This Tutorial Set contains a selection of questions covering material over the semester.

1. Use the method of separation of variables to solve the modified wave equation

\[ \frac{\partial^2 u}{\partial x^2} + \frac{1}{L^2} u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad (1) \]

for the unknown \( u(x, t) \) within the domain \( 0 < x < L \) for positive times \( t > 0 \). Here, \( L \) and \( c \) are known positive constants. This equation is subject to the boundary and initial conditions

\[
\begin{align*}
    u(0, t) &= 0 \quad \text{for all } t \geq 0, \\
    u(L, t) &= 0 \quad \text{for all } t \geq 0, \\
    u(x, 0) &= 0 \quad \text{for all } x \in [0, L], \\
    \frac{\partial u}{\partial t}(x, 0) &= f(x) \quad \text{for all } x \in [0, L],
\end{align*}
\]

where \( f(x) \) is a given function.

(a) Postulating a separable solution of the form \( u(x, t) = X(x) T(t) \), and considering all homogeneous (i.e., zero) conditions, obtain the equations

\[
\begin{align*}
    X''(x) - \left[ \lambda - \frac{1}{L^2} \right] X(x) &= 0 ; \quad X(0) = X(L) = 0, \quad (2) \\
    T''(t) - \lambda c^2 T(t) &= 0 ; \quad T(0) = 0, \quad (3)
\end{align*}
\]

where \( \lambda \) is a separation constant.

(b) Considering the equation (2) for \( X(x) \), show that its eigenvalues are

\[
\lambda = \lambda_n = \frac{1 - n^2 \pi^2}{L^2} \quad ; \quad n = 1, 2, 3, \ldots,
\]

with corresponding eigenfunctions

\[
X_n(x) = \sin \left( \frac{n \pi}{L} x \right) \quad ; \quad n = 1, 2, 3, \ldots.
\]

(c) For choices of \( \lambda \) as given in (b), solve equation (3) to obtain the solutions

\[
T_n(t) = \sin \left( \frac{c \sqrt{n^2 \pi^2 - 1}}{L} t \right) \quad ; \quad n = 1, 2, 3, \ldots
\]

(d) Using the above parts, write down the form of the solution \( u(x, t) \) which satisfies the partial differential equation (1) along with the given homogeneous boundary and initial conditions. Your answer will depend on some number of yet to be determined constants.
(e) Finally, utilise the non-zero initial condition to find the constants in (d) in terms of the function $f(x)$, thereby determining the complete solution to the given initial-boundary value problem.

2. In this problem, we solve a system of ordinary differential equations using two different techniques adapted from our knowledge of solutions of ODEs. Consider the simultaneous ordinary differential equations

$$\frac{dx}{dt} = y + \sin t,$$
$$\frac{dy}{dt} = x + 2 \cos t,$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 0$.

(a) **Laplace transforms approach**: Let $X(s)$ and $Y(s)$ be the Laplace transforms of $x(t)$ and $y(t)$ respectively. By taking Laplace transforms of each of the above equations, obtain two simultaneous equations for $X(s)$ and $Y(s)$. Solve the above simultaneous equations to get

$$X(s) = \frac{3s}{(s^2 + 1)(s^2 - 1)}.$$

Determine $x(t)$ by inverting $X(s)$. Thereby, find $y(t)$.

(b) **Second-order approach**: Differentiate the first equation, and substitute for $dy/dt$ from the second, in order to obtain a second-order constant-coefficient, non-homogeneous ODE for $x(t)$. Solve this using the standard methods (homogeneous and particular solutions), and thereby obtain $x(t)$. Thereafter, find $y(t)$ using any convenient method.

3. Use the Fourier transform to show that the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad t > 0, \quad -\infty < x < \infty,$$
$$u(x, 0) = f(x),$$
$$\frac{\partial u}{\partial t}(x, 0) = 0.$$

can be written in the simple form

$$u(x, t) = \frac{1}{2} \left[ f(x - ct) + f(x + ct) \right].$$
Short answers:

1. (a) 
(b) 
(c) 
(d) 
\[ u(x, t) = \sum_{n=1}^{\infty} \alpha_n \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{c\sqrt{n^2\pi^2 - 1}}{L} \right) \] 
(e) The coefficients in the solution above are given by 
\[ \alpha_n = \frac{2}{c\sqrt{n^2\pi^2 - 1}} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx. \]

2. (a) 
\[ x(t) = \frac{3}{2} (\cosh t - \cos t), \]
\[ y(t) = \frac{3}{2} \sinh t + \frac{1}{2} \sin t. \] 
(b)