

# **Fourth Year Pure Mathematics**

2008 Handbook

School of Mathematics and Statistics  
University of Sydney



[www.maths.usyd.edu.au/u/UG/HM/](http://www.maths.usyd.edu.au/u/UG/HM/)



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Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something completely different.  
Goethe, *Maximen und Reflexionen*

## CHAPTER 1

# The structure of pure mathematics four

### 1.1. Introduction

In linguistics it is increasingly believed that universal features of language are reflections of the structure of the human brain and its perception of the world around us. In a similar fashion, mathematics is a universal language that has been developed to understand and describe how nature and life work. Mathematics, both in structure and development, is inextricably bound to our attempts to understand the world around us and our perceptions of that world. We see this in the mathematical descriptions and formulations of models in the theoretical and applied sciences: from physics, computer science and information theory on the one hand, to engineering, chemistry, operations research and economics on the other.

Just as remarkable is the way in which esoteric and abstract mathematics finds applications in the applied sciences. Indeed, one of the most exciting developments in science over the past decade has been the re-emergence of a dynamic interaction between pure mathematicians and applied scientists, which is bringing together several decades of the relatively abstract and separate development of pure mathematics and the sciences. Examples include the applications of singularity theory and group theory to symmetry-breaking and bifurcation in engineering; number theory to cryptography; category theory and combinatorics to theoretical and computational computer science; and, most spectacular of all, the recent developments of general field theories in mathematical physics based on the most profound work in complex analysis and algebraic geometry. Of course, this interaction is not one way. For example, there is the recent discovery of “exotic” differential structures on  $\mathbb{R}^4$  utilising ideas from Yang-Mills theory.

There are many valid approaches to the study of Pure Mathematics in the final Honours Year. Thus, the course may be regarded as useful in its own right, or may lead on to an M. Sc or Ph.D. or to a teaching position in University or High School. In another direction, what want a solid base from which to continue with studies in computer science or physics, for

example. Finally, you may intend to seek employment with the CSIRO or in the operations research field, or in a financial institution. In the latter circumstances, one well-known advantage of studying mathematics is that mathematics gives training in a particular way of thinking and an analytical approach to problem solving. Mathematicians are highly adaptable (and employable).

The Fourth Year Honours program in Pure Mathematics caters for the various needs described above by offering a highly flexible and adaptable program, which is both interesting and challenging. We offer a combination of core courses, which introduce the major areas of mathematics, together with a smorgasbord of deeper courses which can be arranged to suit your personal requirements.

In brief, the Fourth Year course comprises the equivalent of seven lecture courses, together with an essay project (the equivalent of three lecture courses) and a 40-45 minute talk on the essay project.

A description of the various components of the course is given below. For detailed descriptions of the courses, the essay project, etc., see the appropriate chapter in this Handbook.

## 1.2. The lecture courses

Students are required to be assessed on 7 units of approved lecture courses (or equivalent - see below).

In 2008, the courses may be chosen from:

- a) three PM4 core courses, each worth  $1\frac{1}{2}$  units;
- b) five other PM4 courses, each worth 1 unit.  
(These may presume some knowledge of one or more of the core courses.)
- c) six third year *advanced* courses, each worth 1 unit.  
(Students in Pure Mathematics 4 may take any 3(A) course which they have not previously taken.)
- d) Approved substitutions (up to the value of 2 units) by courses given by other Departments. (See §1.5 below.)
- e) Reading courses arranged with staff members (after consultation with the PM4 coordinator).

**Read carefully the guidelines in §1.5 below.**

In addition, we are negotiating with UNSW over the possibility of sharing some PM4 Courses. (The likely UNSW courses include “Graph Theory” and “Banach Algebras”; the potential difficulties are in timetabling.)

Overall, the lecture courses offered at the level of PM4 and above are intended to introduce students to the major divisions of modern mathematics and provide a knowledge of some of the main ideas needed for the understanding of much of contemporary mathematics, while still reflecting the research interests within the pure mathematics research groups.

The “core” of Fourth Year is considered to include Commutative Algebra, Functional Analysis and Algebraic Topology. Each of these has 3 lectures per week with no tutorial and counts as  $1\frac{1}{2}$  units. Students are strongly advised to take all of the core courses.

**1.3. Pure Mathematics 4/PG Courses for 2008**

SEMESTER I	
Algebraic Topology	Paunescu
Commutative Algebra	Lai
Functional Analysis	Daners
SEMESTER II	
Symmetric Functions	Molev
Partial Differential Equations	Dancer

If you are unsure about the combination of courses you should take, consult with your supervisor or the course coordinator. In any case, you are very welcome to attend all the lecture courses.

### 1.4. Pure Mathematics 3(A) Courses for 2008

SEMESTER I	SEMESTER II
Algebra (3962)	Complex Analysis (3964)
Metric Spaces (3961)	Lebesgue/Fourier Analysis (3969)
	Representation Theory (3966)

### 1.5. Tailor-made courses

Students may also take certain courses of an essentially mathematical nature other than these (for instance, in Applied Mathematics, Econometrics, Formal Logic, Statistics, etc.) **with the approval** of the Course Coordinator. (Normally we require that at least five of the seven course units be chosen from the Pure Mathematics options). Details of the Applied Mathematics 4 and Mathematical Statistics 4 options may be obtained from the Coordinators, Dr. Martin Wechselberger (Applied Maths - Carlaw 628, tel. 9351 3860, *M.Wechselberger@maths.usyd.edu.au*) and Dr Rafał Kulik (Math. Stats. Carlaw 818, tel. 9356 7878, *R.Kulik@maths.usyd.edu.au*).

A number of staff are usually willing to supervise a reading course in their particular area of interest. Consult the course coordinator if you have a special topic in mind that might be acceptable as a reading course. Reading courses are generally a matter between the student and a willing member of the department, subject to the approval of the course coordinator.

If you wish to do a **reading course** in Pure Mathematics, or substitute a course from outside Pure Mathematics, you should ask the lecturer to prepare a **summary** of the course and description of the **assessment**. This should then be submitted to the course coordinator for approval. Provided the guidelines sketched above are followed, and a satisfactory balance overall in the Fourth year courses is maintained, approval will normally be granted.

### 1.6. The essay project

The essay project counts as the equivalent of three PM4 units. Work on the essay project proceeds throughout the year and the finished essay is submitted near the end of the second semester. Note that it is also possible

for the project to be supervised by a member of another department (or jointly).

### **1.7. The talk**

As part of the essay project, students are required to give a talk about their project. Talks are normally scheduled to take place in September.

### **1.8. Mathematics in other languages**

Ability to read mathematics in at least one approved foreign language is no longer a requirement for Pure Mathematics 4. An increasing proportion of mathematical papers are now written in English, there are still a significant number of important mathematical works not written in English. In addition, many older mathematical works written in other languages have not yet been translated into English. For these reasons, students who are seriously thinking of pursuing further studies in mathematics are strongly encouraged to acquire a reading knowledge of mathematics in at least one foreign language. Indeed, such knowledge is often a requirement of M. Sc. and Ph.D. courses in mathematics (especially in the U. S. A.).

In particular, a working knowledge of mathematical French is extremely useful (and relatively easy to acquire). At present, Russian is less useful, as the collapse of the Soviet system has obliged many of the best mathematicians from the former Soviet Union to work in the West, and in any case there are English translations of most of the Russian language mathematics journals. (However, the Carlsaw library still subscribes to the Russian originals in most cases, as they are much cheaper than the translations). On the other hand German may again become the preferred language of publication for mathematicians in an united Germany. Although the mathematical schools of China and Japan are large and increasingly important acquiring a reading knowledge of the languages of these countries is difficult and time-consuming for most adults.

