1. If $\alpha$ is a positive constant, show that the Fourier transform defined by

$$\mathcal{F}\{f(x)\}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

of the function $f(x) = e^{-\alpha|x|}$ is given by

$$\mathcal{F}\{e^{-\alpha|x|}\} = \sqrt{\frac{2}{\pi}} \frac{\alpha}{k^2 + \alpha^2}.$$ 

2. Compute the inverse Fourier transform defined by

$$\mathcal{F}^{-1}\{\hat{f}(k)\}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk$$

of the function $\hat{f}(k) = e^{-\alpha|k|}$.

3. For non-negative integers $n$, prove the following properties of the Fourier transform that facilitate the computation of Fourier transforms:

(a) $\mathcal{F}\{x^n f(x)\} = i^n \hat{f}^{[n]}(k)$

(b) $\mathcal{F}\{f^{[n]}(x)\} = (ik)^n \hat{f}(k)$

You may assume that all Fourier transforms involved are defined.

4. Determine the Fourier transform of the function

$$f(x) = \text{rect}(x/2) = \begin{cases} x, & \text{if } |x| < 1 \\ 0, & \text{otherwise} \end{cases}.$$ 

5. Show that if $f$ is an EVEN, real-valued function, then its Fourier transform $\hat{f}$ is PURELY REAL.

6. † What is the Fourier transform of the function $e^{-i\beta x}$, where $\beta$ is a real constant?

*Hint: Take a good look at the inverse Fourier transform!*
Short answers:

1.

2. \( f(x) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{x^2 + \alpha^2} \)

3.

4. \( \hat{f}(k) = i \sqrt{\frac{2}{\pi}} \left( \frac{\cos(k)}{k} - \frac{\sin(k)}{k^2} \right) \)

5.

6.