

Tutorial 3

1. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_i = F_{i-2} + F_{i-1}$ for all $i \geq 2$. (In Question 3 last week we investigated the sequence (F_i) modulo a prime p .)

(i) Suppose that a Fibonacci number F_n is divisible by some positive integer d , and write $F_{n+1} = k$. Show that $F_n \equiv kF_0 \pmod{d}$ and $F_{n+1} \equiv kF_1 \pmod{d}$, and use induction on i to prove that $F_{n+i} \equiv kF_i \pmod{d}$ for all integers $i \geq 0$. [Hint: $kF_0 = k \cdot 0 = 0$; so the first statement you are asked to prove just says $F_n \equiv 0 \pmod{d}$. As for the second, in fact $F_{n+1} = kF_1 = k$.]

(ii) Use Part (i) and induction on j to prove that if $d|F_n$ then $d|F_{jn}$ for all natural numbers j .

2. Let F_n be as in Question 1. Check that the following matrix formula holds for $n = 1$, $n = 2$ and $n = 3$, and then try to prove it for all positive integers n , using induction:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Use this formula, combined with the same formula with n replaced by $2n$, to show that $F_n(F_{n-1} + F_{n+1}) = F_{2n}$, and $F_n^2 + F_{n+1}^2 = F_{2n+1}$. Verify this for $n \leq 5$ by direct calculation.

3. For each prime p , define the *Fibonacci entry point* of p to be the least integer n such that $p|F_n$ (where F_n is as in Questions 1 and 2). The following table shows the Fibonacci entry points of all the primes up to 61.

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
3	4	5	8	10	7	9	18	24	14	30	19	20	44	16	27	58	15

Use this table and Question 1 above to verify that for $p \leq 61$ the following is true: if $p \equiv \pm 1 \pmod{5}$ then $p|F_{p-1}$, and if $p \equiv \pm 2 \pmod{5}$ then $p|F_{p+1}$. Find all primes p for which $p \not\equiv \pm 1$ and $p \not\equiv \pm 2 \pmod{5}$, and find their Fibonacci entry points.

4. (i) Let p be a prime number, and let $t_1 = 1$. Now define t_i recursively, for $i > 1$, as follows: if $t_i \neq 0$, choose a number s_i such that $s_i t_i \equiv 1 \pmod{p}$ and let t_{i+1} be the residue of $1 + s_i \pmod{p}$. That is to say, t_{i+1} is the unique integer satisfying $0 \leq t_{i+1} < p$ and $(t_{i+1} - 1)t_i \equiv 1 \pmod{p}$. The sequence (t_1, t_2, \dots) stops as soon as we find an integer ℓ such that $t_\ell = 0$. Work out the value of ℓ for the first few prime numbers p , and compare with the table in Question 3. What relationship do you observe? MATH2988 students: prove it.

(ii) Assuming the relationship you observed persists, show that the Fibonacci entry point of a prime number p can never exceed $p + 1$. (Hint: Show that the sequence (t_i) in (i) cannot have any repeated terms, by considering the first repeated term.)

- *5. Find a closed formula for $\sum_{n=0}^{\infty} F_n x^n$ (ignoring questions of convergence).

- *6. Prove that for every $n \in \mathbb{Z}^+$ there exists an $r \in \mathbb{Z}^+$ and integers k_i for $1 \leq i \leq r$ with $k_1 \geq 2$ and $k_{i+1} \geq k_i + 2$ for $1 \leq i < r$, such that $n = \sum_{i=1}^r F_{k_i}$. Prove, furthermore, that the k_i are uniquely determined by n .