MATH2065
Introduction to Partial Differential Equations
Semester 2 – Tutorial Questions (Week 11)

Questions marked with the dagger symbol † are intentionally more challenging.

1. If α is a positive constant, show that the Fourier transform defined by

$$\mathcal{F}\{f(x)\}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

of the function $f(x) = e^{-\alpha|x|}$ is given by

$$\mathcal{F}\left\{e^{-\alpha|x|}\right\} = \sqrt{\frac{2}{\pi}}\, \frac{\alpha}{k^2 + \alpha^2}\,.$$

2. Compute the inverse Fourier transform defined by

$$\mathcal{F}^{-1}\{\hat{f}(k)\}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk$$

of the function $\hat{f}(k) = e^{-\alpha|k|}$.

3. For non-negative integers n, prove the following properties of the Fourier transform that facilitate the computation of Fourier transforms:

   (a) $\mathcal{F}\left\{x^n f(x)\right\} = i^n\, \hat{f}^{[n]}(k)$

   (b) $\mathcal{F}\left\{f^{[n]}(x)\right\} = (ik)^n \hat{f}(k)$

   You may assume that all Fourier transforms involved are defined.

4. Determine the Fourier transform of the function

$$f(x) = x\,\mathrm{rect}\,(x/2) = \begin{cases} x, & \text{if } |x| < 1 \\ 0, & \text{otherwise} \end{cases}\,.$$

5. Show that if f is an EVEN, real-valued function, then its Fourier transform $\hat{f}$ is PURELY REAL.

6. † What is the Fourier transform of the function $e^{-i\beta x}$, where β is a real constant?
   Hint: Take a good look at the inverse Fourier transform!

Short answers:

1.

2. $f(x) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{x^2 + \alpha^2}$

3.

4. $\hat{f}(k) = i\sqrt{\frac{2}{\pi}} \left(\frac{\cos(k)}{k} - \frac{\sin(k)}{k^2} \right)$

5.

6.