Integrable Systems: Lecture 8

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Contents

In this lecture, we derived the Gel'fand-Levitan-Marchenko equation. We followed the exposition from [Dub2009, Section 1.4: *Dressing operator*].

References

[Dub2009] B. A. Dubrovin, *Integrable systems and Riemann surfaces*, 2009. lecture notes, available at the author's web-page.

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$$\overline{\Psi}(x,k) = e^{-ikx} + \int_x^\infty K(x,y)\, e^{-iky}\, dy$$

$k \in \mathbb{R}$

$$\Psi(x,k) = e^{ikx} + \int_x^\infty \overline{K(x,y)}\, e^{iky}\, dy$$

Notice that

$$\overline{\Psi}(x,k) = \Psi(x,-k), \text{ for } k \in \mathbb{R}$$

$$\boxed{\Psi(x,k) = e^{ikx} + \int_x^\infty K(x,y)\, e^{iky}\, dy} \quad (**)$$

It follows: $\overline{K(x,y)} = K(x,y)$

So, K is real!

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$$\varphi = a\bar{\Psi} + b\Psi$$

Multiply by $\dfrac{e^{iky}}{a(k)}$ and integrate over real k.

$$\int_{-\infty}^{\infty}\left(\frac{\varphi(x,k)\,e^{iky}}{a(k)} - e^{ik(y-x)}\right)dk =$$

$$= \int_{-\infty}^{\infty}\left[\bar{\Psi}(x,k) - e^{-ikx} + r(k)\Psi(x,k)\right]e^{iky}\,dk$$

Right-hand side, 1st term:

$$\left(\bar{\Psi}(x,k) - e^{-ikx}\right)e^{iky} = \left(\Psi(x,-k) - e^{-ikx}\right)e^{iky} =$$

$$= \int_x^{\infty} K(x,z)\,e^{ik(y-z)}\,dz \qquad \text{using } (**)$$

Integrate over k's:

$$\int_{-\infty}^{\infty}\left(\int_x^{\infty} K(x,z)\,e^{ik(y-z)}\right)dz\,dk =$$

$$= \int_x^{\infty} K(x,z)\left(\int_{-\infty}^{\infty} e^{ik(y-z)}\,dk\right)dz$$

$$= 2\pi\int_x^{\infty} K(x,z)\,\delta(z-y)\,dz = 2\pi K(x,y)$$

Left-hand side: the function has simple poles at $i\varkappa_s$, analytic elsewhere in the upper half-plane, and diminishes as $|k| \to \infty$

$$\int_{-\infty}^{\infty}\left(\frac{\varphi(x,k)\,e^{iky}}{a(k)} - e^{ik(y-x)}\right)dk =$$

$$= 2\pi i\sum_{s=1}^{n}\frac{\varphi(x,i\varkappa_s)}{a'(i\varkappa_s)}\,e^{-\varkappa_s y}$$

$$= 2\pi i\sum_{s=1}^{n}\frac{b_s\Psi(x,i\varkappa_s)}{a'(i\varkappa_s)}\,e^{-\varkappa_s y}$$

we use $(**)$

$$= 2\pi i\sum_{s=1}^{n}\frac{b_s e^{-\varkappa_s(x+y)}}{a'(i\varkappa_s)} +$$

$$+ 2\pi i\int_x^{\infty} K(x,z)\sum_{s=1}^{n}\frac{b_s e^{-\varkappa_s(z+y)}}{a'(i\varkappa_s)}\,dz$$

Right-hand side, 2nd term:

$$\int_{-\infty}^{\infty} r(k)\,\Psi(x,k)\,e^{iky}\,dk =$$

$$= \int_{-\infty}^{\infty} r(k)\,dk\left(e^{ik(x+y)} + \int_x^{\infty} K(x,z)\,e^{ik(y+z)}\,dz\right)$$

using $(**)$

$$= \int_{-\infty}^{\infty} r(k)\,e^{ik(x+y)}\,dk + \int_x^{\infty} K(x,z)\int_{-\infty}^{\infty} r(k)\,e^{ik(y+z)}\,dk\,dz$$

Collecting the obtained expressions into one equation, and applying the definition of $F(x)$, we will get the Gel'fand-Levitan-Marchenko equation.

We need to prove:

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$$u(x) = -2\frac{d}{dx}K(x,x)$$

$$\Psi(x,k) = e^{ikx} + \int_x^\infty \underbrace{K(x,y)}\,\underbrace{e^{iky}\,dy}$$

We analyze the behaviour as $|k| \to \infty$

Integrate by parts:

$$\Psi(x,k) = e^{ikx} + \left[K(x,y)\frac{e^{iky}}{ik}\right]_x^\infty - \frac{1}{ik}\int_x^\infty K_y(x,y)\,e^{iky}\,dy$$

$$= e^{ikx} - K(x,x)\frac{e^{ikx}}{ik} - \frac{1}{ik}\int_x^\infty K_y(x,y)\,e^{iky}\,dy$$

We substitute this in the Schrödinger equation

$$-\partial_x^2\Psi + u\Psi = k^2\Psi$$

Compare the terms of the form e^{ikx} (no k appears)

$$2\frac{d}{dx}K(x,x)\,e^{ikx} + u\,e^{ikx} = 0$$