The Departments of French and German offer reading courses that enable one to acquire a working reading knowledge of French and German. These courses begin in the first semester and participants must register with the relevant Department before the start of the course.

I fear that I bore you with these details, but I have to let you see my little difficulties, if you are to understand the situation.

Sherlock Holmes, *A Scandal in Bohemia*

## CHAPTER 2

### Entry, administration and assessment

#### 2.1. Entry Requirements for Pure Mathematics 4

Students who have fulfilled the requirements of the faculty in which they are enrolled and satisfied conditions (a), and (b) or (c) below are eligible to enrol in Pure Mathematics 4:

- a) taken 24 credit points of third year mathematics units (see the senior pure and applied mathematics handbook) with at least 16 of these credit points in pure mathematics;
  - b) obtained a distinction average or better in 24 credit points of third year mathematics units
- or;*
- c) obtained a credit average in 24 credit points of third year mathematics units, *including* a credit in at least one Pure Mathematics 3 *Advanced* unit.

*Entry to PM4 is also subject to the approval of the Head of School*

**Note** Since we advise all PM4 students to take the core courses (commutative algebra, algebraic topology and functional analysis), the natural prerequisites for PM4 are: Metric Spaces; Algebra 1; and Lebesgue and Fourier Analysis. Students without this background should expect to do some preliminary reading over the summer (see the course coordinator for advice if necessary).

#### 2.2. Actions to be taken

All students intending to take Pure Mathematics 4 in 2008 should see the *PM4 Course Coordinator*, Dr Laurentiu Paunescu (Carslaw 816, tel. 9351-2969, e-mail: [laurent@maths.usyd.edu.au](mailto:laurent@maths.usyd.edu.au)) at their earliest opportunity, and in any case well before the beginning of the new teaching year. The Course

Coordinator will advise you about choosing a supervisor and a topic for the essay project (see also section 4.2).

### 2.3. Administrative arrangements

The Course Coordinator is in charge of Pure Mathematics 4 and should be consulted about any organisational problems that may arise.

In particular, students should note that the Course Coordinator's permission should be obtained if you wish to substitute courses from outside, or take a reading course or a postgraduate course. In the first instance, however, you should discuss such matters with your supervisor. Provided you can agree, the Course Coordinator's permission would *normally* be a formality.

Please take particular note of the procedure to be followed if you are sick or other circumstances arise that may lead to late submission of your essay (see §4.4). Also note that at the end of first semester a progress report must be given to the Course Coordinator (see Chapter 5).

When we know that you are enrolled for PM4 you will be given a computer account. *The usual way in which messages for PM4 students will be distributed will be via e-mail.* Please remember to check your e-mail regularly. This will become second nature once you start to type up your essays.

### 2.4. Assessment

The possible results for Fourth year are First Class Honours, Second Class Honours division 1, Second Class Honours Division 2, Third Class Honours and No Award (Fail), usually abbreviated I, II-1, II-2, III and F. The last two are rarely awarded.

Each Fourth Year course is assessed at a time and in a manner arranged between the lecturer and the class. Usually, a written examination is held during the exam period immediately following the course; however, some courses are assessed entirely by assignment. It is undesirable to have examinations during term or to have many papers deferred to the end of the year. Each PM3 advanced course is assessed in the usual way. Students are informed about their performance as information becomes available.

Marks provided by the lecturers are scaled according to the lecturer's judgement as to what constitutes a First Class, II-1, etc., performance on each course. The marks from the best  $7/7\frac{1}{2}$  units are counted towards the final

mark. The essay is equivalent to 3 PM4 units; it accounts for 30% of the years assessment.

As well as assessing the Fourth Year performance, the Department is required to make a recommendation for a grade of Honours based on the performance in all subjects over the four years. In exceptional cases, the grade of Honours awarded could differ from the level of performance in the Fourth Year.

### **2.5. Honours grades**

The Faculty of Science has given the following guidelines for assessment of student performance in fourth year.

- 95–100 Outstanding First Class quality of clear Medal standard, demonstrating independent thought throughout, a flair for the subject, comprehensive knowledge of the subject area and a level of achievement similar to that expected by first rate academic journals. This mark reflects an exceptional achievement with a high degree of initiative and self-reliance, considerable student input into the direction of the study, and critical evaluation of the established work in the area.
- 90-94 Very high standard of work similar to above but overall performance is borderline for award of a Medal. Lower level of performance in certain categories or areas of study above.
- Note:* An honours mark of 90+ and a third year WAM of 80+ are necessary but not sufficient conditions, for the award of the Medal. Examiners are referred to the Academic Board Guidelines on the award of Medals found in the general policy pages at the front of the Examiners' Manual.
- 80-89 Clear First Class quality, showing a command of the field both broad and deep, with the presentation of some novel insights. Student will have shown a solid foundation of conceptual thought and a breadth of factual knowledge of the discipline, clear familiarity with and ability to use central methodology and experimental practices of the discipline, and clear evidence of some independence of thought in the subject area. Some student input into the direction of the study or development of techniques, and critical discussion of the outcomes.

75-79 Second class honours, first division - student will have shown a command of the theory and practice of the discipline. They will have demonstrated their ability to conduct work at an independent level and complete tasks in a timely manner, and have an adequate understanding of the background factual basis of the subject. Student shows some initiative but is more reliant on other people for ideas and techniques and project is dependent on supervisor's suggestions. Student is dedicated to work and capable of undertaking a higher degree.

70-74 Second class honours, second division - student is proficient in the theory and practice of their discipline but has not developed complete independence of thought, practical mastery or clarity of presentation. Student shows adequate but limited understanding of the topic and has largely followed the direction of the supervisor.

65-69 Third class honours - performance indicates that the student has successfully completed the work, but at a standard barely meeting honours criteria. The student's understanding of the topic is extremely limited and they have shown little or no independence of thought or performance.

The award of a medal is *not* made just on the basis of a numerical mark or formula. The merits of each eligible candidate are debated by the Board of Examiners of the relevant Faculty.

## 2.6. School Facilities

Pure Mathematics 4 students traditionally enjoy a number of privileges. These include:

- Office space and a desk in the Carslaw Building.
- A computer account with access to e-mail and the World-Wide Web, as well as  $\text{\TeX}$  and laser printing facilities for the preparation of essays and projects.
- A photocopying account paid by the School for essay/project source material.
- After-hours access to the Carslaw Building. (A deposit is payable.)
- A pigeon-hole in room 728 - please inspect it regularly as lecturers often use it to hand out relevant material.
- Participation in the School's social events.

- Class representative at School meetings.

## 2.7. Scholarships, Prizes and Awards

The following scholarships and prizes may be awarded to Pure Mathematics 4 students of sufficient merit. (Note that unless the conditions of the prize state otherwise, as in the David G.A.Jackson Prize and the A.F.U.W. Prize, these prizes are also open to all Honours students in the School of Mathematics and Statistics.)

### 2.7.1. Joye Prize in Mathematics. Value: \$5650

To the most outstanding student completing fourth year honours in the School of Mathematics and Statistics, \$5650 plus medal and shield.

### 2.7.2. George Allen Scholarship in Pure Mathematics. Value:\$400

To a student proceeding to Honours in Pure Mathematics who has shown greatest proficiency in at least 24 credit points of Senior units of study in the School of Mathematics and Statistics.

### 2.7.3. Barker Prize. Value: \$375

Awarded at the fourth (Honours) year examiner's meetings for proficiency in Pure Mathematics, Applied Mathematics or Mathematical Statistics.

