ALGEBRAIC NUMBER THEORY, PURE MATHEMATICS IV, SEMESTER 2, 2018

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Course overview

The course is intended as an introduction to properties of the ring of integers of an algebraic number field, and ramification of primes in such fields. This is the circle of mathematical ideas which led to Wiles' proof of Fermat's Last Theorem, although the course will stop well short of such a proof. The subject matter has intimate connections with algebraic geometry, representation theory and many other branches of mathematics.

Some of the general concepts to be treated include: the theory of modules over a Dedekind domain, the geometry of abelian lattices and Galois theory. This topic is characterised by a mixture of ancient and modern themes. Some results go back to Gauss, about 200 years ago, while others are still under development.

There will be a little dependence on Commutative algebra, but in principle the course will be accessible to people with open minds and basic algebra.

Rough list of contents

(1) Introductory concepts–field extensions, the theorem of a primitive element, perfect fields, Galois groups.
(2) Integral dependence–integral closures of domains and related topics.
(3) Algebraic integers–definitions, properties, examples (quadratic, cyclotomic).
(4) Norms, traces and discriminants–for modules over any integral domain; computation in case where L/K is finite; interaction of these concepts with integrality. The structure of the ring of integers in a number field (it is free, of rank n).
(5) Cyclotomic fields–their integers, discriminants and other properties.
(6) Factorisation in Dedekind domains–factorisation of integers; prime ideals, fractional ideals; the ideal class group of a Dedekind domain; unique factorisation of ideals in a Dedekind domain.
(7) Minkowki's theorem and lattices in $\mathbb{R}^n$–discrete subgroups of topological groups; fundamental domain; Minkowski's theorem. Application to prove finiteness of the number of ideal classes.
(8) Units in an algebraic number field–Dirichlet's theorem and other results; applications to Diophantine equations.
(9) Sums of 2 and 4 squares. Waring's problems.
(10) Factorisation of ideals in extensions–ramification theory via localisation;residual degrees, basic relations, examples.
(11) Ramification–and discriminants; examples of quadratic, cyclotomic and cubic fields.

(12) The Galois case– decomposition and inertia groups.
(13) Quadratic reciprocity–two proofs. Gauss' favourite theorem.
(14) (Time permitting) Extra topics, selected from among: elliptic curves and Fermat, beginning Shimura theory, examples.

Some useful books

Borevich and Shafarevich: Algebraic number theory.
N. Bourbaki: Algèbre commutatif.
Pierre Samuel: Théorie algébrique des nombres.
Serge Lang: Algebraic number theory.
Serge Lang: Algebra.
Jean-Pierre Serre: Cours d'arithmétique.
Takeshi Saito: Fermat's last theorem. (for reference only-most material is beyond the scope of the course)

Assessment

There will be 1 assignment, which in total will count for 20% of the final assessment. The end of semester exam will count for 80% of the final assessment.