Integrable Systems: Lecture 4

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Contents

In this lecture, we continued the study of the scattering theory for the Schrödinger operator. We followed the exposition from Section 1.2: *Elements of scattering theory for the Schrödinger operator* from [Dub2009].

References

[Dub2009] B. A. Dubrovin, *Integrable systems and Riemann surfaces*, 2009. lecture notes, available at the author's web-page.

$a_1' = \frac{1}{2ik} e^{-ikx} f \qquad a_2' = -\frac{1}{2ik} e^{ikx} f$

Wednesday, August 09, 2017 12:08 PM

$\Downarrow$

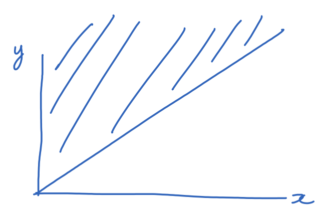
$$a_1(x) = a_1^0 + \frac{1}{2ik}\int_{x_0}^{x} f(y)\, e^{-iky}\, dy$$

$$a_2(x) = a_2^0 - \frac{1}{2ik}\int_{x_0}^{x} f(y)\, e^{iky}\, dy$$

$\Downarrow$

$$\Psi(x) = \frac{1}{2ik}\int_{x_0}^{x} e^{ik(x-y)} u(y)\Psi(y)\, dy - \frac{1}{2ik}\int_{x_0}^{x} e^{ik(y-x)} u(y)\Psi(y)\, dy + a_1^0 e^{ikx} + a_2^0 e^{-ikx}$$

We want: $\Psi \sim e^{ikx}$ as $x \to +\infty$ | We set $x_0 = +\infty$, $a_1^0 = 1$, $a_2^0 = 0$



$$\Psi(x) = e^{ikx} - \int_{x}^{+\infty} \frac{\sin k(x-y)}{k} u(y)\Psi(y)\, dy$$

$$\Psi = f_0 + f_1 + f_2 + \dots$$

$$f_0 = e^{ikx}$$

$$f_{n+1}(x) = -\int_{x}^{+\infty} \frac{\sin k(x-y)}{k} u(y) f_n(y)\, dy$$

We need to prove that the sum is uniformly convergent.

$$|f_n(x)| \leq \frac{1}{n!} U^n(x), \quad U(x) = \frac{1}{k}\int_{x}^{+\infty} |u(y)|\, dy$$

Proved by induction.

$n=0$: $|f_0(x)| = |e^{ikx}| = 1$ ✓

$n=1$: $f_1(x) = -\int_{x}^{+\infty} \frac{\sin k(x-y)}{k} u(y) f_0(y)\, dy$

$$|f_1(x)| \leq \int_{x}^{+\infty} \frac{|\sin k(x-y)|}{|k|} |u(y)|\,|f_0(y)|\, dy \leq \frac{1}{|k|}\int_{x}^{+\infty} |u(y)|\, dy \quad ✓$$

$n=2$: $|f_2(x)| \leq \int_{x}^{+\infty} \frac{1}{|k|} |u(y)|\,|f_1(y)|\, dy \leq \int_{x}^{+\infty} |u(y)|\,|U(y)|\, dy$

Wednesday, August 09, 2017 12:26 PM

$$|\psi(x) - e^{ikx}| \le |f_1| + |f_2| + |f_3| + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} U^n(x)$$

$$= \left| e^{U(x)} - 1 \right| \xrightarrow[\text{as } x \to +\infty]{} 0$$

Thus, we proved that ψ exists and is unique and $\psi \sim e^{ikx}$ as $x \to +\infty$

$\psi_2 = \overline{\psi_1}$ – existence and uniqueness hold for ψ_1 as well

Now we prove that ψ_2 is analytic for $\operatorname{Im} k > 0$.

Denote: $\psi(x) = e^{ikx} \chi(x)$

$$\chi(x) = 1 - \int_x^{+\infty} \frac{1 - e^{2ik(y-x)}}{2ik} u(y)\chi(y)\, dy$$

$$\left| e^{2ik(y-x)} \right| = e^{-2\operatorname{Im}(k)(y-x)} \longrightarrow 0$$
for $|k| \to \infty$

$\psi_2 e^{-ikx} = \chi = 1 + O\left(\frac{1}{k}\right)$ and it is analytic in k

$\square$

We proved that for each k,

there is a unique solution of

$$L\psi = k^2\psi$$

with $\psi \sim e^{ikx}$ as $x \to +\infty$

Similarly, we can prove existence and uniqueness of φ_1, φ_2:

$$\varphi_1 \sim e^{-ikx}$$
$$\varphi_2 \sim e^{ikx} \quad \text{as } x \to -\infty$$

We can prove that φ_1 extends analytically to $\operatorname{Im} k > 0$

Denote: $\psi := \psi_2$ we have $\psi_1 = \overline{\psi}$

$\varphi := \varphi_1$ we have $\varphi_2 = \overline{\varphi}$

Lemma $(\psi, \overline{\psi})$ and $(\varphi, \overline{\varphi})$ are two bases in the space of solutions. $(k \neq 0)$

Proof. $W(\psi, \overline{\psi})$ – does not depend on x

$$\lim_{x \to +\infty} W(\psi, \overline{\psi}) = 2ik \neq 0$$

$\square$

$$(\overline{\psi}, \psi) = (\psi_1, \psi_2)$$
$$(\varphi, \overline{\varphi}) = (\varphi_1, \varphi_2)$$

$$\varphi(x,k) = a(k)\,\overline{\psi}(x,k) + b(k)\,\psi(x,k)$$
$$\overline{\varphi}(x,k) = \overline{a}(k)\,\psi(x,k) + \overline{b}(k)\,\overline{\psi}(x,k)$$

the transition matrix is $A = \begin{pmatrix} a(k) & b(k) \\ \overline{b}(k) & \overline{a}(k) \end{pmatrix}$

__Lemma__ $|a(k)|^2 - |b(k)|^2 = 1$

__Proof__ $\det A = 1$ since
$$\begin{pmatrix} \varphi \\ \overline{\varphi} \end{pmatrix} = A \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix}$$
$\square$

__Lemma__ $a(k)$ can be analytically extended to $\operatorname{Im} k > 0$.

__Proof.__
$$W(\psi, \varphi) = W(\psi, a\overline{\psi} + b\psi) =$$
$$= a\,W(\psi, \overline{\psi}) + b\,\cancel{W(\psi,\psi)} \to 0$$
$$= a(k)\,W(\psi, \overline{\psi}) = -2ik\,a(k)$$
$$\Downarrow$$
$$a(k) = -\frac{1}{2ik} W(\psi, \varphi)$$
$\square$

$\ulcorner W(f,g) = \det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix} \lrcorner$