

Riemannian Geometry – Lecture 15 Riemann Curvature Tensor

Dr. Emma Carberry

September 14, 2015

Sectional curvature

Recall that as a $(1, 3)$ -tensor, the Riemann curvature endomorphism of the Levi-Civita connection ∇ is

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z.$$

It measures the failure of the manifold to be locally isometric to Euclidean space.

Starting with the Riemann curvature tensor, there are various simplifications of this tensor one can define.

An important one is *sectional curvature*, because it

- is the natural generalisation of Gauss curvature of the surface
- completely determines the Riemann curvature tensor

A two-dimensional subspace π of $T_p M$ is called a *tangent 2-plane* to M at p .

Sectional curvature is defined on tangent 2-planes $\pi \subset T_p M$.

We will define the sectional curvature of a tangent 2-plane π in terms of a basis for π .

In order to obtain an expression which is independent of this choice of basis, we will need a normalisation factor.

Definition 15.1. For linearly independent vectors $X, Y \in T_p M$, let

$$\begin{aligned} Q(X, Y) &= (\text{area of the parallelogram with sides } X \text{ and } Y)^2 \\ &= \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2. \end{aligned}$$

Definition 15.2. Given any point $p \in M$ and any tangent 2-plane π to M at p , the sectional curvature of π is the real number

$$\sec(\pi) := \frac{\langle R(X, Y)Y, X \rangle}{Q(X, Y)} = \frac{\text{Rm}(X, Y, Y, X)}{Q(X, Y)},$$

where $\{X, Y\}$ denotes a basis of π . In components $K_{ij} := \sec(\text{span}\{\partial_i, \partial_j\})$

Independence of basis choice

Proposition 15.3. *Sectional curvature $\sec(\pi)$ is well-defined, independent of the choice of basis $\{X, Y\}$ for π .*

Proof: If $\{\tilde{X}, \tilde{Y}\}$ is another basis for π , then

$$\begin{aligned}\tilde{X} &= aX + bY, \\ \tilde{Y} &= cX + dY,\end{aligned}$$

with non-zero determinant $ad - bc$.

An easy computation gives

$$Q(\tilde{X}, \tilde{Y}) = (ad - bc)^2 Q(X, Y).$$

Using symmetry of Rm ,

$$\begin{aligned}\text{Rm}(\tilde{X}, \tilde{Y}, \tilde{Y}, \tilde{X}) &= \text{Rm}(aX + bY, cX + dY, cX + dY, aX + bY) \\ &= (ad - bc) \text{Rm}(X, Y, cX + dY, aX + bY) \\ &= (ad - bc)^2 \text{Rm}(X, Y, Y, X)\end{aligned}$$

so $\sec(\pi)$ is independent of the choice of basis $\{X, Y\}$. □

We have seen that the sectional curvature of M is a real-valued function defined on the set of all tangent 2-planes of M .

Although sectional curvature seems simpler than the curvature tensor Rm , it encapsulates the same information.

Lemma 15.4. *Given $\sec(\pi)$ for all tangent 2-planes $\pi \subset T_p M$, we can algebraically determine the curvature tensor Rm at p .*

Proof: The idea is to use the symmetry of the Riemann curvature tensor. Set $X, Y, Z, W \in T_p M$, and consider the polynomial in t defined by

$$\begin{aligned}f(t) &= \text{Rm}(X + tW, Y + tZ, Y + tZ, X + tW) \\ &\quad - t^2(\text{Rm}(X, Z, Z, X) + \text{Rm}(W, Y, Y, W))\end{aligned}$$

Due to the symmetries in each term, we can write f in terms of sectional curvatures (and the function Q which is given by the inner product).

The coefficient of t^2 in $f(t)$ is

$$\text{Rm}(X, Y, Z, W) + \text{Rm}(X, Z, Y, W) + \text{Rm}(W, Z, Y, X) + \text{Rm}(W, Y, Z, X).$$

