1. The general solution has been covered in lectures.
   
   (a) \( x = 2e^{3t} \).
   
   (b) \( y = 10e^{5(1-z)} \).
   
   (c) The transformed equation is \( \frac{dX}{dt} = -3X \), with solution \( X = Ae^{-3t} \) or \( 4-3x = Ae^{-3t} \).
   
   Hence \( A = 4 \) and the solution is \( x = \frac{4}{3}(1 - e^{-3t}) \).

2. (a) While we can solve \( y' - 7y = 0 \) using standard separation of variables in the form \( \frac{dy}{y} = -7dt \) and then integration (see Q1 above), here’s an alternative approach which borrows from the approach for second-order equations.

   This ODE has characteristic equation \( \lambda - 7 = 0 \), and hence a solution of the form \( \exp(7x) \) works. (Note: the form \( \exp(7x) \) just means \( e^{7x} \), and can be used for clearer display purposes.) The general solution is therefore \( y(x) = C_1 \exp(7x) \), where \( C_1 \) is an arbitrary constant.

   (b) \( y'' + y' - 2y = 0 \) has characteristic equation \( \lambda^2 + \lambda - 2 = 0 \), which gives \( \lambda = 1 \) and \( \lambda = -2 \). The two solutions corresponding to these are \( \exp(x) \) and \( \exp(-2x) \), leading to the general solution, with \( C_1 \) and \( C_2 \) being arbitrary constants,

   \[ y(x) = C_1 \exp(x) + C_2 \exp(-2x) \, . \]

   (c) The characteristic equation for \( y'' - 3y' = 0 \) is \( \lambda^2 - 3\lambda = 0 \), with roots \( \lambda = 0 \) and \( \lambda = 3 \). The general solution is therefore

   \[ y(x) = C_1 \exp(0x) + C_2 \exp(3x) = C_1 + C_2 \exp(3x) \, . \]

   (d) \( y'' - 4y = 0 \) gives the characteristic equation \( \lambda^2 - 4 = 0 \), and hence \( \lambda = \pm 2 \), leading to

   \[ y(x) = C_1 \exp(2x) + C_2 \exp(-2x) \, . \]

   (e) \( y'' + 4y = 0 \) gives a characteristic equation \( \lambda^2 + 4 = 0 \), and hence \( \lambda = \pm 2i \). These complex conjugate solutions have no real part. Hence there is no exponential part to the solution; only sines and cosines. The two corresponding solutions are \( \sin 2x \) and \( \cos 2x \), leading to the general solution

   \[ y(x) = C_1 \sin 2x + C_2 \cos 2x \, . \]

   (f) The characteristic equation for \( y'' + 6y' + 9y = 0 \) is \( \lambda^2 + 6\lambda + 9 = 0 \), which gives \( (\lambda + 3)^2 = 0 \). This therefore has one repeated root, and the corresponding solution is \( \exp(-3x) \). Hence, \( x \) times this is another solution; i.e., \( x \exp(-3x) \). Therefore,

   \[ y(x) = C_1 \exp(-3x) + C_2 x \exp(-3x) \, . \]
(g) \(y'' - 6y' + 25y = 0\) has characteristic equation \(\lambda^2 - 6\lambda + 25 = 0\), whose roots are \(3 \pm 4i\). This gives the two solutions \(\exp(3x) \cos 4x\) and \(\exp(3x) \sin 4x\), and thus
\[
y(x) = C_1 \exp(3x) \cos 4x + C_2 \exp(3x) \sin 4x.
\]

(h) \(\hat{y}'' + 2y'' - y' - 2y = 0\) is a third-order ODE, but there's no reason not to expect our technique to work. The characteristic equation, obtained by substituting a solution of the form \(y(x) = \exp(\lambda x)\) directly into the ODE, is
\[
\lambda^3 + 2\lambda^2 - \lambda - 2 = 0.
\]
By inspection, we see that \(\lambda = 1\) is a solution to this, which means that \(\lambda - 1\) will be a factor. Factoring this out, we obtain
\[
(\lambda - 1) (\lambda^2 + 3\lambda + 2) = 0.
\]
The quadratic factor above also factors to \((\lambda + 1) (\lambda + 2)\), and thus we get the three possible \(\lambda\) values +1, −1 and −2. The general solution must be
\[
y(x) = C_1 \exp(x) + C_2 \exp(-x) + C_3 \exp(-2x),
\]
where we need three arbitrary constants \(C_1, C_2\) and \(C_3\) since this is third-order.