### 2.7.4. Ashby Prize. Value: \$250

Offered annually for the best essay, submitted by a student in the Faculty of Science, that forms part of the requirements of Pure Mathematics 4, Applied Mathematics 4 or Mathematical Statistics 4.

### 2.7.5. Norbert Quirk Prize No IV. Value: \$225

Awarded annually for the best essay on a given mathematical subject by a student enrolled in a fourth year course in mathematics (Pure Mathematics, Applied Mathematics or Mathematical Statistics) provided that the essay is of sufficient merit.

**2.7.6. David G.A.Jackson Prize.**

Value: \$200

Awarded for creativity and originality in any undergraduate Pure Mathematics unit of study.

**2.7.7. Australian Federation of University Women (NSW) Prize in Mathematics.**

Value: \$100

Awarded annually, on the recommendation of the Head of the School of Mathematics and Statistics, to the most distinguished woman candidate for the degree of BA or BSc who graduates with first class Honours in Pure Mathematics, Applied Mathematics or Mathematical Statistics.

**2.7.8. University Medal.**

Awarded to Honours students who perform outstandingly. The award is subject to Faculty rules, which require a Faculty mark of 90 or more in Pure Mathematics 4 and a Third year WAM of 80 or higher. More than one medal may be awarded in any year.

I dislike arguments of any kind. They are always vulgar,  
and often convincing.  
Oscar Wilde, *The importance of being earnest*

## CHAPTER 3

### Course descriptions

The three “core” Fourth Year courses (Commutative Algebra, Algebraic Topology and Functional Analysis) are 3 lecture per week and count  $1\frac{1}{2}$  units. The other five courses offered in 2008 are 2 lecture per week courses and count as 1 unit. Some of the Semester II courses have one of the core courses as a prerequisite. In addition, all 3(A) courses, not previously examined, are be available for credit. Each 3(A) course is run at 3 lectures per week (plus tutorial(s)) and counts as 1 unit. For substitutions by courses not given by Pure Mathematics see Section 1.3.

#### 3.1. Fourth Year Courses — Semester I

##### Algebraic Topology — L. Paunescu

Algebraic topology has advanced more rapidly than any other branch of mathematics during the twentieth century. Its influence on other branches, such as algebra, number theory, algebraic geometry, differential geometry, and analysis has been enormous.

The typical problems of topology such as whether  $\mathbb{R}^m$  is homeomorphic to  $\mathbb{R}^n$  or whether the projective plane can be embedded in  $\mathbb{R}^3$  or whether we can choose a continuous branch of the complex logarithm on the whole of  $\mathbb{C} \setminus \{0\}$  may all be interpreted as asking whether there is a suitable continuous map. The goal of Algebraic Topology is to construct invariants by means of which such problems may be translated into algebraic terms. The homotopy groups  $\pi_n(X)$  and homology groups  $H_n(X)$  of a space  $X$  are two important families of such invariants. The homotopy groups are easy to define but in general are hard to compute; the converse holds for the homology groups.

We begin with simplicial homology theory. Then we define singular homology theory, and over several weeks develop the properties which are summarized in the Eilenberg-Steenrod axioms. (These give an axiomatic

characterization of homology for reasonable spaces). We then apply homology to various examples, and conclude this part of the course with two or three lectures on cohomology and differential forms on open subsets of  $\mathbb{R}^n$ . The final third of the course shall be devoted to the fundamental group, its relationship with covering space theory and elementary ideas of combinatorial group theory.

**Prerequisite:** Metric spaces.

**Assessment:** Assignment (20%) and final exam (80%).

**References:** *Algebraic Topology: A First Course* by M. J. Greenberg and J. Harper, Benjamin/Cummings (1981)  
*Algebraic Topology* by Allan Hatcher, Cambridge UP (2002).

There are also about 70 pages of notes (by Dr Hillman) available through the School web site.

### Commutative Algebra — K. Lai

One of the most significant mathematical innovations of the 20th century was the development of “context-free geometry”. The key idea of this is that the study of the simultaneous solutions of polynomial equations such as  $x^4 + y^4 + z^4 - 5x^2y^2 = 0$  may be carried out in a way which is independent of the domain in which the variables  $x, y, z, \dots$  lie. For example they may lie in  $\mathbb{R}, \mathbb{C}, \mathbb{Z}$  or  $\mathbb{F}_q$ , and prior to these developments the study of solutions in those domains would have been regarded as separate disciplines. Thus new common ground now exists between geometry over various domains. There is also now a much better understanding of much studied concepts such as the “multiplicity” of a higher order intersection of curves or their higher dimensional analogues.

The foundations for these spectacular advances were laid largely by the Paris school of Grothendieck in the 1950’s and 60’s building on the work of many mathematicians over many centuries, but most importantly on that of Hilbert in the 1920’s. Its basis is the abstraction of geometry by algebra, and this course, on commutative algebra, is intended to be an introduction to the basic ideas of the subject. The course will cover the basic concepts of commutative algebra, and illustrate them by giving geometric interpretations as far as possible.

**Course content:**

Selected topics from

1. Localisation.
2. Flatness.
3. Polynomials. Nullstellensatz.
4. Power series. DVR.

**Prerequisites:** A thorough knowledge of linear algebra. The third year advanced algebra courses would be a distinct advantage.

**Assessment:** Assignments, and a written examination.

**References:** David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag 1994.

M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley 1969.

N. Bourbaki, *Algèbre commutative*, Ch. 8–9, Masson, NY 1983.

S. Lang, *Algebra (3rd edn)*, Addison-Wesley 1993. (For background in algebra).

### Functional Analysis —D. Daners

Functional analysis is one of the indispensable tools in the modern theory of partial differential equations, mathematical physics, probability theory etc. It grew out of the idea that functions with certain properties (for instance continuous functions on an interval) can be considered as a “point” in a “vector space,” giving the field its name.

Modern functional analysis is the study of infinite dimensional vector spaces, and linear transformations acting between them. It can be thought of linear algebra in infinite dimensional spaces. Unlike in  $\mathbb{R}^n$  a topology compatible with the vector space structure is no longer unique, and linear transformations need not be continuous. This leads to a very rich theory.

In this course we introduce Banach and Hilbert spaces, the most immediate generalisations of  $\mathbb{R}^n$ . Moreover we consider linear transformations between such spaces, and investigate their properties. The theory of eigenvalues and eigenvectors extends to a “spectral theory.” We illustrate the theory with many examples.

**Prerequisites:** Metric spaces, Lebesgue/Fourier Analysis, a knowledge of  $L_p$ -spaces is of advantage.

**Assessment:** Assignments and exam.

**Reference:** K. Yosida, *Functional Analysis*, Springer Verlag (1980).

### 3.2. Fourth Year Courses — Semester II

#### Partial Differential Equations—N. Dancer

The course is an introduction to the modern theory of partial differential equations. The major types of equations will be studied including elliptic, parabolic and if time permits also hyperbolic equations. We will look at weak solutions, maximum principles, existence and uniqueness of solutions to linear and non-linear equations. Assumed knowledge is the functional analysis from first semester and some knowledge of the Lebesgue integral and Fourier Transforms.

**Assessment:** Assignments and final exam.

**Reference:** L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998 (Library 515.353/25)

#### Symmetric Functions—A. Molev

Classical families of symmetric polynomials and relations between them, Schur polynomials, Jacobi-Trudi determinants, Gessel-Viennot method;

Littlewood-Richardson coefficients, plactic monoid, combinatorial algorithms;

Double Schur polynomials, their vanishing properties, new classes of symmetric functions.

