Lemma 22.1 (Rescaling lemma). For $X \in T_pM$ and $c \in \mathbb{R} \setminus \{0\}$, if the geodesic $\gamma_X$ is defined on the interval $(-\epsilon, \epsilon)$ then the geodesic $\gamma_{cX}$ is defined on interval $(-\epsilon c, \epsilon c)$ and

$$\gamma_{cX}(t) = \gamma_X(ct).$$

Proof: Define $\tilde{\gamma} : (\frac{-\epsilon}{c}, \frac{\epsilon}{c}) \to M$ by

$$\tilde{\gamma}(t) = \gamma_X(ct).$$

Then writing $\tilde{\nabla}_{\dot{\gamma}}$ for covariant differentiation along $\tilde{\gamma},$ since $\dot{\tilde{\gamma}}(t) = c\dot{\gamma}(ct)$ and $\dot{\tilde{\gamma}}(0) = cX$

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\tilde{\gamma}}(t) = (\ddot{\gamma}^k(t) + \Gamma_{ij}^k(\tilde{\gamma}(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)) \partial_k$$

$$= c^2\nabla_{\dot{\gamma}}\dot{\gamma}_X(ct)$$

$$= 0.$$

Define $\mathcal{E} \subset TM$ by

$$\mathcal{E} = \{(p, X) \in TM \mid \gamma_X \text{ is defined on an interval containing } [0, 1]\},$$

and denote by $\mathcal{E}_p$, the intersection of $\mathcal{E}$ with the set $T_pM$. Then the exponential map is

$$\exp : \mathcal{E} \to M$$

$$X \mapsto \gamma_X(1).$$
Proposition 22.2.  
1. For each \( p \in M \) and \( X \in T_pM \), \( \gamma_X(t) = \exp(tX) \) whenever these are defined.

2. \( \mathcal{E} \) is an open subset of \( TM \) and for each \( p \in M \), we have \( 0 \in \mathcal{E}_p \).

3. For each \( p \in M \), \( \mathcal{E}_p \) is a star-shaped region about 0. That is, whenever \( X \in \mathcal{E}_p \) then also \( cX \in \mathcal{E}_p \) for all \( c \in [0,1] \).

4. In fact for each \( p \in M \) there exists a neighbourhood \( V \) of \( p \) in \( M \) and \( \bar{\epsilon} > 0 \) such that for every \( (q, X) \in TV \) with \( |X| < \bar{\epsilon} \), we have \( (q, X) \in \mathcal{E} \).

5. \( \exp \) is a smooth map.

6. For each \( p \in M \) there exists \( \epsilon > 0 \) such that

\[
\exp_p : \{ X \in T_pM \mid |X| < \epsilon \} \to M
\]

is a diffeomorphism onto its image.

Proof:

(1) is simply the rescaling lemma with \( t = 1 \):

\[
\exp(cX) = \gamma_{cX}(1) = \gamma_X(c).
\]

(2) Clearly \( \mathcal{E} \) contains the 0 section of \( TM \).

To prove that \( \mathcal{E} \) is open, we will use the stronger version of the existence and uniqueness theorem. From Theorem [21.10] we have:

For each \( p \in M \) there exists an open neighbourhood \( U \) of \( p \) in \( M \) and an open neighbourhood \( \mathcal{U} \) of \( (p, 0) \) in \( TU \) together with \( \epsilon > 0 \) and a smooth map

\[
\gamma : (-\epsilon, \epsilon) \times \mathcal{U} \to TU
\]

such that

\[
t \mapsto \gamma(t, q, X)
\]

is the unique trajectory of the geodesic vector field \( G \) which for \( t = 0 \) passes through the point \( q \) with velocity vector \( X \in T_qM \).

(3) This follows immediately from the rescaling lemma; since

\[
\gamma_{cX}(t) = \gamma_{c}(ct)
\]
is defined whenever \(ct \in [0, 1]\) and \(c \in [0, 1]\), it is certainly defined whenever \(t \in [0, 1]\).

(4) Taking a compact subset of the open neighbourhood \(U\) of \(p\) and then an open subset \(V\) of this compact set, then there exists \(\delta > 0\) such that

\[
\{(q, X) \in TV \mid X \in T_qM \text{ satisfies } |X| < \delta\} \subset U.
\]

Then for \(q \in V\), the geodesic \(\gamma_X(t)\) is defined whenever \(|t| < \epsilon\) and \(|X| < \delta\).

By the rescaling lemma then the geodesic \(\gamma_Y\) is defined whenever \(|t| < 2\) and \(|Y| < \frac{\epsilon \delta}{2}\), where \(Y = \frac{\epsilon X}{2}\).

(5): The exponential map \((p, X) \mapsto \exp(p, X)\) is the evaluation of the smooth map \(\gamma(t, p, X)\) at \(t = 1\), and hence is smooth.

(6): The differential of \(\exp_p\) at \(0 \in T_pM\) is just the identity map, since

\[
d(\exp_p)_0(X) = \frac{d}{dt}(\exp_p(tX))\bigg|_{t=0} = \frac{d}{dt}(\gamma(1, p, tX))\bigg|_{t=0} = \frac{d}{dt}(\gamma(t, p, X))\bigg|_{t=0} = X.
\]

Then by the inverse function theorem, the exponential map is locally a diffeomorphism.

\[
\square
\]

**Definition 22.3.** If \(V\) is a neighbourhood of \(0\) in \(T_pM\) such that \(\exp_p\) restricted to \(V\) is a diffeomorphism onto its image, then we call \(\exp_p(V) =: U\) a **normal neighbourhood** of \(p\).

A choice of orthonormal basis \(E_1, \ldots, E_n\) for \(T_pM\) is equivalent to an isomorphism

\[
E : T_pM \rightarrow \mathbb{R}^n
\]

\[
X^i E_i \rightarrow (X^1, \ldots, X^n)
\]

and so on a normal neighbourhood \(V\) of \(p \in M\) we have local coordinates

\[
E \circ \exp_p^{-1} : V \rightarrow \mathbb{R}^n.
\]

Such coordinates are called **normal coordinates** centered at \(p\).

Normal coordinates are extremely useful, in particular they yield a very simple representation for geodesics.
Lemma 22.4. Let \((M, g)\) be a Riemannian manifold and \((V, \{\phi^i\})\) be a normal coordinate chart centred at \(p \in M\).

For any \(X = X^i \partial_i \in T_p M\), the geodesic \(\gamma_X\) is given in normal coordinates by

\[
\gamma_X(t) = \phi^{-1}(tX^1, \ldots, tX^n).
\]

Question 22.5. Prove this.