Extra Solutions 3

4. (*i*) Since $t_1 = 1 = F_1 = F_2$ we observe that $F_i t_i \equiv F_{i+1} \pmod{p}$ holds when $i = 1$. Proceeding inductively, assume that this congruence is satisfied for some value of i for which t_{i+1} is defined. Since t_{i+1} satisfies $(t_{i+1} - 1)t_i \equiv 1 \pmod{p}$ it follows that $t_i t_{i+1} \equiv 1 + t_i \pmod{p}$, and so $F_i t_i \equiv F_{i+1} \pmod{p}$ gives

$$F_{i+1}t_{i+1} \equiv F_i t_i t_{i+1} \equiv F_i(1 + t_i) \equiv F_i + F_i t_i \equiv F_i + F_{i+1} \equiv F_{i+2} \pmod{p}.$$

By induction the congruence $F_i t_i \equiv F_{i+1} \pmod{p}$ is valid for all positive integers i such that t_i exists.

Suppose that $t_\ell = 0$. If $1 \le i < \ell$ then $p \nmid t_i$, and so $p|F_i t_i$ if and only if $p|F_i$. That is, $F_{i+1} \equiv 0 \pmod{p}$ if and only if $F_i \equiv 0 \pmod{p}$. Since $F_1 \not\equiv 0 \pmod{p}$ a trivial induction yields that $F_i \not\equiv 0 \pmod{p}$ for all $i \in \{1, 2, \ldots, \ell\}$. But $F_{\ell+1} \equiv F_\ell t_\ell \equiv 0 \pmod{p}$, and so $\ell + 1$ is the Fibonacci entry point of p.

5. Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$. Then $xf(x) = \sum_{n=0}^{\infty} F_n x^{n+1} = \sum_{n=1}^{\infty} F_{n-1}x^n$. Another dose of the same medicine gives $x^2 f(x) = \sum_{n=2}^{\infty} F_{n-2}x^n$. Now

$$\begin{aligned} xf(x) + x^2 f(x) &= F_0 x^1 + \sum_{n=2}^{\infty} F_{n-1}x^n + \sum_{n=2}^{\infty} F_{n-2}x^n \\ &= \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2})x^n \qquad \text{(since } F_0 = 0\text{)} \\ &= \sum_{n=2}^{\infty} F_n x^n \\ &= f(x) - F_0 x^0 - F_1 x^1. \end{aligned}$$

Since $F_0 = 0$ and $F_1 = 1$ this gives $f(x)(1 - x - x^2) = x$; hence

$$f(x) = \frac{x}{1 - x - x^2}.$$

This argument is valid if $f(x)$ is regarded as a so-called "formal power series" in the symbol x; there is no implication that it is possible to replace x by any number and get anything valid. However, if we do regard x as a real variable and define $f(x) = x/(1-x-x^2)$ (whenever $x \ne \frac{-1\pm\sqrt{5}}{2}$), then by repeatedly differentiating $f(x)(1 - x - x^2) = x$ you can show that

$$f^{(n)}(x)(1 - x - x^2) - nf^{(n-1)}(x)(1 + 2x) - n(n - 1)f^{(n-2)}(x) = 0,$$

and by putting $x = 0$ deduce that

$$\tfrac{1}{n!}f^{(n)}(0) = \tfrac{1}{(n-1)!}f^{(n-1)}(0) + \tfrac{1}{(n-2)!}f^{(n-2)}(0).$$

Since, moreover, $f(0) = 0$ and $f'(0) = 1$, it follows by induction that F_n is the coefficient of x^n in the Taylor series expansion for $f(x)$ about $x = 0$. The ratio test can be used to show that the series converges when $|x| < (\sqrt{5} - 1)/2$.

6. Let us first prove that for all $r \in \mathbb{Z}^+$ the following statement, P_r, is true:

> P_r: *if* $(k_i)_{1 \le i \le r}$ *is any sequence of integers such that* $k_1 \ge 2$ *and* $k_{i+1} \ge k_i + 2$ *for all* $i \in \mathbb{Z}^+$ *such that* $1 \le i \le r$, *then* $\sum_{i=1}^{r} F_{k_i} < F_{k_r+1}$.