**Assessment:** One assignment and exam.

**Reference:** W. Fulton, "Young tableaux. With applications to representation theory and geometry". London Mathematical Society Student Texts, 35, Cambridge University Press, Cambridge, 1997.

I. G. Macdonald, "Symmetric Functions and Hall Polynomials", Oxford University Press, Oxford, 1995.

*A man ought to read just as the inclination leads him; for  
what he reads as a task will do him little good*  
Samuel Johnson

## CHAPTER 4

### **The essay**

#### **4.1. Introduction**

The essay project has several objectives. First and foremost, it is intended to provide an essentially open-ended framework whereby you may pursue, develop and discover your interests in mathematics unencumbered by syllabus and the prospect of eventual written examination. Basic to this process is the use of the library and communication with others, most especially your supervisor. The writing of the essay is a most valuable part of the project. The very act of writing is an invaluable aid to comprehension. A good essay should be carefully organised, clear, readable by others, laid out well, properly referenced and convey the essential ideas. Attainment of such writing skills is of great benefit whether or not you elect to stay in mathematics.

One point should, perhaps, be emphasized: the essay project is *not* intended to be a contribution to original research; however, the essay must clearly demonstrate that you understand and have mastered the material. Originality in presentation or view in the essay is required.

#### **4.2. Choosing a supervisor and topic**

Choosing a supervisor and topic are the first two things that you should do, and are really not two choices, but one. It is recommended that you begin in the long vacation (preceding your fourth year) by seeking out members of staff and asking them about their interests and topics they would be keen on supervising. (See also Chapter 5 below). It is a good idea to ask them about their particular method of supervising and other questions important to you. Do not feel you must settle for the first person you talk to!

All staff members, lecturer and above, are potential supervisors.

There is not necessarily any correlation between supervising style and lecturing style. Also, the subject a lecturer taught you may not be their real area of interest. You should try to decide on a supervisor and topic *before*

the start of first semester. Most Department members will be available during the last two weeks of the long vacation and so, if you have not arranged a topic and supervisor at the beginning of the long vacation, you will probably have to organise your supervisor and topic during these last two weeks.

Changes in supervisor and/or topic are possible during the year (the earlier the better!). If you do change supervisors then you must notify the fourth year coordinator.

It is a good idea to have a provisional topic and supervisor in mind at the beginning of the long vacation. Your potential supervisor can then suggest some reading over the vacation and, if you have second thoughts about the topic or supervisor, it is then easy to change before the first semester starts.

### 4.3. Essay content and format

The essay must start with an introduction describing the objective and contents of the essay. The essay may end with a summary or conclusion; however, this is optional. Should you wish to make any acknowledgements, they should appear on a separate page, following the introduction.

You should aim at the best scholarly standards in providing bibliographic references. In particular, clear references to cited works should be made, where appropriate, throughout the text. Furthermore, it is not acceptable to base large portions of your essay on the existing literature and whenever part of your essay closely follows one of your sources this must be *explicitly acknowledged* in the text. References should not appear in the bibliography unless they are referred to in the text. For the format of the references see the appendix.

The essay should be clear, coherent, self contained and something that others (your fellow students and other non-specialists in the topic) can read with profit. The essay should not exceed (the equivalent of) 60 pages one and a half spaced type of normal T<sub>E</sub>X font size (i.e., as on this sheet). About 40–50 pages would normally be acceptable. Students are asked to try to keep their essays within these limits; overly long essay may be penalized. Supervisors should advise their students accordingly.

Take pains over style: especially clarity, precision and grammar. Aim at readability for the non-specialist. Avoid starting sentences with symbols. Aim for succinct statements of theorems and lemmas. Break up long proofs

into lemmas. Cross reference previous results and notation as this markedly improves readability.

Finally, the essay must be typed or printed and prepared in accordance with the instructions listed in the appendix. These days most Fourth Year students prepare their essays using a word-processing program such as  $\text{\LaTeX}$ .

#### **4.4. Submission of essay, assessment, corrections**

Three copies of the essay should be given to the Course Coordinator for marking *not later than the second Friday following the mid-semester break in second semester.*

Students should be aware that at least two weeks is set aside for marking of the essay and if the essay is not marked before the examiners meeting at the end of November then it will **not** count towards the final mark obtained in Fourth Year.

If, during the year, illness or other personal circumstances seem likely to increase the probability of late submission of the essay, such matters should be reported to the Course Coordinator through your supervisor. *Do not present such evidence at the last minute!*

Each essay will be read independently by at least two members of the Department. (The number of readers will depend on the staff available). One of the readers may or may not be the candidate's supervisor. The markers may suggest corrections should be made to the manuscript. If corrections are required, a final corrected copy of the essay should be given to the Course Coordinator as a Departmental record of the essay. If no corrections are required, one of the markers' copies will normally be kept by the department and the remaining two copies returned to the candidate.

#### **4.5. Time management and progress reports**

At the end of the first semester you should write a summary (approximately one page in length) of your essay project and progress and give this to the Course Coordinator. Here are some rough guidelines and deadlines:-

- Select supervisor and topic - Before beginning of first semester
- Reading, discussion and understanding - first semester
- Start work on first draft - beginning of second semester

- Final proofreading - mid-semester break

The essay should be submitted by the second Friday following the second semester break.

*Do not underestimate the time it takes you to do the actual writing.* Often it is not until you start writing that you will settle on a final view, or realise that you have misunderstood a particular part of the theory. Allow yourself sufficient time both for the typing and proof reading of the manuscript.

#### **4.6. Your supervisor**

To get the most benefit from the course, you should work closely with your supervisor. To this end, you may set up a regular hour each week to meet and discuss progress and problems with your essay project. Alternatively, you might come to some more informal arrangement.

You can expect your supervisor to:

- Help you select or modify your topic;
- Direct you to useful sources on your topic;
- Explain difficult points;
- Provide feedback on whether you are going in the right direction;
- Advise you on other course matters.

#### **4.7. Use of the library**

The Mathematics Library is located on the eighth floor of the Carlaw Building (Hours (subject to change): 9:00 - 5:00, Monday to Friday). It is among the most complete mathematical libraries in Australia. A photocopier is available to copy journal articles, although not all projects require extensive use of these.

The most useful source is *Mathematical Reviews*, which is available on-line at <http://ams.rice.edu/mathscinet>. *Mathematical Reviews* is updated monthly and contains reviews of a paragraph or so, describing the contents of recent books and research articles, grouped together by research area. It also has extensive subject and author search facility. Every search for information on what has been done in a given area should begin with *Mathematical Reviews*. Your supervisor can help you learn how to effectively use reference sources like *Mathematical Reviews* and, together with

the Librarian, will gladly assist you with any other problem concerning library use.

Every Friday the new mathematical periodicals and books are displayed (near the photocopier). It is a worthwhile and enjoyable habit to glance at each new journal and book to see if it contains a relevant or interesting article.

A magazine of general interest that frequently has excellent articles on developments in contemporary mathematics is the monthly *Mathematical Intelligencer*. Two other journals that usually carry expository articles of high quality are the *American Mathematical Monthly* and *L'Enseignement Mathématique*. (The majority of the articles in the latter journal are in English).

The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact.

A. Whitehead, *Science and the Modern World*

## CHAPTER 5

### Sample essay topics

Here are some topics or areas of interest suggested by members of the Department for 2008. Please note that this list is *not* intended to be complete, the topics suggested are perhaps best regarded as a guide to the likely interests of the proposer, and other staff members are willing to act as essay supervisors. The topics are grouped according to the Research Group to which each staff member belongs.

