MATH2065: INTRO TO PDEs
Ordinary Differential Equations: Summary

Homogeneous Linear ODEs

The general second-oder linear homogeneous ordinary differential equation is

$$y''(x) + a(x)\, y'(x) + b(x)\, y(x) = 0\,.$$

- Its ***general solution*** takes the form

  $$y_h(x) = C_1\, y_1(x) + C_2\, y_2(x)\,,$$

  where C_1 and C_2 are constants, and $y_1(x)$ and $y_2(x)$ are two independent solutions.
- The general solution depends on *two* constants in general, since this is a *second*-order ODE.
- It is in general difficult to find $y_1(x)$ and $y_2(x)$, if a and b are functions of the independent variable x.
- Since this is difficult, we will only deal with a simpler case in MATH2065! (See below.)

Homogeneous Linear Constant-Coefficient ODEs

Consider the case where the coefficients above are ***constant***; i.e.,

$$y''(x) + a\, y'(x) + b\, y(x) = 0\,, \tag{1}$$

where a and b are real constants.

- To solve, guess a solution of the form $y(x) = \exp(\lambda x)$, where λ is not yet known. (Note: $\exp(\lambda x)$ means $e^{\lambda x}$.) Substituting leads to the ***characteristic equation*** (also called the ***auxilliary equation***)

  $$\lambda^2 + a\,\lambda + b = 0\,.$$

- The characteristic equation is quadratic, and therefore there are three different situations.

  (i) Two distinct roots: Suppose the two distinct roots are λ_1 and λ_2. This means that both $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ are possible solutions. These are independent, and hence the general solution is

  $$y_h(x) = C_1\, \exp(\lambda_1 x) + C_2\, \exp(\lambda_2 x)\,.$$

  (ii) A repeated root: In this case, we have only *one* solution $\exp(\lambda x)$. It turns out that a second, independent, solution in this case is $x\, \exp(\lambda x)$. Thus, the general solution is

  $$y_h(x) = C_1\, \exp(\lambda x) + C_2\, x\, \exp(\lambda x)\,.$$

  (iii) Complex conjugate roots: Suppose these roots are $\alpha \pm i\,\beta$. Then it turns out that two independent solutions are $\exp(\alpha x)\cos(\beta x)$ and $\exp(\alpha x)\sin(\beta x)$. The general solution is therefore

  $$y_h(x) = C_1 \exp(\alpha x)\cos(\beta x) + C_2 \exp(\alpha x)\sin(\beta x)\,.$$

Inhomogeneous (Nonhomogeneous) Ordinary Differential Equations

The equation

$$y''(x) + a\,y'(x) + b\,y(x) = R(x)\,, \tag{2}$$

is the general form for a second-order, linear, constant-coefficient, inhomogeneous (nonhomogeneous) ODE. The function $R(x)$ is the *inhomogeneity.* To solve this:

(i) First, solve the *homogeneous* version of the equation (2) by setting $R(x) \equiv 0$; the technique for equation (1) will yield the ***homogeneous solution*** in the form

$$y_h(x) = C_1\,y_1(x) + C_2\,y_2(x)\,.$$

(ii) Determine a ***particular solution*** $y_p(x)$ to (2) (by using, say, undetermined coefficients as described below).

(iii) The general solution to (2) is then

$$y(x) = y_h(x) + y_p(x)\,.$$

Method of Undetermined Coefficients

Used for finding particular solutions $y_p(x)$ to equations of the form (2). *Guess* the following type of form for $y_p(x)$, substitute into (2), and then equate coefficients to find the ***undetermined coefficients***.

$R(x)$	$y_p(x)$
$k\ \exp(\alpha x)$	$C\ \exp(\alpha x)$
$P_n(x)$ polynomial degree n	$a_n x^n + \cdots + a_1 x + a_0$
$k\ \cos(\omega x)$ or $k\ \sin(\omega x)$	$A\ \cos(\omega x) + B\ \sin(\omega x)$
$k\ \exp(\alpha x)\cos(\omega x)$ or $k\ \exp(\alpha x)\sin(\omega x)$	$A\ \exp(\alpha x)\cos(\omega x) + B\ \exp(\alpha x)\sin(\omega x)$
$k\,P_n(x)\ \exp(\alpha x)$	$\exp(\alpha x)\,[a_n x^n + \cdots + a_1 x + a_0]$

Modification Rule: If even one term in the guessed form is part of the homogeneous solution, multiply *everything* by x. If the same is still true, multiply everything by x again.