1. Use the method of undetermined coefficients to find a particular solution to each of the following inhomogeneous differential equations:

(a) \( y'' - 4y = e^x \)
(b) \( y'' - 9y = x + 18 \)
(c) \( y'' - y' + 2y = -2 \sin x \)
(d) \( y'' + y = x^2 \)
(e) \( y'' - 9y = 2e^{3x} \)

2. Solve the following initial value problems.

(a) \( y'' + y' - 6y = 5e^x \), with initial conditions \( y(0) = 0 \) and \( y'(0) = 2 \)
(b) \( y'' + 3y' + 2y = xe^{4x} \), with initial conditions \( y(0) = 0 \) and \( y'(0) = 1 \)
(c) \( y'' - 4y' + 3y = 2 \sin x \), with initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \)

3. Show that if \( y_1(x) \) is a solution of \( y'' + ay' + by = f_1(x) \) and if \( y_2(x) \) is a solution of \( y'' + ay' + by = f_2(x) \), then the function \( y_1(x) + y_2(x) \) is a solution of \( y'' + ay' + by = f_1(x) + f_2(x) \).

Use this result to obtain a particular solution of the following ODEs.

(a) \( 3y'' + 10y' + 3y = x + \cos x \)
(b) \( y'' + 5y' + 6y = \sin x + 2e^x \)
(c) \( y'' - 4y' + 3y = e^x + e^{2x} \)

4. (Resonance) Consider a block and spring with mass \( m \), spring constant \( k \), damping coefficient \( \gamma \), and imposed forcing \( F(t) \). Recall that its motion can be represented by \( m\ddot{x} + \gamma \dot{x} + kx = F(t) \).

where \( x(t) \) represents the position of the block measured from its equilibrium. For the purposes of this problem, consider the particular case where \( m = 1, \gamma = 0, k = 4 \) and \( F(t) = \cos \omega t \), where the frequency \( \omega \) is a positive constant.

(a) Recall that the solution \( x(t) \) can be represented as \( x(t) = x_h(t) + x_p(t) \), where \( x_h \) and \( x_p \) are respectively the homogeneous and the particular solutions to the ODE. Determine \( x_h(t) \).
(b) Determine \( x_p(t) \). [Hint: does your solution work for all positive \( \omega \)s, or do you have to consider some special cases?]

(c) † Based on your solutions to (a) and (b) above, describe the behaviour of the block at large times. Is there a “special” value of the forcing frequency \( \omega \) at which something different happens?

Short answers:

1. (a) \( y_p = -\frac{1}{3}e^x \)
   (b) \( y_p(x) = -\frac{1}{9}x - 2 \)
   (c) \( y_p(x) = -\cos x - \sin x \)
   (d) \( y_p(x) = x^2 - 2 \)
   (e) \( y_p = \frac{1}{2}xe^{3x} \)

2. (a) \( y(x) = \frac{7}{5}e^{2x} - \frac{3}{20}e^{-3x} - \frac{5}{4}e^x \)
   (b) \( y(x) = -\frac{37}{36}e^{-2x} + \frac{26}{25}e^{-x} + (\frac{1}{30}x - \frac{11}{900})e^{4x} \)
   (c) \( y(x) = -\frac{2}{5}e^{3x} + e^x + \frac{1}{5}\sin x + \frac{2}{3}\cos x \)

3. (a) \( y_p = \frac{1}{3}x - \frac{10}{9} + \frac{1}{10}\sin x \)
   (b) \( y_p = \frac{1}{6}e^x + \frac{1}{10}\sin x - \frac{1}{10}\cos x \)
   (c) \( y_p = -\frac{1}{2}xe^x - e^{2x} \)

4. (a) \( x_h(t) = C_1 \cos (2t) + C_2 \sin (2t) \)
   (b) \[
x(t) = \begin{cases} 
\frac{1}{4 - \omega^2} \cos (\omega t) , & (\omega \neq 2) \\
\frac{t}{4} \sin (2t) , & (\omega = 2)
\end{cases}
\]
   (c) \[
x(t) = \begin{cases} 
C_1 \cos (2t) + C_2 \sin (2t) + \frac{1}{4 - \omega^2} \cos (\omega t) & (\text{if } \omega \neq 2) \\
C_1 \cos (2t) + C_2 \sin (2t) + \frac{t}{4} \sin (2t) & (\text{if } \omega = 2)
\end{cases}
\]