#### 5.1. Algebra Research Group

##### A/Prof David Easdown — Carslaw 619.

- a) *Computational geometry and perceptrons* Perceptrons are a primitive parallel computing device, described in an interesting book by Minsky and Papert. Some topics in the book could make a good essay, by filling out details (and correcting mistakes), for example, theorems about computational power of perceptrons, and convergence when training a perceptron to recognize patterns. The mathematics involves a mixture of algebra, logic and analysis. There are interesting open questions about polynomials, the answers to which may make some proofs constructive.
- b) *Transformation representations of semigroups* There is a well developed theory of permutation groups. However very little is known about transformation semigroups, and, in particular, how a given semigroup might be represented “efficiently” by transformations. There are a few published and unpublished results that could be the core of an essay, and possibly lead to new results.
- c) *Minimal representations of inverse semigroups by partial one-one mappings* Inverse semigroups are an abstraction of collections of partial one-one mappings of a set closed under composition and inversion. When the mappings are total then these become permutation groups. In concretely representing a given inverse semigroup

one would like to minimize the size of the set on which the partial mappings act. A discussion of the general problem, its reformulation in special cases, and even detailed solutions in particular examples would make a rounded essay.

- d) *Idempotents of semigroups* The set of idempotents may be thought of as forming the “skeleton” of a semigroup, and such skeletons have been characterized axiomatically (Easdown 1985) as biordered sets, and may be visualized (analogous to Hasse diagrams for lattices). An essay could describe the axiomatics and known techniques for constructing biordered sets. No-one has exhaustively constructed (using a machine) biordered sets of small order, so there are opportunities to break new ground.
- e) *Fundamental semigroups* These are “basic building blocks” in semigroup theory (as simple groups are in group theory), and there is a technique for constructing them from biordered sets (putting “flesh on the skeleton”), which relies on being able to calculate automorphism groups of biordered sets. The basic theory (much of which is still unpublished) could make a good essay, which might also include calculations of automorphism groups of some interesting biordered sets.
- f) *Idempotents of rings* It is well known that no finite set of axioms can describe the multiplicative semigroups of rings. However it is not known whether their sets of idempotents (biordered sets) can be axiomatically described. An essay might include the above result for multiplicative semigroups of rings, and describe known necessary conditions for their idempotents, including calculations in interesting special cases.
- g) *Braid semigroups* In 1925 Artin defined a braid group and found a presentation for it, which uses just the Coxeter relations describing the symmetric group minus the relations that the generators are involutions. If one considers braids with strings “missing” one obtains an inverse semigroup, which has a nice presentation, again which can be obtained by deleting certain relations from a presentation for the symmetric inverse semigroup of all partial one-one mappings. This was discovered recently by Easdown and Lavers. A discussion of this theorem and related questions would make an interesting and topical essay.
- h) *Group and semigroup algebras* The matrix representation theory of a finite group or semigroup is encoded in its algebra formed by taking linear combinations of group or semigroup elements with scalars from some field. Maschke’s Theorem says that if the characteristic

of the field doesn't divide the order of the group then the representation theory is well-behaved, reflected in the semisimplicity of the group algebra. Modular representation theory for groups deals with the situation where this condition on the field fails. The semisimplicity of semigroup algebras appears to be even more delicate, and the discussion of any of a number of recent results would make a fine essay.

- i) *Equational axioms for regular languages* Bloom and Esik (1993) find a list of equations that axiomatize the set of recognizable languages over a given alphabet as a universal algebra with 2 binary, 1 unary and 2 nullary operations. Their beautiful proof exploits the standard theory of minimal automata recognizing regular languages, and the fact that the behaviour of an automaton can be expressed using matrix multiplication. Their list of equations is necessarily infinite. Redko (1964) and Conway (1971) proved that no finite list of equations can suffice. An essay could give an account of these results with as much detail as space allows.

### **Dr Anthony Henderson — Carslaw 805.**

I would be happy to supervise an Honours essay in some area of representation theory or related parts of combinatorics and geometry. The following topics could be tailored to the student's prior knowledge.

a) *Symmetric functions*. This is a crucial area of combinatorics which has connections to representations of the symmetric group and polynomial representations of the general linear group. An especially interesting feature is the operation of plethysm and its generalizations.

b) *Representations of quivers*. This theory systematizes linear algebra problems such as classifying pairs of linear maps between vector spaces of a fixed dimension. A highlight is Gabriel's Theorem, which reveals the astonishing connection with reflection groups and Lie algebras.

c) *Partition combinatorics*. A partition is merely a finite nonincreasing sequence of positive integers. Many sets of interest in representation theory are in bijection with the set of partitions (or collections of partitions) satisfying certain properties. Equalities derived from representation theory then imply results in combinatorics and vice versa.

d) *Representations of Lie algebras*. Beyond the classical foundations, there has been exciting recent research on such problems as that of finding canonical bases for various kinds of irreducible representations.

e) *Finite groups of Lie type*. The leading example of this class is the general linear group over a finite field, whose character table was computed by Green in one of the seminal papers of the second half of the twentieth century.

f) *Schubert varieties*. The study of these varieties, their intersections and singularities, is of great importance in algebraic geometry. It is now deeply embedded in representation theory as well, via the ubiquitous Kazhdan-Lusztig polynomials, a master key to all the above topics.

### **A/Prof. Bob Howlett — Carslaw 523.**

I would be happy to supervise a fourth year essay on any topic related to my areas of research expertise. These are all various branches of algebra, involving group theory and/or representation theory. The four main areas are as follows.

- a) *Representation theory of finite groups of Lie type* Lie theory encompasses a large variety of topics that are in one way or another related to the work of the 19th century mathematician Sophus Lie on continuous transformation groups. The central objects of study in this area are these days known as *Lie algebras*, although in the 19th century they were called *infinitesimal groups*. It turns out that there are seven types of simple Lie algebras, imaginatively called types *A*, *B*, *C*, *D*, *E*, *F* and *G*. As a finite group theorist, I am interested in finite groups of these same types; they are basically obtained by replacing the field of complex numbers by various finite fields. These finite groups of Lie type include matrix groups (such as the general linear group, orthogonal groups and symplectic groups) as well as the slightly more mysterious groups obtained from the so-called exceptional Lie algebras (types *E*, *F* and *G*).
- b) *Finite Coxeter groups and Iwahori-Hecke algebras* Associated with each simple Lie algebra over the complex field is a finite group known as the *Weyl group* of the Lie algebra. It turns out that these Weyl groups can be realized as finite groups generated by reflections acting on Euclidean space. The Canadian mathematician H. S. M. Coxeter studied and classified finite Euclidean reflection groups, and these are now known as Coxeter groups in his honour. They include a few groups that are not Weyl groups. There is a beautiful and extensive theory of Coxeter groups, and it seems that almost all questions in Lie theory lead back to questions about Coxeter groups.

It has also been discovered that each Coxeter group has a “deformation”, or “ $q$ -analogue”, known as an Iwahori-Hecke algebra, and it is profitable to study these at the same time as one studies Coxeter groups.