3. (a) We have already found the general solution to \(y' - 7y = 0\) above; it is \(y(x) = C_1 \exp(7x)\) for some constant \(C_1\). Putting in the initial condition \(y(0) = 1\), we find that \(C_1 = 1\). The solution to the initial value problem is thus \(y(x) = \exp(7x)\).

(b) Again, we have found the general solution to \(y'' - 3y' = 0\) above, which is \(y(x) = C_1 + C_2 \exp(3x)\). Now, we need \(y(0) = 0\), and hence \(C_1 + C_2 = 0\). Secondly, we have an initial condition on \(y'\), and so we compute \(y'(x) = 3C_2 \exp(3x)\). Applying \(y'(0) = 1\), we have \(3C_2 = 1\), and thus \(C_2 = 1/3\). Then \(C_1 = -C_2 = -1/3\), leading to the solution
\[
y(x) = \frac{1}{3} \left[ \exp(3x) - 1 \right].
\]

(c) The general solution to \(y'' + 6y' + 9y = 0\), from the previous problem, is
\[
y(x) = C_1 \exp(-3x) + C_2 x \exp(-3x).
\]
Since \(y(0) = 0\), we obtain that \(C_1 = 0\). Thus, \(y(x) = C_2 x \exp(-3x)\). Then, \(y'(x) = C_2 [x(-3) \exp(-3x) + \exp(-3x)]\). We also need to satisfy \(y'(0) = 1\), and thus \(C_2 = 1\). The solution is therefore
\[
y(x) = x \exp(-3x).
\]

(d) The ODE \(y'' - 6y' + 25y = 0\) is another we have solved in the previous problem:
\[
y(x) = C_1 \exp(3x) \cos(4x) + C_2 \exp(3x) \sin(4x).
\]
Applying \(y(0) = 0\), we find that \(C_1 = 0\), and so we can discard the first of the above two terms. Differentiating the remaining term,
\[
y'(x) = C_2 \exp(3x) [4 \cos(4x) + 3 \sin(4x)],
\]
and applying \(y'(0) = 4\), we get \(C_2 = 1\). Thus, the solution is
\[
y(x) = \exp(3x) \sin(4x).
\]
4. The verification that these solutions work is left as a simple exercise in differentiation. Since there are two different solutions $y_1(t) = t^3$ and $y_2(t) = t^3 \ln t$, and this is a linear equation, the guess for the general solution is a linear combination of these two. In other words, it is

$$y(t) = C_1 t^3 + C_2 t^3 \ln t,$$

where $C_1$ and $C_2$ are arbitrary constants.

5. (a) The definitions of these functions are

$$\cosh x := \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x := \frac{e^x - e^{-x}}{2}.$$  

The sketch of $\cosh x$ “looks like” the sketch of the function $y = x^2 + 1$, in that it is even, takes the value 1 at $x = 0$, and increases without bound as $|x|$ gets large. (However, it isn’t quite like $x^2 + 1$ in detail, since the increase is exponential for large $|x|$). The sketch of $\sinh x$ “looks like” the sketch of the function $y = x^3$, in that it is odd, zero at $x = 0$, and increases to $+\infty$ as $x \to +\infty$, and decreases to $-\infty$ as $x \to -\infty$. (Again, this analogy is not correct in detail, since the behaviour for large $|x|$ is exponential.) Numerically generated graphs of these functions appears below:

(b) By adding and subtracting the expressions in part (a) for $\sinh x$ and $\cosh x$, we can obtain

$$e^x = \cosh x + \sinh x \quad \text{and} \quad e^{-x} = \cosh x - \sinh x.$$  

We can use the above observation to write

$$y(x) = C_1 e^{px} + C_2 e^{-px} = C_1 (\cosh px + \sinh px) + C_2 (\cosh px - \sinh px) = (C_1 + C_2) \cosh px + (C_1 - C_2) \sinh px = D_1 \cosh px + D_2 \sinh px,$$

where $D_1 = C_1 + C_2$ and $D_2 = C_1 - C_2$ are themselves arbitrary constants since $C_1$ and $C_2$ are.