1. Determine the eigenvalues \( \lambda \) and corresponding eigenfunctions of the differential equation

\[
\frac{d^2\phi}{dx^2} + \lambda \phi = 0
\]

with boundary conditions \( \phi(0) = 0 \) and \( \frac{d\phi}{dx}(L) = 0 \). Analyze the three cases with real \( \lambda \):\n\( \lambda > 0 \), \( \lambda = 0 \), \( \lambda < 0 \).

2. Solve the heat equation \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \) with boundary conditions \( u(0, t) = 0 \) and \( u(L, t) = 0 \) and initial conditions

\[
u(x, 0) = 3 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L}.
\]

3. As the previous problem but with initial condition

\[
u(x, 0) = \begin{cases} 
1, & 0 < x \leq L/2, \\
2, & L/2 < x < L,
\end{cases}
\]

4. Solve the heat equation \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \) with boundary conditions \( \frac{\partial u}{\partial x}(0, t) = 0 \) and \( \frac{\partial u}{\partial x}(L, t) = 0 \) for \( t > 0 \) and three different initial conditions given by

(a) \( u(x, 0) = \begin{cases} 
0, & 0 < x \leq L/2, \\
1, & L/2 < x < L,
\end{cases} \)

(b) \( u(x, 0) = 6 + 4 \cos \frac{3\pi x}{L} \).

(c) \( u(x, 0) = -3 \cos \frac{8\pi x}{L} \).

5. In this problem, we apply the idea of separation of variables that we have learnt mainly within the context of the heat equation, to the wave equation. Consider the initial-boundary value problem

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

in the domain \( 0 < x < L, t > 0 \), subject to the boundary and initial conditions

\[
u(0, t) = 0 \quad \text{for} \quad t \geq 0, \\
u(L, t) = 0 \quad \text{for} \quad t \geq 0, \\
u(x, 0) = 0 \quad \text{for} \quad 0 \leq x \leq L, \\
\frac{\partial u}{\partial t}(x, 0) = 3 c \sin \frac{4\pi x}{L} \quad \text{for} \quad 0 \leq x \leq L.
\]
Here, the unknown is $u = u(x,t)$, and $c$ and $L$ are positive constants. This models a string tied at the endpoints $x = 0$ and $x = L$, and given an initial vertical velocity $3c \sin \frac{4\pi x}{L}$ at time zero when it is in a purely horizontal position.

Solve the given initial-boundary value problem using separation of variables.

**Short answers:**

1. 

$$\lambda_n = p^2 = \left( \frac{\pi/2 + n\pi}{L} \right)^2 \quad n = 0, 1, 2, 3, \ldots$$

$$\phi_n(x) = \sin \left( \frac{\pi/2 + n\pi}{L} x \right) \quad n = 0, 1, 2, \ldots$$

2. 

$$u(x,t) = 3 \sin \frac{\pi x}{L} \exp \left( -\frac{k\pi^2 t}{L^2} \right) - \sin \frac{3\pi x}{L} \exp \left( -\frac{9k\pi^2 t}{L^2} \right).$$

3. 

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - 2(-1)^n + \cos \frac{n\pi}{2}}{n} \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}.$$

4. (a) $A_0 = \frac{1}{2}$, $A_n = -\frac{2}{n\pi} \sin \frac{n\pi}{2}$ for $n \neq 0$.

(b) $A_0 = 6$, $A_3 = 4$, all other $A_n$ are zero.

(c) $A_8 = -3$, all other $A_n$ are zero.

5. 

$$u(x,t) = \frac{3L}{4\pi} \sin \frac{4\pi x}{L} \sin \frac{4\pi ct}{L}.$$