- c) *Infinite Coxeter groups* Although finite Euclidean reflection groups provided the original motivation for studying Coxeter groups, there are also infinite Coxeter groups, and they are becoming ever more important in generalizations and extensions of Lie theory. Most of my work in recent years has been concerned with advancing the theory of infinite Coxeter groups, attempting to find the correct ways to extend the well-established theory of finite Coxeter groups.
- d) *Representation theory of finite soluble groups* Unlike topics (a), (b) and (c) above, this topic is not derived from Lie theory. The basic problem in representation theory of finite groups is to find ways of constructing and describing the irreducible representations of any given finite group  $G$ . Here a “representation” of  $G$  means a homomorphism from  $G$  to a group of linear operators on a vector space, and the representation is “irreducible” if there are no non-trivial proper invariant subspaces. It is hard to find general methods that apply to all finite groups, but for soluble groups a theory can be developed. The crucial fact is that if  $G$  is soluble then  $G$  has a series of normal subgroups  $\{1\} = G_0 < G_1 < \dots < G_n = G$  such that for all  $i$  the quotient group  $G_i/G_{i-1}$  is Abelian. There are theorems that describe the relationships between the irreducible representations of a group  $G$  and those of  $N$  and  $G/N$ , where  $N$  is a normal subgroup of  $G$ . This topic is known as *Clifford theory*. Repeated use of Clifford theory provides a method for proving many different theorems about representations of soluble groups.

For more information, see Dr Howlett’s web page:

<http://www.maths.usyd.edu.au/~u/bobh/>.

### **Professor Gus Lehrer — Carlaw 813.**

There are several possible topics, which come from two basic themes.

- a) The first theme is that of *symmetries of algebraic varieties*; the solution sets of polynomial equations often have symmetries (for example when the defining polynomials are permuted by a group). This leads to actions on the homology of interesting topological spaces,

such as discriminant varieties and toric varieties, the latter being defined by monomials associated with polyhedral cones. Topics available here include: computing actions via rational points; geometry of certain classical algebraic varieties; topological and geometric problems arising from reflection groups.

- b) The second theme is the *representation theory of associative algebras*. There are many semisimple algebras (with easily described representations) which may be deformed into non-semisimple ones by variation of parameters. These occur in algebra (Brauer, Hecke algebras), in mathematical physics (Temperley-Lieb algebras) and topology (BMW algebras, etc). “Cellular theory” allows one to reduce deep questions about these deformations to (usually hard) problems in linear algebra. There are several possibilities for essay topics in this area.

#### **A/Prof Andrew Mathas — Carslaw 635.**

I would be happy to supervise a fourth year essay on any topic in representations theory, or combinatorics. My main research interests are the representation theory of the symmetric groups and related algebras (such as Hecke algebra, Ariki Koike algebras, Schur algebras, general linear groups, Brauer algebras, Solomon’s descent algebras...), with an emphasis of the non-semisimple case—which is where things start to get interesting, and more difficult!

Possible topics include:

- a) *The modular representation theory of finite groups* In characteristic zero every representation of a finite group can be decomposed, in a unique way, as a direct sum of irreducible representations. For fields of positive characteristic this is no longer the case, but nevertheless the number of times that a given irreducible module can arise as a composition factor of a representation is uniquely determined. Possible projects in this area range from classifying the number of irreducible representations of a finite group, to studying the Brauer and Green correspondences.
- b) *Representations of symmetric groups* The representation theory of the symmetric group is a rich and beautiful subject which involves a lot of algebra and combinatorics. Possible projects here include character formulae, classifying homomorphisms, computing decomposition matrices, Murphy operators, the Jantzen sum formula...

- c) *Brauer algebras* The Brauer algebras arise naturally from the representation theory of the symplectic and orthogonal groups, but they can also be understood from a purely combinatorial viewpoint in terms of a “diagram calculus”. Possible topics in this area include character formulae, classifying semisimplicity, branching theorems,...
- d) *Seminormal forms* For many algebras it is possible to give “nice” generating matrices for the irreducible representations in the semisimple case. These explicit matrix representations are called seminormal forms. The study of the seminormal forms, and the resulting character formulae, for one or more algebras would make an interesting essay topic.

**Dr Alex Molev — Carslaw 531.**

- a) *Quasideterminants and their applications.* In this project we would consider matrices whose entries are elements of an arbitrary (not necessarily commutative) ring. The classical problem is to define an analog of the determinant of such a matrix. About a decade ago Gelfand and Retakh came up with natural noncommutative analogs of the determinant called quasideterminants. We would examine noncommutative analogs of the classical theorems on minors and determinants and their applications.
- b) *Lie algebras and quantum groups.* This project would deal with basic properties of the matrix Lie algebras and their “quantizations” known as quantum groups. The algebraic structure and representations of these objects will be investigated in the project.

**Dr Bill Palmer — Carslaw 521.**

Possible topics include:

- a) *Bhaskar Rao designs* A Bhaskar Rao design is a matrix with group element entries, which has certain very interesting properties. There are many open questions concerning the existence of Bhaskar Rao designs over finite groups. I am currently working on the existence problem for Bhaskar Rao designs over dihedral groups.
- b) *Latin squares* J. Dénes and A. D. Keedwell’s recent book *Latin Squares* (North-Holland, 1991), reveals some interesting topics concerning the existence of Latin squares and the connections between these squares and codes and non-associative binary systems.
- c) *Complete mappings, transversals and Latin squares*

- d) A *transversal* of the Cayley table of a group of order  $n$  is a set of  $n$  cells, one in each row, one in each column, such that no two of the cells contain the same symbol. A *complete mapping* of a group is equivalent to a transversal in a Cayley table of the group. For some large families of finite groups, complete mappings are known to exist. A long-standing conjecture of Hall and Paige: *A finite group  $G$  whose Sylow 2-subgroup is non-cyclic possesses a complete mapping* is the subject of current interest. An interested student would find the book: *Orthomorphism Graphs of Groups*, Anthony B. Evans, Springer-Verlag, 1992, and its review in *Mathematics Reviews* well worth reading.
- e) *Designs, in general* The recent 1996 survey: *CRC Handbook of Combinatorial Designs* by Charles J. Colbourn and Jeffrey H. Dinitz, contains excellent up-to-date summaries of exciting advances in research in design theory. An interested student would be sure to find a worthwhile essay topic in these 748 pages.

**A/Prof Ruibin Zhang — Carlaw 722.**

I work on the representation theory of Lie algebras and quantum groups, and applications of these algebraic structures in quantum physics. I am happy to supervise any projects in this general area. Some possible essay topics are:

- a) *Affine Kac-Moody Algebras and Vertex Operators* Level 1 representations of affine Kac-Moody algebras can be realized on Fock spaces of quantum fields by using vertex operators. These representations played important roles in quantum field theory, and the vertex operator construction gave birth to the subject of vertex operator algebras. Level 1 representations of quantum affine algebras can also be constructed similarly.
- b) *Quantum Groups and Deformations of Universal Enveloping Algebras* Quantum groups are a special type of deformations of universal enveloping algebras of Lie algebras. A distinctive property of quantum groups is their braided structure, namely, the existence of universal R-matrices satisfying the Yang-Baxter equation. For the universal enveloping algebras of the finite dimensional semi-simple Lie algebras, it is possible to classify all the deformations with braiding.

## 5.2. Analysis Research Group

### Dr Donald Cartwright — Carlaw 620.

- a) *Analysis on trees* A homogeneous tree  $T$  is an infinite graph, in which there are no "loops", and in which each vertex has the same number of neighbours. Because  $T$  is highly symmetric, it has a very large group of automorphisms, and many nice representations of this group can be defined in a natural way. Research in this area has been very active over the last decade, but the subject is quite accessible. Preliminary reading: *Harmonic Analysis and Representation Theory for Groups acting on Homogeneous Trees*, by A. Figà-Talamanca and C. Nebbia (Cambridge University Press, 1991).
- b) *Analysis on the hyperbolic plane* The "hyperbolic plane" is the unit disc in the ordinary plane, endowed with a different idea of distance: *hyperbolic distance*. It is an example of a "symmetric space", which means roughly that there is a large group of (hyperbolic-)distance preserving transformations acting on it. These give rise to beautiful tessellations of the disc which inspired many designs by the artist Escher. An interplay of geometry, complex analysis, number theory and a little bit of group theory. Preliminary reading: Chapter III of *Harmonic Analysis on Symmetric Spaces and Applications. I* by A. Terras, Springer-Verlag (1985).
- c) *Random walks on groups* The classical example is of a point moving on the group  $\mathbb{Z}$  of integers, at each time hopping one to the right or one to the left with probability  $1/2$ . One studies what happens "in the long run". Considering the natural extension to more general groups leads to an interesting interplay between analysis and group theory, which has received much attention in recent years. Some background in probability and measure theory would be useful, but not essential. Preliminary reading: *Random Walks and Electric Networks* by P. Doyle and L. Snell, Mathematical Association of America (1984).