Recall that Rm is skew-symmetric in each of the first and last 2 entries

$$\begin{cases} \text{Rm}(X, Y, Z, W) = -\text{Rm}(Y, X, Z, W), \\ \text{Rm}(X, Y, Z, W) = -\text{Rm}(X, Y, W, Z) \end{cases}$$

and it is symmetric between the first pair and last pair

$$\text{Rm}(X, Y, Z, W) = \text{Rm}(Z, W, X, Y).$$

Hence the coefficient of t^2 in $f(t)$ is

$$2\text{Rm}(X, Y, Z, W) - 2\text{Rm}(Z, X, Y, W). \quad (1)$$

Interchanging X and Y , we similarly obtain that the expression

$$2\text{Rm}(Y, X, Z, W) - 2\text{Rm}(Z, Y, X, W),$$

is determined by the sectional curvatures of M at p . Again using symmetry of Rm , this is

$$-2\text{Rm}(X, Y, Z, W) + 2\text{Rm}(Y, Z, X, W). \quad (2)$$

Recall also the algebraic Bianchi identity:

$$\sum_{\text{all cyclic permutations of } (X, Y, Z)} \text{Rm}(X, Y, Z, W) = 0.$$

Using this,

$$(1) - (2) = 6\text{Rm}(X, Y, Z, W),$$

so the Riemann curvature tensor is determined by the sectional curvature. \square

Definition 15.5. A Riemannian manifold M has *constant curvature* if its sectional curvature function is constant, that is there exists a constant k such that $\text{sec}_p(\pi) = k$ for all $p \in M$ and all tangent planes $\pi \subset T_p M$.

Remark 15.6. Let $p \in M$. If the sectional curvature function at p is zero (that is, $\text{sec}_p = 0$), then the curvature $\text{Rm} = 0$ at p .

Hence, a Riemannian manifold (M, \mathbf{g}) is flat if and only if the sectional curvature is identically zero.

Let us consider the special case when our Riemannian manifold is a surface.

In that case we had already an intrinsic notion of curvature, namely the Gauss curvature.

Proposition 15.7. For a Riemannian surface (M, \mathbf{g}) , the sectional curvature of M is equal to its Gauss curvature.

Proof: By definition, for $p \in M^2$,

$$\begin{aligned} \sec(T_p M) &= \frac{\text{Rm}_{2112}}{(g_{11}g_{22} - g_{12}^2)} \\ &= \frac{\langle (\nabla_{\partial_2}(\nabla_{\partial_1}\partial_1) - \nabla_{\partial_1}(\nabla_{\partial_2}\partial_1)), \partial_2 \rangle}{\det g}, \end{aligned}$$

which is a formula for the Gauss curvature. □

Exercise 15.8. Prove this expression, using

- the definition of Gauss curvature $K(p)$ as the determinant of the shape operator $-dN_p$, where N is the Gauss map
- the resulting expression $K = \frac{\det(\text{II})}{\det \text{I}}$ where I and II denote the 1st and 2nd fundamental forms. (You should remind yourself how to obtain this expression.)
- the fact (which you should make sure you understand) that

$$\nabla_{\partial_i}(\partial_j) = \Gamma_{ij}^k \partial_k + h_{ij} N$$

where h_{ij} denotes the coefficients of the 2nd fundamental form.

We will next study some spaces which have constant sectional curvature, for which the following remark will prove helpful.

Remark 15.9. Let (M, \mathbf{g}) be a Riemannian manifold and consider the $(0, 4)$ -tensor Q defined by

$$Q(X, Y, Z, W) = \langle X, W \rangle \langle Y, Z \rangle - \langle Y, W \rangle \langle X, Z \rangle$$

for $X, Y, W, Z \in \mathcal{X}(M)$. Then M has constant sectional curvature equal to K_0 if and only if $\text{Rm} = K_0 Q$, where Rm denotes the Riemann curvature tensor.

Proof:

Suppose M has constant sectional curvature K_0 and take $p \in M$ and linearly independent $X, Y \in T_p M$. Then

$$\begin{aligned} \text{Rm}(X, Y, Y, X) &= K_0(|X|^2 |Y|^2 - \langle X, Y \rangle^2) \\ &= K_0 Q(X, Y, Y, X). \end{aligned}$$

As in the above proof, Lemma 15.4, this then implies that

$$\text{Rm}(X, Y, Z, W) = K_0 Q(X, Y, Z, W)$$

for all $X, Y, Z, W \in \mathcal{X}(M)$, as required.

The converse is trivial. □

Corollary 15.10 (Constant Sectional Curvature). *Let (M, g) be a Riemannian manifold, $p \in M$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of $T_p M$. Then the following are equivalent:*

1. $\sec(\pi) = K_0$ for all 2-planes $\pi \subset T_p M$
2. $\text{Rm}_{ijkl} = K_0(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$.
3. $\text{Rm}_{ijji} = -\text{Rm}_{ijij} = K_0$ for all $i \neq j$ and $\text{Rm}_{ijkl} = 0$ otherwise.