MATH2065
Introduction to Partial Differential Equations
Semester 2 – Tutorial Solutions (Week 1)

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1. The general solution has been covered in lectures.

   (a) $x = 2e^{3t}$.

   (b) $y = 10e^{5(1-z)}$.

   (c) The transformed equation is $dX/dt = -3X$, with solution $X = Ae^{-3t}$ or $4-3x = Ae^{-3t}$. Hence $A = 4$ and the solution is $x = \frac{4}{3}(1 - e^{-3t})$.

2. (a) While we can solve $y' - 7\,y = 0$ using standard separation of variables in the form $\frac{dy}{y} = -7dt$ and then integration (see Q1 above), here's an alternative approach which borrows from the approach for second-order equations.
   This ODE has characteristic equation $\lambda - 7 = 0$, and hence a solution of the form $\exp(7x)$ works. (Note: the form $\exp(7x)$ just means e^{7x}, and can be used for clearer display purposes.) The general solution is therefore $y(x) = C_1 \exp(7x)$, where C_1 is an arbitrary constant.

   (b) $y'' + y' - 2\,y = 0$ has characteristic equation $\lambda^2 + \lambda - 2 = 0$, which gives $\lambda = 1$ and $\lambda = -2$. The two solutions corresponding to these are $\exp(x)$ and $\exp(-2x)$, leading to the general solution, with C_1 and C_2 being arbitrary constants,

   $$y(x) = C_1 \exp(x) + C_2 \exp(-2x)\ .$$

   (c) The characteristic equation for $y'' - 3\,y' = 0$ is $\lambda^2 - 3\lambda = 0$, with roots $\lambda = 0$ and $\lambda = 3$. The general solution is therefore

   $$y(x) = C_1 \exp(0x) + C_2 \exp(3x) = C_1 + C_2 \exp(3x)\ .$$

   (d) $y'' - 4\,y = 0$ gives the characteristic equation $\lambda^2 - 4 = 0$, and hence $\lambda = \pm 2$, leading to

   $$y(x) = C_1 \exp(2x) + C_2 \exp(-2x)\ .$$

   (e) $y'' + 4\,y = 0$ gives a characteristic equation $\lambda^2 + 4 = 0$, and hence $\lambda = \pm 2i$. These complex conjugate solutions have no real part. Hence there is no exponential part to the solution; only sines and cosines. The two corresponding solutions are $\sin 2x$ and $\cos 2x$, leading to the general solution

   $$y(x) = C_1 \sin 2x + C_2 \cos 2x\,.$$

   (f) The characteristic equation for $y'' + 6\,y' + 9\,y = 0$ is $\lambda^2 + 6\lambda + 9 = 0$, which gives $(\lambda + 3)^2 = 0$. This therefore has one repeated root, and the corresponding solution is $\exp(-3x)$. Hence, x times this is another solution; i.e., $x \exp(-3x)$. Therefore,

   $$y(x) = C_1 \exp(-3x) + C_2\, x \exp(-3x)\ .$$

(g) $y'' - 6\,y' + 25\,y = 0$ has characteristic equation $\lambda^2 - 6\lambda + 25 = 0$, whose roots are $3 \pm 4i$. This gives the two solutions $\exp(3x)\cos 4x$ and $\exp(3x)\sin 4x$, and thus

$$y(x) = C_1\,\exp(3x)\,\cos(4x) + C_2\,\exp(3x)\sin(4x)\ .$$

(h) † $y''' + 2\,y'' - y' - 2\,y = 0$ is a *third-order* ODE, but there's no reason not to expect our technique to work. The characteristic equation, obtained by substituting a solution of the form $y(x) = \exp(\lambda x)$ directly into the ODE, is

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0\,.$$

By inspection, we see that $\lambda = 1$ is a solution to this, which means that $\lambda - 1$ will be a factor. Factoring this out, we obtain

$$(\lambda - 1)\left(\lambda^2 + 3\lambda + 2\right) = 0\,.$$

The quadratic factor above also factors to $(\lambda + 1)(\lambda + 2)$, and thus we get the three possible λ values $+1$, -1 and -2. The general solution must be

$$y(x) = C_1\,\exp(x) + C_2\,\exp(-x) + C_3\exp(-2x)\ ,$$

where we need *three* arbitrary constants C_1, C_2 and C_3 since this is third-order.

3. (a) We have already found the general solution to $y' - 7\,y = 0$ above; it is $y(x) = C_1\,\exp(7x)$ for some constant C_1. Putting in the initial value $y(0) = 1$, we find that $C_1 = 1$. The solution to the initial value problem is thus $y(x) = \exp(7x)$.