### Dr Adrian Nelson — Carlaw 526.

Current research interests include various topics in number theory, for example those concerning zeta functions and Galois module structure. A possible essay topic would be on zeta functions and regularized determinants. See C. Deninger, *Local L-factors of motives and regularized determinants*, Invent. Math, 107, 135-150 (1992).

### 5.3. Computational Algebra Research Group

**Professor John Cannon — Carslaw 618.**

- a) *Computational Number Theory* For example:
  - Primality testing and factorization
  - Constructive algebraic number theory
  - Computation of Galois groups
- b) *Computational Group Theory*
  - Algorithmic methods for finitely presented groups
  - Algorithmic methods for permutation groups
  - Computational representation theory
  - Constructive invariant theory
- c) *Computational Differential Algebra* For example:
  - The Risch algorithm for indefinite integration

### 5.4. Geometry Research Group

**Dr Emma Carberry—Carslaw 609.**

My primary research areas are differential geometry and integrable systems, although I also often use methods from algebraic geometry in my work. I have listed some specific areas below in which I would be happy to supervise an essay; if you have other geometric interests please feel free to contact me for further ideas. I am on leave during 2007, so the best way to contact me at the moment is via e-mail: [carberry@maths.usyd.edu.au](mailto:carberry@maths.usyd.edu.au).

**Minimal Surfaces :** Physically, minimal surfaces model soap films: they locally solve the problem of finding the least area surface with a given boundary. They have been extensively studied and have a rich theory, with many interesting examples and generalisations. They are an active area of current research. There are various possibilities here for a project; ranging from the very explicit (some of the most exciting research here involves finding new examples), to the more theoretical. David Hoffman's expository article "The computer-aided discovery of new embedded minimal surfaces" in *Math Intelligencier*, 9 (1987), no 3 and Robert Osserman's book "A Survey of Minimal Surfaces" are good places in which to get a feel for this area.

**Calibrations :** The notion of a calibrated geometry was introduced in a seminal paper by Harvey and Lawson 25 years ago. In these geometries, one studies special submanifolds which are globally area minimising (this is much stronger than the local condition that characterises minimal surfaces). The first non-classical example is special Lagrangian geometry, which plays an important part in mirror symmetry and is currently a hot research area. One can also use the octonions to define three more calibrated geometries, termed exceptional geometries due to their relationship with exceptional Lie groups. This area requires some background in differential geometry, such as that provided by MATH 3968.

**Spectral Curves :** There is an important class of differential equations which can be written in a particularly simple form, called Lax form. For example, the equations describing a minimal surface in a compact Lie group or symmetric space can be written in this form. It is a beautiful fact that solutions to such differential equations on the complex plane (satisfying a finiteness condition) are in one-to-one correspondence with purely algebro-geometric data, consisting of an algebraic curve and a line bundle. The curve is called a spectral curve, and this correspondence gives one powerful new tools with which to attack the original geometric problem. This area requires some background in complex analysis, such as that provided by MATH 3964. References include Philip Griffiths' article "Linearizing flows and a cohomological interpretation of Lax equations" *American Journal of mathematics*, 107 (1985), no 6, 1445–1484 (1986). and the section by Hitchin in the book "Integrable Systems - Twistors, Loop Groups, and Riemann Surfaces" by Hitchin, Segal and Ward.

**Quaternionic Holomorphic Geometry :** When studying the geometry of surfaces, one usually works locally as there are few global tools available. A couple of years ago it was observed that surfaces in  $S^4$  could be studied more globally, using quaternionic analogs of standard complex analytic results. The quaternions enter the picture since  $S^4$  is isomorphic to the quaternionic projective line, and one can use these tools to study surfaces in  $\mathbb{R}^3$  simply by embedding  $\mathbb{R}^3$  in  $S^4$ . This new theory is being used to study conformal immersions of surfaces, and in particular to attack the Willmore conjecture. A good reference is the book "Conformal geometry of surfaces in  $S^4$  and quaternions" by Burstall, Ferus, Leschke, Pedit and Pinkall, available online at [front.math.ucdavis.edu](http://front.math.ucdavis.edu). Some background in complex analysis and differential geometry would be needed for this project.

**Curves and their Jacobians :** Algebraic curves (smooth algebraic curves are also called compact Riemann surfaces) and line bundles on them are utilised in many areas of mathematics. Important theorems include the Riemann-Roch theorem and the Abel-Jacobi theorem. A good reference is Philip Griffiths' book " An Introduction to Algebraic Curves".

**Dr Jonathan Hillman — Carslaw 617.**

Dr. Hillman will not be available for supervision in 2008.

**Dr Laurentiu Paunescu — Carslaw 816.**

I am interested in the applications of singularity theory to differential equations, and in using the combinatorics of Toric Modifications in investigating the equisingularity problem. My main research interests are:

- a) *Singularities of complex and real analytic functions*
- b) *Stratified Morse theory*
- c) *Toric resolution of singularities*

**5.5. Non-Linear Analysis Research Group**

**Professor Norm Dancer — Carslaw 717.**

- a) *Nonlinear analysis*
- b) *Nonlinear ordinary differential equations*
- c) *Nonlinear partial differential equations*
- d) *Bifurcation theory*

**Dr Daniel Daners — Carslaw 715.**

Areas of interest:

- a) partial differential equations (linear or nonlinear)
- b) ordinary differential equations (linear or nonlinear)
- c) bifurcation theory
- d) analytic semigroup theory and abstract evolution equations. (This is a theory of "ordinary differential equations" in infinite dimensional spaces with applications to partial differential equations).

Please see me to negotiate a topic of your interest or for suggestions for specific projects related to the above areas.

Extraordinaire comme les mathématiques vous aident à  
vous connaître.

Samuel Beckett, *Molloy*

## CHAPTER 6

### The talk

#### 6.1. General remarks

Before the essay is submitted at the end of Second Semester, each student gives a talk on their essay project. In past years these talks have taken place in the weeks of September leading up to the mid-semester break.

The aim of the talk is to provide training in the explanation to others of the purpose and nature of a project, within definite time limits. In recent years we have allowed forty minutes for each talk, plus five minutes for questions.

All members of the Department, Fourth Year and postgraduate students are invited to the Fourth Year talks.

*No explicit grade is given for the talk*

#### 6.2. Preparing the talk

The purpose of your talk is to convey to your *fellow students* (and the academic staff) what you are working on. They probably know very little about your essay topic; this comment may also apply to the academic staff. Do not make the talk too long or ambitious. Aim to convey the essence of your project to the audience rather than trying to impress the audience; after all, it is unlikely that you can cover the whole of your project in 40 minutes!

The key to giving a successful mathematical talk is: “Keep it simple!” *One* idea, illustrated by one or two examples, is enough for a good talk. A special case often conveys more than a general, all-encompassing theorem. For example, to give the flavour of general fields, a detailed study of a simple, but unfamiliar field, such as  $GF(9)$ , might be appropriate.

