Integrable Systems:
Lecture 6

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Contents

In this lecture, we described the behaviour in time of the scattering data corresponding to a potential satisfying the KdV. We followed the exposition from Section 1.2: *Elements of scattering theory for the Schrödinger operator* from [Dub2009].

References

[Dub2009] B. A. Dubrovin, *Integrable systems and Riemann surfaces*, 2009. lecture notes, available at the author's web-page.

Wednesday, August 16, 2017 12:02 PM

$$L = -\partial_x^2 + \alpha\,\delta(x)$$
$$\alpha \in \mathbb{R}$$

$$\alpha\,\psi(0) = \psi'(0_+) - \psi'(0_-)$$

<u>the discrete spectrum</u> $\lambda < 0$ $\quad \varkappa = \sqrt{-\lambda} > 0$

$$\psi(x) = \begin{cases} A e^{-\varkappa x}, & x > 0 \\ e^{\varkappa x}, & x < 0 \end{cases}$$

ψ is continuous at 0. $\quad \psi(0_+) = \psi(0_-)$

$$A = 1$$

The solutions of the Schrödinger equation

$$L\psi = \lambda\psi$$

will satisfy: $\quad -\psi'' = \lambda\psi$ on $(-\infty, 0)$

on $(0, +\infty)$

$$\alpha = \psi'(0_+) - \psi'(0_-)$$
$$= A\cdot(-\varkappa) - \varkappa = -\varkappa(A+1) = -2\varkappa$$
$$\Rightarrow \varkappa = -\frac{\alpha}{2}$$

If $\alpha > 0$, then the discrete spectrum is empty.

If $\alpha < 0$, then the discrete spectrum:

$$\boxed{\begin{array}{c} \varkappa_1 = -\frac{\alpha}{2} \\ \hline b_1 = 1 \end{array}}$$

continuous spectrum $\quad \lambda > 0 \quad k = \sqrt{\lambda}$

$$\psi = \begin{cases} e^{-ikx}, & x < 0 \\ A e^{-ikx} + B e^{ikx}, & x > 0 \end{cases}$$

$$\psi(0_+) = \psi(0_-) \qquad \psi(0_+) = A + B, \quad \psi(0_-) = 1 \quad \Rightarrow \boxed{A + B = 1}$$

$$\alpha\,\psi(0) = \psi'(0_+) - \psi'(0_-)$$
$$\alpha = A(-ik) + B(ik) - (-ik) \quad \Rightarrow \boxed{B - A + 1 = \frac{\alpha}{ik}}$$

$$\Rightarrow \quad A = 1 - \frac{\alpha}{2ik}, \quad B = \frac{\alpha}{2ik}$$

$$\psi(x) = \left(1 - \frac{\alpha}{2ik}\right) e^{-ikx} + \frac{\alpha}{2ik} e^{ik|x|}$$

Similarly,

$$\psi(x) = \left(1 - \frac{\alpha}{2ik}\right) e^{ikx} + \frac{\alpha}{2ik} e^{ik|x|}$$

$$\varphi = a\overline{\psi} + b\psi \quad \Rightarrow \quad a(k) = 1 - \frac{\alpha}{2ik}$$
$$b(k) = \frac{\alpha}{2ik}$$

the reflection coefficient:

$$\boxed{r(k) = \frac{b}{a} = -\frac{i\alpha}{2k + i\alpha}}$$

Scattering map

Wednesday, August 16, 2017 12:26 PM

potential $u(x) \rightsquigarrow$ scattering data

Theorem If $u(x,t)$ satisfies the KdV, then:

1° $\dot{r}(k) = 8ik^3 r(k)$

2° $\dot{\varkappa}_s = 0$

3° $\dot{b}_s = 8\varkappa_s^3 b_s$.

Similarly,
$$\dot{\varphi} + A\varphi = 4ik^3\varphi$$
$$\dot{\bar{\psi}} + A\bar{\psi} = 4ik^3\bar{\psi}$$
$$\dot{\bar{\varphi}} + A\bar{\varphi} = -4ik^3\bar{\varphi}$$

$\varphi = a\bar{\psi} + b\psi$

$$\dot{\varphi} = \dot{a}\bar{\psi} + a\dot{\bar{\psi}} + \dot{b}\psi + b\dot{\psi}$$

$$\dot{\varphi} + A\varphi = \dot{a}\bar{\psi} + a\dot{\bar{\psi}} + \dot{b}\psi + b\dot{\psi} + A(a\bar{\psi} + b\psi)$$
$$= \dot{a}\bar{\psi} + \dot{b}\psi + a(\dot{\bar{\psi}} + A\bar{\psi}) + b(\dot{\psi} + A\psi)$$

$$4ik^3(a\bar{\psi} + b\psi) = \dot{a}\bar{\psi} + \dot{b}\psi + 4ik^3 a\bar{\psi} - 4ik^3 b\psi$$

$4ik^3 \cdot a = \dot{a} + 4ik^3 a \implies \dot{a} = 0$

$4ik^3 b = \dot{b} - 4ik^3 b \implies \dot{b} = 8ik^3 b$

$\implies \dot{r} = 8ik^3 r$

Proof $k \in \mathbb{R}$, $\lambda = k^2$

$$\dot{L} = [L, A] \qquad A = 4\partial_x^3 - 6u\partial_x - 3u_x$$

$L\psi = \lambda\psi \qquad \dot{L}\psi + L\dot{\psi} = \dot{\lambda}\psi + \lambda\dot{\psi}$

$\dot{\lambda} = 0$ - by definition

$[L, A]\psi + L\dot{\psi} = \lambda\dot{\psi}$

$LA\psi - AL\psi + L\dot{\psi} = \lambda\dot{\psi}$

$LA\psi - \lambda A\psi + L\dot{\psi} = \lambda\dot{\psi}$

$L(\dot{\psi} + A\psi) = \lambda(\dot{\psi} + A\psi)$

$\implies \dot{\psi} + A\psi$ - an eigen-function

ψ - Jost solution

$$\dot{\psi} + A\psi = \alpha\bar{\psi} + \beta\psi$$

$\psi \sim e^{ikx}$ as $x \to +\infty$

$\dot{\psi} \sim 0 \qquad A\psi \sim -4ik^3 e^{ikx}$

$$\alpha\bar{\psi} + \beta\psi \sim \alpha e^{-ikx} + \beta e^{ikx}$$

$\implies \alpha = 0 \quad \beta = -4ik^3$

$$\dot{\psi} + A\psi = -4ik^3\psi$$

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b_s: $\dot{\varphi}_s + A\varphi_s = 4\varkappa_s^3 \varphi_s$

We get the desired relation in the same way.

$\square$

KdV equation

Wednesday, August 16, 2017 12:51 PM

$$u_t - 6uu_x + u_{xxx} = 0$$

$u_0(x) = u(x,0)$ - given

inverse scattering:

$$\begin{tikzcd}
u_0(x) \arrow[r, squiggly, "\text{scattering}"] & \kappa_s^0,\ b_s^0,\ r^0(k) \arrow[d, "\text{evolution in } t \text{, given by linear differential equations}"] \\
u(x,t) & \kappa_s,\ b_s,\ r(k) \arrow[l, squiggly, "\text{inverse scattering}"']
\end{tikzcd}$$