TIME ALLOWED: One and a half hours

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 30% of the total examination;
there are 24 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 70% of the total examination;
there are 7 questions; the marks for each question are shown;
all questions may be attempted;
working must be shown.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE
EXAMINATION ROOM.
Multiple Choice Section

In each question, choose at most one option.
Your answers must be entered on the Multiple Choice Answer Sheet.

1. The degree sequence for a graph $G$ is: $(1, 2, 2, 3, 4, 4)$. The size of a spanning tree is:
   (a) 6; (b) 5; (c) 3; (d) 2; (e) 4.

2. Which one of the following sequences is graphic?
   (a) $(2, 3, 3, 3, 4, 5)$; (b) $(2, 3, 3, 3, 4, 7)$;
   (c) $(1, 2, 3, 4, 5, 6)$; (d) $(2, 3, 3, 3, 3, 4)$;
   (e) none are graphic.

3. The size of the complete graph $K_{11}$ is:
   (a) 10; (b) 110; (c) 55; (d) 11; (e) none of the above.

4. Let $m$ and $n$ be positive integers. The size of the complete bipartite graph $K_{m,n}$ is:
   (a) $m + n$; (b) $m + n - 1$; (c) $mn$; (d) $mn - 1$; (e) $\frac{mn(mn - 1)}{2}$.

5. The number of spanning trees for the complete graph $K_{11}$ is:
   (a) $11^9$; (b) 10; (c) $9^{11}$; (d) 55; (e) $10^{11}$.

6. A vertex sequence $bcgefadb$ in $H$ is:
   (a) a path; (b) cycle; (c) of length 4;
   (d) trail; (e) an Eulerian circuit.

7. The size of the $k$-cube $Q_k$ is:
   (a) $2^k$; (b) $k$; (c) $k2^{k-1}$; (d) $k^2$; (e) $2k^2$. 
8. The length of an Eulerian circuit in an Eulerian graph of order \( v \) and size \( e \) is:
   (a) \( v - 1 \); (b) \( e - 1 \); (c) \( v \); (d) \( e \); (e) none of the above.

9. The length of a Hamiltonian cycle in a Hamiltonian graph of order \( v \) and size \( e \) is:
   (a) \( v - 1 \); (b) \( e - 1 \); (c) \( v \); (d) \( e \); (e) none of the above.

![Diagram of a graph with vertices and edges labeled](image)

10. Consider the graph \( L \) which is drawn above. The size of the graph obtained by removing the vertex \( u \) from the graph \( K \) is:
   (a) 6; (b) 7; (c) 5; (d) 4; (e) 8.

11. Consider the graph \( L \) which is drawn above question 10. The order of the graph obtained by removing the edge \( e \) from the graph \( L \) is:
   (a) 6; (b) 7; (c) 5; (d) 4; (e) 8.

12. From a simple graph \( G \) of order 10 and size 42, a graph \( G \cdot e \) is constructed by a contraction of \( G \) by an edge \( e \). The order and size of \( G \cdot e \) are respectively:
   (a) 10 and 42; (b) 9 and 42; (c) 9 and 41;
   (d) 10 and 41; (e) none of the above.

13. Let \( G \) be a simple graph of order \( n \geq 3 \). For \( G \) the correct statement of Ore’s Theorem is:
   (a) “If \( \deg(v) + \deg(w) \leq n \) for all pairs of adjacent vertices \( v \) and \( w \), then \( G \) is Hamiltonian”;
   (b) “If \( \deg(v) + \deg(w) \geq n \) for each pair of adjacent vertices \( v \) and \( w \), then \( G \) is Hamiltonian”;
   (c) “If \( \deg(v) + \deg(w) \geq n \) for all pairs of non-adjacent vertices \( v \) and \( w \), then \( G \) is Hamiltonian”;
   (d) “If \( \deg(v) + \deg(w) \leq n \) for all pairs of non-adjacent vertices \( v \) and \( w \), then \( G \) is Hamiltonian”;
   (e) “If \( \deg(v) + \deg(w) \geq n \) for some pair of non-adjacent vertices \( v \) and \( w \), then \( G \) is Hamiltonian”;

14. The number of different Hamiltonian cycles in the complete graph $K_{13}$ is:

(a) 12; (b) $13!$; (c) $12!$; (d) $\frac{12!}{2}$; (e) 6.

15. The number of different labelled trees of order 15 is:

(a) $15^2$; (b) $15^{13}$; (c) $13^2$; (d) $13^{15}$; (e) none of the above.

16. Consider the Prüfer sequence (10, 10, 10, 3, 9, 8, 7, 1, 2, 3, 1, 10) for the labelled tree $T$. The degree in $T$ of the vertex with the label 8 is:

(a) 1; (b) 2; (c) 3; (d) 4; (e) 5.

17. The chromatic number of the complete bipartite graph $K_{3,5}$ is:

(a) 2; (b) 3; (c) 5; (d) 14; (e) 15.

18. If the chromatic polynomial $P(t)$ for the simple graph $G$ is given by

$P(t) = t^5 - 7t^4 + 18t^3 - 20t^2 + 8t$, the size of $G$ is:

(a) 5; (b) 4; (c) 18; (d) 20; (e) 7.