Keep in mind that the audience is swept along with you and that they cannot go back to earlier stages of your talk like when they are reading an article.

You are not giving a lecture, so although some definitions may be appropriate, lengthy technical proofs should be avoided. It is also not a good idea to over-develop the theory at the expense of examples: a well-chosen example is worth ten thousand theorems. Finally, try and relate your content to other areas of mathematics or applications; this can make the talk much more interesting for the general audience.

You should aim your talk at a general mathematical audience and *avoid* directing it at the odd specialist in your topic in the audience. Thus a good talk is judged by one criterion: you have given the audience, especially your fellow Fourth Year students, a good idea of your project and its significance.

Discuss the talk with your supervisor.

Having chosen the topic for your talk, prepare a written outline. Some people write their talk out in full, while others prefer to use only a written outline and allow improvisations. As it is probably your first talk of this kind, it is advisable to do a full dress rehearsal the previous evening; so find a blackboard and an overhead project and go through the complete talk. This will help you in judging the timing of your talk properly: it takes much longer to say things than you probably realize. If you can, find a sympathetic listener to give you feedback. Your listener does not have to be mathematically literate: a good talk is almost as much about theatre and presentation as it is about mathematics.

### 6.3. Overhead projectors

Decide if an overhead projector is appropriate. This allows preparation of complicated figures or tables ahead of time, or the inclusion of photocopies of published material in your exposition. Beware, however, that although the speaker can by this means pass a vast amount of information before the eyes of the audience very quickly, the audience will probably not take it all in. It is important either to write clearly and in large letters and to refer explicitly to each line (say by gradually revealing line-by-line using covering paper) or, in the case of a diagram or complicated formula, to allow your audience time to absorb its detail.

If you are going to use  $\text{T}_{\text{E}}\text{X}$  to create slides then *make sure that you use large enough fonts*; the easiest way to do this is to use a  $\text{L}_{\text{A}}\text{T}_{\text{E}}\text{X}$  package such as `foiltex`, or a program such as powerpoint.

Life is good for only two things, discovering mathematics  
and teaching mathematics.

Siméon Poisson

## CHAPTER 7

### **Your future and mathematics**

As a fourth year student you are a member of the mathematics department and you should take advantage of the facilities it offers. The University of Sydney has one of the top mathematics research departments in the country, and it ranks very highly internationally in several areas. There are also a number of prominent international (short and long term) visitors to the department who give seminar talks within the department. It pays to keep an eye on `scnews` (the School's web based bulletin board), for upcoming seminar announcements.

The academic staff, the many postdocs and the visitors to the department are all usually very happy to talk mathematics talk with interested students: all you have to do is find the courage to ask!

Fourth year students are also very welcome to join the staff and postgraduates in the use of the tea room; this can be a good place to meet other people in the department.

#### **7.1. The colloquium and other seminars**

Most Fridays during the year, a Colloquium is held at 2:00pm in either Carlaw Lecture Theatre 375 or in the 'Red Centre' UNSW. Topics vary, but the intention is to provide a one-hour talk on a subject of contemporary mathematical interest to a general audience. Fourth Year students are encouraged to attend the Colloquium and indeed are welcome to any seminar run in the Department. For a schedule of upcoming seminars, see the Department notice board on Level 7, Carlaw, or *read your e-mail*.

There are also a number of other active seminars in the department; notably, in algebra, computational algebra algebraic geometry seminar and category theory.

## 7.2. After fourth year, what then?

Recent graduates have found employment in a wide variety of occupations: computer related jobs, teaching (University or School), positions in insurance and finance. To find out more about where maths can take you:

<http://www.careers.usyd.edu.au>

<http://www.amsi.org.au>

<http://www.austms.org.au/Jobs>

<http://www.ice-em.org.au/careers.html>

Here we shall just outline briefly the postgraduate degree options. For more information consult the departments web pages.

## 7.3. Higher degrees

A result of II-2 or better is the minimum requirement for entry into a higher degree at Sydney. However it should be noted that one should not normally contemplate continuing without a result of at least II-1. Anyone intending to undertake a higher degree should consult with the Mathematics Postgraduate Coordinator (A/Prof Andrew Mathas, Carslaw 635) as soon as possible. The usual practice is to enrol for an M. Sc in the first instance and later to convert to a Ph.D if it is desired to continue.

## 7.4. Scholarships and other support

Scholarships, prizes and travel grants are available both for study at Sydney and for study elsewhere. Full details can be found in the University Calendar and from the Scholarships Office (Administration Building). Intending applicants should obtain application forms from the Scholarships Office as soon as possible. *The closing dates for some scholarships can be as early as September*

If you are considering further study at an Australian University, you should apply for an Australian Postgraduate Research Award. (even for an M. Sc by coursework). For study at a university in Britain or Canada, apply for a University of Sydney travelling Scholarship and also apply to the chosen university for employment as a Graduate Assistant.

### 7.5. Further study in another subject

As mentioned in the introduction to this booklet, it is quite possible to do Fourth Year Pure Mathematics and then continue with a higher degree in another subject. Within Australia, prerequisites vary from Department to Department and for those intending to follow this path it is advisable to consult with the Department concerned to determine an appropriate choice of Fourth Year topics. If you are intending to continue with Postgraduate studies in another field *outside* Australia, do check prerequisites. Provided you have done third year courses in the subject at Sydney, you will *probably* not encounter significant problems over prerequisites.

It has long been an axiom of mine that the little things are infinitely the most important.

Sherlock Holmes, *A Case of Identity*

## APPENDIX A

### Instructions on preparing the Manuscript

Essays must be typed using  $\LaTeX$  (or  $\TeX$ ), or a commercial word processing program such as word. Amongst professional mathematicians  $\LaTeX$  has become the standard; it produces better quality output than any word processing programs program — at least when it comes to mathematics. The downside to  $\LaTeX$  is that it takes some time to learn.

The fourth year coordinator will give an introduction to using  $\TeX$  and  $\LaTeX$  before the beginning of second semester. For those wishing to use  $\LaTeX$  A/Prof Mathas has written a  $\LaTeX$  class file which takes care of the basic layout of the essay; for information see

<http://www.maths.usyd.edu.au/~u/mathas/pm4/>

If you decide not to use this  $\LaTeX$  class file, then your document must satisfy the following requirements.

- a) A margin of at least 2.5cm must be left at the top, bottom, left- and right-hand side of each page. The margin is determined by the last letter or character in the longest line on the page.
- b) All pages must be numbered (in a consistent way), except for the title page.
- c) Avoid excessive use of footnotes. They are rarely necessary in mathematics.
- d) Diagrams should be created using appropriate software; hand drawn diagrams are not acceptable.
- e) Theorem Propositions, etc. should be labelled consistently throughout the document.
- f) *References* A consistent scheme should be adopted. Thus, references may be numbered: [1], [2],... and referred to as such in the text. Alternatively, by author's initial: [A], [F1], [F2],...

Sample references for the bibliography are given below:

[10] C. Chevalley, *Theory of Lie Groups*, Princeton Univ. Press, Princeton (1946).

[15] H. Grauert, On Levi's problem and the embedding of real analytic manifolds, *Ann. Math.* 68 (1958), 460-472.

[28] J. N. Mather, Stratifications and mappings, in *Dynamical Systems*, Academic Press, New York (1973), 195-232.

Note: instead of using italics, the titles of books may be underlined, or placed in quotation marks. Similarly the titles of journals may be italicised or underlined.

References should be listed alphabetically.

Notes on the use of  $\text{\TeX}$  and  $\text{\LaTeX}$  are available from Dr Howlett and the web page above.