Integrable Systems: Lecture 5

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Contents

In this lecture, we introduced the scattering data for the Schrödinger operator. We followed the exposition from Section 1.2: *Elements of scattering theory for the Schrödinger operator* from [Dub2009].

References

[Dub2009] B. A. Dubrovin, *Integrable systems and Riemann surfaces*, 2009. lecture notes, available at the author's web-page.

Recall:

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KdV equation
$u_t - 6uu_x + u_{xxx} = 0$

Schrödinger operator:
$L = -\partial_x^2 + u(x)$

isospectral property: $L_t = [L, A]$

for $\psi \in \mathcal{L}_2(-\infty, +\infty)$

Jost solutions: $L\psi = \lambda\psi$

for $\lambda > 0$ $k^2 = \lambda$

$$\begin{cases} \psi \sim e^{ikx} \\ \bar{\psi} \sim e^{-ikx} \end{cases} \quad x \to +\infty$$

$$\begin{cases} \varphi \sim e^{-ikx} \\ \bar{\varphi} \sim e^{ikx} \end{cases} \quad x \to -\infty$$

We proved Jost solutions exist and ψ, φ can be analitically extended in k to $\operatorname{Im} k > 0$.

$\varphi = a\bar{\psi} + b\psi$

We proved that $a(k)$ also extends analytically to $\operatorname{Im} k > 0$.

Lemma $a(k) = 1 + O\left(\frac{1}{k}\right)$ as $|k| \to +\infty$

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Proof $a(k) = -\frac{1}{2ik} W(\psi, \varphi)$

$$\psi = e^{ikx} + e^{ikx} O\left(\frac{1}{k}\right)$$

$$\varphi = e^{-ikx} + e^{-ikx} O\left(\frac{1}{k}\right)$$

$$\psi' = ike^{ikx} + e^{ikx} O(1) \qquad \text{as } |k| \to +\infty$$

$$\varphi' = -ike^{-ikx} + e^{-ikx} O(1)$$

$$W(\psi, \varphi) = \psi\varphi' - \psi'\varphi = -2ik + O(1)$$

$$\Rightarrow a(k) = 1 + O\left(\frac{1}{k}\right)$$

□

Corollary All zeroes of $a(k)$ are on the imaginary axis.

Proof. The discrete spectrum of L is real. $L\psi = \lambda\psi$ $\psi \in L_2$

$$\Rightarrow \lambda \in \mathbb{R}$$

$$k^2 = \lambda < 0$$

□

Corollary $a(k)$ can have at most finitely many zeroes in the upper half-plane.

Proposition (relating zeroes of a with the discrete spectrum of L)

$a(k) = 0 \iff$ there is a solution of $L\psi = k^2\psi$ which is exponentially decaying as $x \to \pm\infty$.

Proof. $a(k) = 0 \iff W(\psi, \varphi) = 0$

$\iff \psi, \varphi$ are proportional

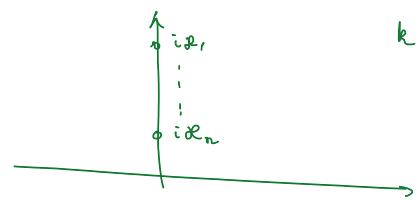
$\varphi \sim e^{-ikx}$ as $x \to -\infty$

$\psi \sim e^{ikx}$ as $x \to +\infty$

$$|e^{-ikx}| = e^{\operatorname{Im}(k)\cdot x} \longrightarrow 0 \text{ as } x \to -\infty \text{ since } \operatorname{Im} k > 0$$

$$|e^{ikx}| = e^{-\operatorname{Im}(k) x} \longrightarrow 0 \text{ as } x \to +\infty$$

□



$\varkappa_1 > \varkappa_2 > \dots > \varkappa_n > 0$

the eigen-functions: $\varphi_s(x) = \varphi(x, i\varkappa_s)$

$$\varphi_s(x) = \begin{cases} e^{\varkappa_s x}, & x \to -\infty \\ b_s e^{-\varkappa_s x}, & x \to +\infty \end{cases}$$

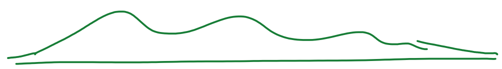
$b_s \in \mathbb{R}$

It can be proved that

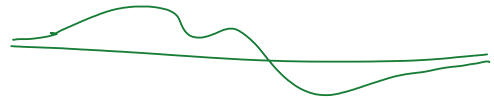
$(-1)^{s-1} b_s > 0$

$b_1 > 0 \quad b_2 < 0 \quad b_3 > 0$ - - - - - -

φ_1



φ_2 - one zero



φ_3 - two zeroes - - -

Scattering data

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1° reflection coefficient $r(k) := \frac{b(k)}{a(k)}$, $k \in \mathbb{R}$

2° $\varkappa_1, \ldots, \varkappa_n$

3° $b_1, \ldots, b_n$

Example

$$L = -\partial_x^2 + \alpha\delta(x), \quad \alpha \in \mathbb{R}$$

δ - Dirac delta-function

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$L\psi = \lambda\psi \quad / \int_{-\varepsilon}^{\varepsilon}$$

$$\int_{-\varepsilon}^{\varepsilon} \left(-\partial_x^2 \psi + \alpha\delta(x)\psi(x)\right) dx = \lambda \int_{-\varepsilon}^{\varepsilon} \psi(x)\,dx$$

ψ - continuous $\quad \varepsilon \to 0$

$$\alpha\psi(0) = \psi'(0_+) - \psi'(0_-)$$