Integrable Systems: Lecture 3

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8 August 2017

Contents

In this lecture, we started to study the scattering theory for the Schrödinger operator. We followed the exposition from Section 1.2: *Elements of scattering theory for the Schrödinger operator* from [Dub2009].

References

[Dub2009] B. A. Dubrovin, *Integrable systems and Riemann surfaces*, 2009. lecture notes, available at the author's web-page.

Elements of the scattering theory for the Schrödinger operator

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scattering $\quad \psi \sim e^{\pm ikx}$ $\quad$ potential $u(x)$

from $x=+\infty$ to $x=-\infty$

Proposition 2 $\quad \det A = 1$

Proof. Wronskian $\quad W(\varphi_1,\varphi_2) = \det\begin{pmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{pmatrix}$

$$W' = \underbrace{\det\begin{pmatrix}\varphi_1' & \varphi_2'\\ \varphi_1' & \varphi_2'\end{pmatrix}}_{0} + \underbrace{\det\begin{pmatrix}\varphi_1 & \varphi_2\\ \varphi_1'' & \varphi_2''\end{pmatrix}}_{0} = 0$$

$\Rightarrow W = \text{const.}$

$$W(\varphi_1,\varphi_2) = \begin{vmatrix} e^{-ikx} & e^{ikx}\\ -ik e^{-ikx} & ik e^{ikx}\end{vmatrix} = 2ik$$

$$W(\psi_1,\psi_2) = 2ik$$

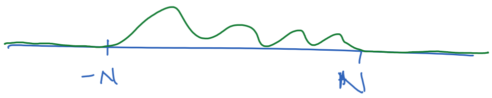
$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = A\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad \begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = A\begin{pmatrix}\psi_1'\\ \psi_2'\end{pmatrix}$$

$$\begin{pmatrix}\varphi_1 & \varphi_1'\\ \varphi_2 & \varphi_2'\end{pmatrix} = A\begin{pmatrix}\psi_1 & \psi_1'\\ \psi_2 & \psi_2'\end{pmatrix} \quad / \det$$

$\Rightarrow \det A = 1$ $\quad \square$

First, consider the case of compact support potential.

$$u(x) = 0 \quad \text{for } |x| > N$$



- consider the differential equation:

$$-\psi'' + u(x)\psi = k^2\psi(x)$$

I basis: $\begin{cases}\varphi_1 = e^{-ikx}\\ \varphi_2 = e^{ikx}\end{cases}$ for $x < -N$

II basis: $\begin{cases}\psi_1 = e^{-ikx}\\ \psi_2 = e^{ikx}\end{cases}$ for $x > N$ $\quad$ for all $k \in \mathbb{R}$

The scattering matrix

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix} a_{11}(k) & a_{12}(k)\\ a_{21}(k) & a_{22}(k)\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$$

Proposition 1 $\quad a_{22} = \overline{a_{11}}, \quad a_{12} = \overline{a_{21}}$

Proof $\quad \overline{\varphi_1} = \varphi_2, \quad \overline{\psi_1} = \psi_2$

$$\varphi_1 = a_{11}\psi_1 + a_{12}\psi_2$$
$$\overline{\varphi_1} = \overline{a_{11}}\,\overline{\psi_1} + \overline{a_{12}}\,\overline{\psi_2}$$
$$\varphi_2 = \overline{a_{11}}\psi_2 + \overline{a_{12}}\psi_1$$
$$\varphi_2 = a_{21}\psi_1 + a_{22}\psi_2 \quad \square$$

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We wish to extend our considerations to non-localised potentials $u(x)$, decaying at $|x| \to \infty$.

Assumptions: $u(x)$ - smooth real function

$|u(x)| \longrightarrow 0$ when $|x| \to \infty$

$$\int_{-\infty}^{\infty} (1+|x|)\, |u(x)|\, dx < \infty$$

$$-\psi'' + u\psi = \lambda\psi$$

Let $\lambda > 0$ k: $k^2 = \lambda$

Definition The Jost solutions are:

$$(*) \begin{cases} \psi_1(x,k) = \psi_1 \sim e^{-ikx} + o(1) & \text{when } x \to +\infty \\ \psi_2(x,k) = \psi_2 \sim e^{ikx} + o(1) & \text{when } x \to +\infty \end{cases}$$

Lemma For every $k \in \mathbb{R}$, there are exactly two solutions with the asymptotical behaviour (*). Moreover, ψ_2 extends analytically to the upper half plane in k ($\operatorname{Im} k > 0$) and $\psi_2(x,k)\, e^{-ikx} = 1 + O\left(\frac{1}{k}\right)$ when $|k| \to +\infty$.

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Proof. $\psi'' + k^2 \psi = u \psi$

Picard's method

1° step homogeneous equation: $\psi'' + k^2 \psi = 0$

$$\psi = a_1 e^{ikx} + a_2 e^{-ikx}$$

2° step variation of constants $\psi'' + k^2 \psi = f(x)$

$a_1 = a_1(x)$, $a_2 = a_2(x)$

from the equation: $a_1' (e^{ikx})' + a_2' (e^{-ikx})' = f$

$$a_1' e^{ikx} + a_2' e^{-ikx} = 0$$