(b) Again, we have found the general solution to $y'' - 3\,y' = 0$ above, which is $y(x) = C_1 + C_2\exp(3x)$. Now, we need $y(0) = 0$, and hence $C_1 + C_2 = 0$. Secondly, we have an initial condition on y', and so we compute $y'(x) = 3C_2\exp(3x)$. Applying $y'(0) = 1$, we have $3C_2 = 1$, and thus $C_2 = 1/3$. Then $C_1 = -C_2 = -1/3$, leading to the solution

$$y(x) = \frac{1}{3}\left[\exp(3x) - 1\right]\ .$$

(c) The general solution to $y'' + 6\,y' + 9\,y = 0$, from the previous problem, is

$$y(x) = C_1\,\exp(-3x) + C_2\,x\,\exp(-3x)\ .$$

Since $y(0) = 0$, we obtain that $C_1 = 0$. Thus, $y(x) = C_2 x\exp(-3x)$. Then, $y'(x) = C_2\left[x(-3)\exp(-3x) + \exp(-3x)\right]$. We also need to satisfy $y'(0) = 1$, and thus $C_2 = 1$. The solution is therefore

$$y(x) = x\,\exp(-3x)\ .$$

(d) The ODE $y'' - 6\,y' + 25\,y = 0$ is another we have solved in the previous problem:

$$y(x) = C_1\,\exp(3x)\,\cos(4x) + C_2\,\exp(3x)\sin(4x)\ .$$

Applying $y(0) = 0$, we find that $C_1 = 0$, and so we can discard the first of the above two terms. Differentiating the remaining term,

$$y'(x) = C_2\,\exp(3x)\left[4\cos(4x) + 3\sin(4x)\right]\ ,$$

and applying $y'(0) = 4$, we get $C_2 = 1$. Thus, the solution is

$$y(x) = \exp(3x)\sin(4x)\ .$$

4. The verification that these solutions work is left as a simple exercise in differentiation. Since there are two different solutions $y_1(t) = t^3$ and $y_2(t) = t^3 \ln t$, and this is a linear equation, the guess for the general solution is a linear combination of these two. In other words, it is

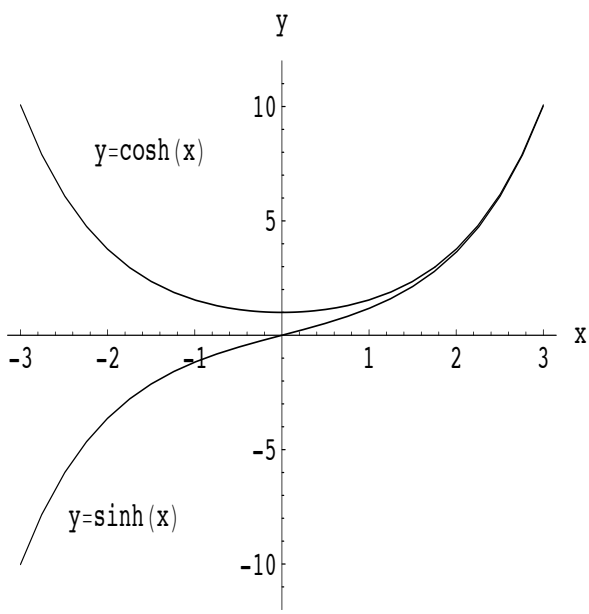
$$y(t) = C_1\, t^3 + C_2\, t^3 \ln t\,,$$

where C_1 and C_2 are arbitrary constants.

5. (a) The definitions of these functions are

$$\cosh x := \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x := \frac{e^x - e^{-x}}{2}\,.$$

The sketch of $\cosh x$ "looks like" the sketch of the function $y = x^2+1$, in that it is even, takes the value 1 at $x = 0$, and increases without bound as $|x|$ gets large. (However, it isn't quite like $x^2 + 1$ in detail, since the increase is exponential for large $|x|$). The sketch of $\sinh x$ "looks like" the sketch of the function $y = x^3$, in that it is odd, zero at $x = 0$, and increases to $+\infty$ as $x \to +\infty$, and decreases to $-\infty$ as $x \to -\infty$. (Again, this analogy is not correct in detail, since the behaviour for large $|x|$ is exponential.) Numerically generated graphs of these functions appears below:



(b) By adding and subtracting the expressions in part (a) for $\sinh x$ and $\cosh x$, we can obtain

$$e^x = \cosh x + \sinh x \qquad \text{and} \qquad e^{-x} = \cosh x - \sinh x\,.$$

We can use the above observation to write

$$\begin{aligned} y(x) &= C_1 e^{px} + C_2 e^{-px} \\ &= C_1\,(\cosh px + \sinh px) + C_2\,(\cosh px - \sinh px) \\ &= (C_1 + C_2)\cosh px + (C_1 - C_2)\sinh px \\ &= D_1 \cosh px + D_2 \sinh px\,, \end{aligned}$$

where $D_1 = C_1 + C_2$ and $D_2 = C_1 - C_2$ are themselves arbitrary constants since C_1 and C_2 are.