] = L_t \iff u$ satisfying KdV $\square$

$L_t = [L, A]$ - Lax representation of KdV

Corollary (isospectral property)

Let λ be an eigen-value of the Schrödinger operator, and ψ the eigenfunction,

$$L\psi = \lambda\psi.$$

$$(\psi, \psi) := \int_{-\infty}^{\infty} |\psi|^2 dx < \infty \qquad \psi \in L^2(-\infty, \infty)$$

Then $\dot\lambda = 0$. $\quad \left[\dot\lambda = \frac{\partial}{\partial t}\lambda\right]$

$$(\psi, \varphi) := \int_{-\infty}^{\infty} \psi \overline{\varphi}\, dx$$

Proof. $\quad L\psi = \lambda\psi \quad / \frac{\partial}{\partial t}$

$$\dot L\psi + L\dot\psi = \dot\lambda\psi + \lambda\dot\psi$$

$$[L, A]\psi + L\dot\psi = \dot\lambda\psi + \lambda\dot\psi$$

$$LA\psi - AL\psi + L\dot\psi = \dot\lambda\psi + \lambda\dot\psi$$

$$LA\psi - \lambda A\psi + L\dot\psi = \dot\lambda\psi + \lambda\dot\psi$$

$$L(A + \partial_t)\psi = \dot\lambda\psi + \lambda(A + \partial_t)\psi \quad / \text{ inner product with } \psi$$

$$(L(A+\partial_t)\psi, \psi) = \dot\lambda(\psi, \psi) + \lambda((A+\partial_t)\psi, \psi)$$

$$(L(A+\partial_t)\psi, \psi) = \dot\lambda(\psi, \psi) + ((A+\partial_t)\psi, L\psi)$$

L - self-conjugate $\qquad (L\psi, \varphi) = (\psi, L\varphi)$

$$\cancel{((A+\partial_t)\psi, L\psi)} = \dot\lambda(\psi, \psi) + \cancel{((A+\partial_t)\psi, L\psi)}$$

$$\dot\lambda(\psi, \psi) = 0 \implies \dot\lambda = 0$$

$\square$

KdV - as a dynamical system

$u(x)$

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λ $\quad L\Psi = \lambda\Psi \quad \Psi \in L^2$

$L - \lambda I$ - not injective

λ is in the discrete spectrum of L

λ is a first integral of the KdV