Integrable Systems: Lecture 7

Milena Radnović

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Contents

In this lecture, we showed that the inverse scattering can be reduced to a linear problem. We followed the exposition from [Dub2009, Section 1.3: *Inverse scattering*] and [NMPZ1984, Section 1.1: *Elements of the scattering theory. The inverse problem in the quantum theory of scattering*].

References

[Dub2009] B. A. Dubrovin, *Integrable systems and Riemann surfaces*, 2009. lecture notes, available at the author's web-page.

[NMPZ1984] S. Novikov, S. V. Manakov, L. P. Pitaevskiĭ, and V. E. Zakharov, *Theory of solitons*, Contemporary Soviet Mathematics, Consultants Bureau [Plenum], New York, 1984. The inverse scattering method; Translated from the Russian.

Inverse scattering

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1° $F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk + \sum_{s=1}^{n} \frac{b_s e^{-\varkappa_s x}}{i a'(i\varkappa_s)}$

2° Solve the Gel'fand-Levitan-Marchenko equation in $K = K(x,y)$:

$$K(x,y) + F(x+y) + \int_{-\infty}^{\infty} K(x,z) F(z+y) dz = 0$$

3° $u(x) = -2 \frac{d}{dx} K(x,x)$

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$$\varphi(x,k) = a(k)\bar{\Psi}(x,k) + b(k)\Psi(x,k) \quad \Big/ \cdot \frac{e^{ikx}}{a(k)}$$

$$\underbrace{\varphi(x,k)\frac{e^{ikx}}{a(k)}}_{\substack{\text{analytic in}\\ \text{Im}\,k>0,\\ \text{except having simple poles at } i\varkappa_s}} = \underbrace{\bar{\Psi}(x,k)e^{ikx}}_{\text{analytic in Im}\,k<0} + r(k)\Psi(x,k)e^{ikx}$$

$k \in \mathbb{R}$

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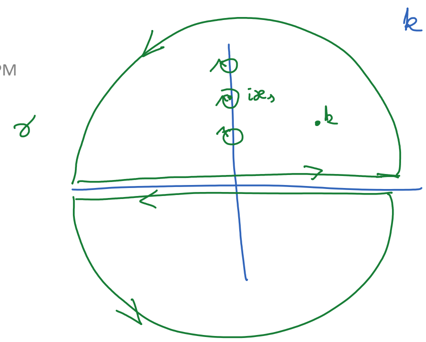
Define:

$$\Phi(x,k) := \begin{cases} \dfrac{\varphi(x,k)\,e^{ikx}}{a(k)}, & \text{Im}\,k>0 \\ \bar{\Psi}(x,k)\,e^{ikx}, & \text{Im}\,k<0 \end{cases}$$

- properties of Φ:
  - simple poles at $i\varkappa_s$
  - "step" at $k \in \mathbb{R}$
  - $\Phi \longrightarrow 1$ as $|k| \longrightarrow \infty$
  - analytic everywhere else



(*)
$$\Phi(x,k)-1=\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi(x,k')-1}{k'-k}\,dk'$$
$$=\sum_{s=1}^{n}\frac{\Gamma_s(x)}{k-i\varkappa_s}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{r(k')\Psi(x,k')e^{ik'x}}{k'-k}\,dk'$$

$\Gamma_s(x)$ - the residue of $\Phi(x,k)$ at $i\varkappa_s$

$$\Gamma_s(x)=\frac{\varphi(x,i\varkappa_s)\,e^{-\varkappa_s x}}{a'(i\varkappa_s)}=\frac{b_s\,\psi(x,i\varkappa_s)\,e^{-\varkappa_s x}}{a'(i\varkappa_s)}=$$
$$=\frac{b_s\,\Psi(x,-i\varkappa_s)\,e^{-\varkappa_s x}}{a'(i\varkappa_s)}=\frac{b_s\,\Phi(x,-i\varkappa_s)\,e^{-2\varkappa_s x}}{a'(i\varkappa_s)}$$

we will plug Φ from (*) into this expression

Take the limit $\operatorname{Im} k\to 0_-$ in (*):

$$\Psi(x,-k)e^{ikx}=1+\sum\frac{\Gamma_s(x)}{k-i\varkappa_s}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{r(k')\Psi(x,k')e^{ik'x}}{k'-k}\,dk'$$

$$\Gamma_s(x)=\frac{b_s e^{-2\varkappa_s x}}{a'(i\varkappa_s)}\left(1+i\sum_m\frac{\Gamma_m(x)}{\varkappa_m+\varkappa_s}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{r(k')\Psi(x,k')e^{ik'x}}{k'+i\varkappa_s}\,dk'\right)$$

We have got $(n+1)$ linear relations in $(n+1)$ functions: $\Gamma_1,\dots,\Gamma_n,\Psi$.

We want to deduce the GLM equation.

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$\bar{\Psi}(x,k)e^{ikx}$ - analytic for $\operatorname{Im} k<0$

and $\sim 1+O\left(\frac{1}{k}\right)$ as $|k|\to\infty$

$\Rightarrow \bar{\Psi}(x,k)e^{ikx}-1=\int_0^\infty A(x,y)\,e^{-iky}\,dy$

$$\bar{\Psi}(x,k)=e^{-ikx}+\int_0^\infty A(x,y)\,e^{-ik(x+y)}\,dy$$

change: $y'=x+y$

$$=e^{-ikx}+\int_x^\infty A(x,y'-x)\,e^{-iky'}\,dy'$$

$$=e^{-ikx}+\int_x^\infty A(x,y-x)\,e^{-iky}\,dy$$

$K=A(x,y-x)$

$$\bar{\Psi}(x,k)=e^{-ikx}+\int_x^\infty K(x,y)\,e^{-iky}\,dy$$