Partial Differential Equations (PDEs)

We now move on to the main topic of this course, which is partial differential equations. First, some revision on partial differentiation.

Partial Differentiation

Let $u = u(x, y, z)$

Then $u_x \equiv \frac{\partial u}{\partial x}$ means differentiate u with respect to x, treating y and z as constants.

Examples

1) $u = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = 2y, \quad \text{etc.}$$

2) $u = \ln(x^2 + 2xy + z)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + 2xy + z}(2x + 2y)$$

Here, we have used the chain rule:

If $u = f[g(x,y,z)]$ then $\frac{\partial u}{\partial x} = \frac{df}{dg}\frac{\partial g}{\partial x}$.

3) $u = xyz + h(z)$ where $h(z)$ is any function of z.

$$\frac{\partial u}{\partial x} = yz$$

Note that $\frac{\partial u}{\partial x}$ is uniquely defined, whereas u was not since it contained the arbitrary function $h(z)$.

4) $u = x e^{yz}$

$$\frac{\partial u}{\partial x} = e^{yz} \quad , \quad \frac{\partial u}{\partial y} = x z e^{yz}$$

We can also calculate mixed second derivatives:

$$u_{xy} \equiv \frac{\partial^2 u}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = z e^{yz}$$

$$u_{yx} \equiv \frac{\partial^2 u}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = z e^{yz}$$

The result

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

is a general property (provided certain continuity conditions are satisfied) and will be true for all functions encountered in this course.

<u>Basic Ideas about PDEs.</u>

The ordinary DE (ODE)

$$\frac{du}{dx} = 1 \quad , \quad u = u(x)$$

has solution $u = x + C \quad , \quad C =$ arbitrary constant

Contrast this with the PDE

$$\frac{\partial u}{\partial x} = 1 \, , \qquad u = u(x, y)$$

which has solution

$$u = x + f(y),$$

where $f(y)$ is an <u>arbitrary function</u> of y.

Another example

ODE: $\dfrac{d^2u}{dx^2} = 0 \Rightarrow \dfrac{du}{dx} = A \Rightarrow u = Ax + B$

PDE: $\dfrac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \dfrac{\partial u}{\partial x} = f(y) \Rightarrow u = x f(y) + g(y)$

Thus we see that the solution of PDEs will involve arbitrary functions, and not just arbitrary constants as is the case for ODEs.

Ex 1. Show that $u = x^2 y - \frac{1}{2} x y^2 + f(x) + g(y)$, where $f(x)$ and $g(y)$ are arbitrary functions, is a solution of
$\dfrac{\partial^2 u}{\partial x \partial y} = 2x - y$.

Ex 2 Show that $u = e^{-8t} \sin 2x$ is a solution of
$\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$ subject to $u(0,t) = 0$, $u(\pi, t) = 0$, $u(x,0) = \sin 2x$.

Partial Differential Equation. (PDE)

A partial differential equation is any equation containing one or more partial derivatives. In general, it will involve:

One dependent variable, u, say.

Two or more independent variables, $x, y, \ldots, t, \ldots$, say.

Thus $u = u(x, y, \ldots, t, \ldots)$

Examples.

1) Heat conduction along an insulated rod:

$$\boxed{k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}}$$

where $u(x,t)$ = temperature at point x at time t

k = constant (conductivity)

[This is also known as the diffusion equation, in which case k is the diffusion coefficient.]

2) Vibration of a stretched string:

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}}$$

where $u(x,t)$ = displacement of point x at time t

c = constant (phase velocity)

[This is called the one-dimensional wave equation.]

3) Steady-state heat conduction in a flat plate:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

where $u(x,y)$ = temperature at point (x,y)

[This is Laplace's equation, and has many other applications.]

These are the main equations that we will study in this course (there are, of course, many others - one of the most famous is Schrödinger's equation for the quantum-mechanical wave function.)

As we have seen, the general solution of a PDE involves arbitrary functions. To find a unique solution, we need additional conditions on $u(x, y, \ldots, t \ldots)$. These usually take the form of initial conditions (ICs), specifying u when $t=0$, and boundary conditions (BCs), specifying u at given values of $x, y, \ldots$

Thus a complete problem consists of

$$\boxed{\text{PDE plus ICs plus BCs}}$$