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Nalini Joshi Room 629

The Gelfand - Levitan - Marchenko equation

$$K(x,\tau) + B(x+\tau) + \int_x^\infty K(x,s)\, B(s+\tau)\, ds = 0 \qquad \text{for } \tau \geq x.$$

where $$B(z) = -i \sum_{k=1}^{N} \gamma_k e^{i\zeta_k x} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} R(\xi)\, e^{i\xi z}\, d\xi$$

$$R(\xi) = \frac{b(\xi)}{a(\xi)} \quad \text{is the reflection coefficient}$$

and $$\gamma_k = \frac{b_k}{\dot{a}_k} = \frac{i}{\int_{-\infty}^{\infty} (\psi_k(x))^2\, dx}, \qquad \dot{a}_k = \frac{d}{d\zeta} a(\zeta)\Big|_{\zeta=\zeta_k}$$

$\psi_k = \psi(x, \zeta_k)$, where ζ_k is a zero of $a(\zeta)$
(these are all simple zeroes, which all lie on the imaginary axis in ζ-plane)

We will need to know that ψ is a Jost function, which satisfies

$$\psi_{xx} + (u(x,\cdot) + \lambda)\psi = 0 \qquad \lambda = \zeta^2$$

$$\psi(x,\zeta) \sim e^{i\zeta x} \quad \text{as } x \to +\infty$$

One-Soliton Case

Assume $u(x,0) = 2k^2 \operatorname{sech}^2(kx)$

By choosing

$$u(x,t) = k^2\, U(\underbrace{kx}_{\xi}, \underbrace{k^3 t}_{\eta}), \quad \text{a scaling transformation}$$

Note. $u_t + 6uu_x + u_{xxx} = 0$ becomes

$$\cancel{k^2} \cdot U_\eta \cdot \cancel{k^3} + 6\, \cancel{k^4}\, U \cdot U_\xi \cdot \cancel{k} + \cancel{k^2}\, U_{\xi\xi\xi} \cdot \cancel{k^3} = 0$$

The KdV again for $U(\xi,\eta) \longrightarrow$ "scaling invariance"

$\Rightarrow$ w.l.o.g. can take $k=1$: $u(x,0) = 2\operatorname{sech}^2(x)$.

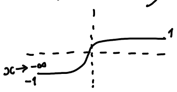
Exercise: Show that the equation

$$\psi_{xx} + (2\operatorname{sech}^2 x + \zeta^2)\psi = 0$$

has the general solution

$$\psi_{gen}(x,\zeta) = A e^{-i\zeta x}(\tanh x + i\zeta) + B e^{i\zeta x}(\tanh x - i\zeta)$$

for arbitrary consts A & B.



$$\Rightarrow \psi_{gen}(x,\zeta) \sim \begin{cases} A e^{-i\zeta x}(-1+i\zeta) + B e^{i\zeta x}(-1-i\zeta) & x \to -\infty \\ A e^{-i\zeta x}(1+i\zeta) + B e^{i\zeta x}(1-i\zeta) & \text{as } x \to +\infty \end{cases}$$

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The Jost functions satisfy

$$\varphi \sim e^{-i\zeta x} \quad x \to -\infty \quad \Rightarrow \quad B = 0, \; A = \frac{1}{-1+i\zeta}$$

$$\psi \sim e^{i\zeta x} \quad x \to +\infty \quad \Rightarrow \quad A = 0, \; B = \frac{1}{1-i\zeta}$$

$$\bar{\psi} \sim e^{-i\zeta x} \quad x \to +\infty \quad \Rightarrow \quad B = 0, \; A = \frac{1}{1+i\zeta}$$

We have $\varphi = a(\zeta)\bar{\psi} + b(\zeta)\psi$

$$\Rightarrow \frac{e^{-i\zeta x}}{-1+i\zeta}(\tanh x + i\zeta) = a(\zeta) \cdot \frac{e^{-i\zeta x}}{1+i\zeta}(\tanh x + i\zeta) + b(\zeta) \cdot \frac{e^{i\zeta x}}{1-i\zeta}(\tanh x - i\zeta)$$

$$\Rightarrow \begin{cases} b(\zeta) \equiv 0 \\ a(\zeta) = \dfrac{-(1+i\zeta)}{(1-i\zeta)} \end{cases} \quad \Rightarrow \text{ simple zero at } \zeta_1 = i$$

$$\dot{a}(\zeta_1) = -\frac{i}{2} \quad \text{(Check!)}$$

$$\Rightarrow R(\zeta) \equiv 0$$

We also need b_1: relate $\varphi(x, \zeta_1)$ to $\psi(x, \zeta_1)$, with $\varphi(x,\zeta_1) = \varphi_1 = \frac{1}{2}\operatorname{sech} x$ and $\psi(x,\zeta_1) = \psi_1 = \frac{1}{2}\operatorname{sech} x$.

$$\varphi_1 = b_1 \psi_1 \quad \Rightarrow \quad b_1 = 1.$$

$$\gamma_1 = \frac{b_1}{\dot{a}_1} = \frac{1}{-i/2} = 2i$$

Time evolution: $b_1(t) = b_1(0)\, e^{8i\zeta_1^3 t} = e^{8t}$

$$\gamma_1(t) = 2i\, e^{8t}$$

$$\Rightarrow B(z) = -i\gamma_1 e^{i\zeta_1 z} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} 0 \cdot e^{i\xi z}\, d\xi = 2e^{8t} \cdot e^{-z}$$

$\Rightarrow$ The GLM eqn becomes (for time t)

$$K(x,\tau) + 2e^{-(x+\tau)+8t} + 2e^{8t}\int_x^{\infty} K(x,s)\, e^{-(s+\tau)}\, ds = 0, \quad \text{for } \tau > x.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \quad s \quad \tau$$

Separation of variables:
$$K(x,\tau) = L(x)\, e^{-\tau}$$

$$\Rightarrow L(x)\cancel{e^{-\tau}} + 2e^{-x\cancel{-\tau}+8t} + 2e^{8t}\int_x^{\infty} L(x)\, e^{-s}\, e^{-s\cancel{-\tau}}\, ds = 0$$

$$\Rightarrow L(x)\left(1 + 2e^{8t} \cdot \left[\frac{e^{-2s}}{-2}\right]_x^{\infty}\right) + 2e^{-x+8t} = 0$$

$$\Rightarrow L(x) = -\frac{2e^{-x+8t}}{1+e^{-2x+8t}}$$

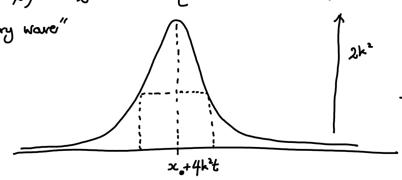
$$\Rightarrow K(x,x) = -2\frac{e^{-2x+8t}}{1+e^{-2x+8t}} = \frac{-2}{e^{2x-8t}+1}$$

$$\Rightarrow u(x,t) = 2\frac{d}{dx}\left(K(x,x)\right) = \cdots = 2\operatorname{sech}^2(x-4t)$$

Or using scaling invariance of the KdV and arbitrary shift in x by x_0

$$\Rightarrow u(x,t) = 2k^2 \operatorname{sech}^2\left[k(x - 4k^2 t - x_0)\right]$$

a "solitary wave"



$\longrightarrow$ travels to the right as t increases with speed $4k^2$.

Remember this is a reflectionless potential.