MATH2065
Introduction to Partial Differential Equations
Semester 2 – Tutorial Questions (Week 13)

This Tutorial Set contains a selection of questions covering material over the semester.

1. Use the method of separation of variables to solve the modified wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{L^2}\, u = \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} \tag{1}$$

for the unknown $u(x,t)$ within the domain $0 < x < L$ for positive times $t > 0$. Here, L and c are known positive constants. This equation is subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0 \quad \text{for all} \quad t \geq 0\,, \\ u(L,t) &= 0 \quad \text{for all} \quad t \geq 0\,, \\ u(x,0) &= 0 \quad \text{for all} \quad x \in [0,L]\,, \\ \frac{\partial u}{\partial t}(x,0) &= f(x) \quad \text{for all} \quad x \in [0,L]\,, \end{aligned}$$

where $f(x)$ is a given function.

(a) Postulating a separable solution of the form $u(x,t) = X(x)\,T(t)$, and considering all homogeneous (i.e., zero) conditions, obtain the equations

$$X''(x) - \left[\lambda - \frac{1}{L^2}\right] X(x) = 0 \;\; ; \;\; X(0) = X(L) = 0\,, \tag{2}$$

$$T''(t) - \lambda\, c^2\, T(t) = 0 \;\; ; \;\; T(0) = 0, \tag{3}$$

where λ is a separation constant.

(b) Considering the equation (2) for $X(x)$, show that its eigenvalues are

$$\lambda = \lambda_n = \frac{1 - n^2\,\pi^2}{L^2} \;\; ; \;\; n = 1, 2, 3, \cdots\,,$$

with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\,\pi}{L} x\right) \;\; ; \;\; n = 1, 2, 3, \cdots\,.$$

(c) For choices of λ as given in (b), solve equation (3) to obtain the solutions

$$T_n(t) = \sin\left(\frac{c\,\sqrt{n^2\pi^2 - 1}}{L}\, t\right) \;\; ; \;\; n = 1, 2, 3, \ldots$$

(d) Using the above parts, write down the form of the solution $u(x,t)$ which satisfies the partial differential equation (1) along with the given homogeneous boundary and initial conditions. Your answer will depend on some number of yet to be determined constants.

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(e) Finally, utilise the non-zero initial condition to find the constants in (d) in terms of the function $f(x)$, thereby determining the complete solution to the given initial-boundary value problem.

2. In this problem, we solve a system of ordinary differential equations using two different techniques adapted from our knowledge of solutions of ODEs. Consider the simultaneous ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= y + \sin t\,, \\ \frac{dy}{dt} &= x + 2\,\cos t\,, \end{aligned}$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 0$.

(a) Laplace transforms approach: Let $X(s)$ and $Y(s)$ be the Laplace transforms of $x(t)$ and $y(t)$ respectively. By taking Laplace transforms of each of the above equations, obtain two simultaneous equations for $X(s)$ and $Y(s)$. Solve the above simultaneous equations to get

$$X(s) = \frac{3s}{(s^2+1)(s^2-1)}\,.$$

Determine $x(t)$ by inverting $X(s)$. Thereby, find $y(t)$.

(b) Second-order approach: Differentiate the first equation, and substitute for dy/dt from the second, in order to obtain a second-order constant-coefficient, non-homogeneous ODE for $x(t)$. Solve this using the standard methods (homogeneous and particular solutions), and thereby obtain $x(t)$. Thereafter, find $y(t)$ using any convenient method.

3. Use the Fourier transform to show that the solution to the wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}; \quad t > 0, \quad -\infty < x < \infty\,, \\ &u(x,0) = f(x)\,, \\ &\frac{\partial u}{\partial t}(x,0) = 0\,. \end{aligned}$$

can be written in the simple form

$$u(x,t) = \frac{1}{2}\left[f\left(x - ct\right) + f\left(x + ct\right)\right]\,.$$

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Short answers:

1. (a)

   (b)

   (c)

   (d)

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\,\pi\,x}{L}\right) \sin\left(\frac{c\sqrt{n^2\pi^2-1}}{L}\,t\right)$$

   (e) The coefficients in the solution above are given by

$$\alpha_n = \frac{2}{c\sqrt{n^2\,\pi^2-1}} \int_0^L f(x)\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,\mathrm{d}x.$$

2. (a)

$$x(t) = \frac{3}{2}\left(\cosh t - \cos t\right)\,,$$
$$y(t) = \frac{3}{2}\sinh t + \frac{1}{2}\sin t\,.$$

   (b)