The proof is by induction on r. The result is trivial for $r = 1$ since it says that $F_{k_1} < F_{k_1+1}$ if $k_1 \ge 2$. (I suppose that this fact, also, should be proved by induction.)

Now suppose that $r > 1$ and assume, inductively, that P_{r-1} is true. Let $(k_i)_{1 \le i \le r}$ be a sequence of integers such that $k_1 \ge 2$ and $k_{i+1} \ge k_i + 2$ for all $i \in \mathbb{Z}^+$ such that $1 \le i < r$. The inductive hypothesis yields that $\sum_{i=1}^{r-1} F_{k_i} < F_{k_{r-1}+1}$, and adding F_{k_r} to both sides gives

$$\begin{aligned}\sum_{i=1}^{r} F_{k_i} &< F_{k_{r-1}+1} + F_{k_r} \\ &\le F_{k_r-1} + F_{k_r} \\ &= F_{k_r+1}\end{aligned}$$

where the second line follows from the fact that the Fibonacci sequence is increasing and $k_{r-1} + 1 \le k_r - 1$ by hypothesis. By induction, P_r holds for all positive integers r, as claimed.

It follows that if $n \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} F_{k_i}$ for some sequence of integers k_i satisfying the conditions in P_r above, then $F_{k_r} \le n < F_{k_r+1}$. Thus F_{k_r} is the largest Fibonacci number not exceeding n. We now use induction to prove that every positive integer n can be uniquely expressed in the desired form.

The equation $1 = F_2$ expresses 1 in the desired form, and clearly all other expressions $\sum_{i=1}^{r} F_{k_i}$ satisfying the conditions in P_r give a value greater than 1. So the result holds when $n = 1$.

Now let $n > 1$ and assume that all smaller positive integers have unique expressions of the desired form. Let F_k be the largest Fibonacci number not exceeding n (which exists since, for example, $F_k \ge F_{n+2} > n$ whenever $k \ge n + 2$).

If $n = F_k$ then putting $r = 1$ and $k_1 = k$ gives an expression for n of the desired form (since clearly $k > 2$). Moreover, if also $n = \sum_{i=1}^{r} F_{k_i}$ with $r \ge 2$ and the k_i satisfying the conditions of P_r, then $F_{k_r} < n$ (since F_{k_r} is one of the terms in the sum, and there is at least one other,) and $n < F_{k_r+1}$ by P_r. But $F_{k_r} < F_k < F_{k_r+1}$ is obviously impossible; so in this case $n = F_k$ is the unique expression for n in the desired form.

On the other hand, if $F_k < n$ then by the inductive hypothesis the positive integer $n - F_k$ has an expression of the required form, and it follows that we may write

$$n - F_k = \sum_{i=1}^{r-1} F_{k_i} \qquad (\$)$$

where $r - 1 \ge 1$ and the k_i satisfy $k_1 \ge 2$ and $k_{i+1} \ge k_i + 2$. Now put $k_r = k$, and suppose, for a contradiction, that $k_r \le k_{r-1} + 1$. Then $k - 1 \le k_{r-1}$, and so $F_{k-1} \le F_{k_r} \le n - F_k$ (by (\$)). But this gives $n \ge F_k + F_{k-1} = F_{k+1}$, contrary to the fact that F_k is the largest Fibonacci number not exceeding n. So $k_r \ge k_{r-1} + 2$, and hence the sequence $(k_i)_{1 \le i \le r}$ satisfies the requirements of P_r.

It remains to prove the uniqueness in the case $F_k < n$. But as we have seen above, if $n = \sum_{i=1}^{r} F_{k_i}$ is an expression for n of the required form then F_{k_r} is the largest Fibonacci number not exceeding n. So $k = k_r$, and since $1 \le n - F_k < n$ the inductive hypothesis guarantees that the equation (\$) above has a unique solution of the required form. So all the k_i are uniquely determined, as required.