19. A simple graph $G$ consists of two components $G_1$ and $G_2$. The component $G_1$ is of order 4 and size 6 and the component $G_2$ is of order 5 and size 8. The degree of the chromatic polynomial $P_G(t)$ for $G$ is:

(a) 20; (b) 6; (c) 9; (d) 4; (e) 48.

20. Let $F$ be a forest of order 15, consisting of a tree of order 5 and tree of order 10. The chromatic number of $F$ is:

(a) 3; (b) 15; (c) 5; (d) 2; (e) 10.

21. $K^*$ is the dual graph for $K$. The order of $K^*$ is:

(a) 7; (b) 2; (c) 3; (d) 6; (e) 4.

22. The number of edges on the boundary of each face of the dodecahedron graph is:

(a) 20; (b) 15; (c) 5; (d) 12; (e) 10.
23. Which of the following sentences is a correct statement of Robbins’ Theorem on orientable graphs?
   (a) “A graph $G$ is orientable if and only if $G$ is disconnected”;
   (b) “A graph $G$ is orientable if and only if $G$ is 3-connected”;
   (c) “A graph $G$ is orientable if and only if $G$ is connected and has at least one bridge”;
   (d) “A graph $G$ is orientable if and only if $G$ is connected and has no bridges;”
   (e) “A graph $G$ is orientable if and only if $G$ is disconnected and has at least two bridges”.

24. Let $A$ be an adjacency matrix for a labelled digraph $D$. The sum of the entries in row $i$ is:
   (a) the degree of vertex $i$ in $D$;
   (b) the indegree of vertex $i$ in $D$;
   (c) the sum of the degrees of vertices adjacent to vertex $i$ in $D$;
   (d) the outdegree of vertex $i$ in $D$;
   (e) none of the above.

End of Multiple Choice Section
Make sure that your answers are entered on the Multiple Choice Answer Sheet
Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. (a) [3 Marks] Let $u$ and $v$ be (not necessarily distinct) vertices of a graph $G$. Carefully define the terms: a $uv$-walk; $uv$-trail and a $uv$-path.

(b) [3 Marks] Prove that every $uv$-walk contains a $uv$-path.

(c) [3 Marks] Prove that if the graph $G$ is disconnected then its complement $\overline{G}$ is connected.

(d) [3 Marks] Let $G$ be a simple graph of order 12 and size 28. The degree of each vertex of $G$ is either 3 or 5. Find the degree sequence of $G$.

(e) [3 Marks] The graphs $G$ and $H$, with vertex-sets $V(G)$ and $V(H)$, are drawn below. Determine whether or not $G$ and $H$ drawn below are isomorphic. If they are isomorphic, give a function $g : V(G) \rightarrow V(H)$ that defines the isomorphism. If they are not, explain why they are not.

2. Consider the graph $G$ which is drawn below:

(a) [1 Marks] Copy and complete Euler’s Theorem: “A connected graph is Eulerian ...

(b) [2 Marks] Use Euler’s Theorem to show that $G$ drawn above is Eulerian.

(c) [3 Marks] Describe and apply Fleury’s algorithm for finding an Eulerian circuit in $G$. Use vertex $u$ as the starting point in your construction.
3. (a) [2 Marks] Let $F$ be a forest of order $n$ which consists of $k$ trees. Find the size of $F$.

(b) [3 Marks] Describe Krusal’s algorithm for finding a minimum weight spanning tree for an edge weighted graph. Use the Krusal’s algorithm to find a minimum weight spanning tree for the weighted graph above. Write down the weight of the minimum weight spanning tree.

(c) [3 Marks] Make a copy of the graph $L$. Use Dijkstra’s algorithm to add to your copy a label to each vertex, showing the least weight of all paths from $A$ to that vertex. (in your working, show all temporary and permanent labels.) Hence determine all least weight paths from $A$ to $J$.

4. (a) [2 Marks] Sketch a labelled graph whose adjacency matrix $A$ is given by:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

(b) [4 Marks] Form the Prüfer sequence for the given labelled tree, $T$, and draw the labelled tree whose Prüfer sequence is $(3, 2, 2, 2, 4)$.

(c) [2 Marks] State the Matrix-Tree Theorem.

(d) [3 Marks] Use the Matrix-Tree Theorem to determine the number of spanning trees for the graph $G$ drawn below.
5. (a) [1 Marks] Define the chromatic number \( \chi(G) \) of the graph \( G \).

(b) [3 Marks] Let \( G \) be any graph with maximum vertex-degree \( \Delta(G) \). Prove that \( \chi(G) \leq \Delta(G) + 1 \) for any graph \( G \).

(c) [3 Marks] State Brooks’ Theorem. Applying Brooks’ Theorem, or otherwise, find the chromatic number of the graph \( G \) which is shown below:

\[
G:
\]

6. (a) [1 Marks] Define the chromatic index (or edge chromatic number) of a graph.

(b) [2 Marks] State Vizing’s theorem on chromatic index.

(c) [3 Marks] Find, supplying your reasons, the chromatic index for the graph given below:

\[
G:
\]

(d) [5 Marks] State the two reduction formulae for chromatic polynomials of simple graphs. Using one of these formulae, or otherwise, find the chromatic polynomial for the following graph.

\[
\]

7. (a) [3 Marks] State and prove Euler’s Formula for a connected plane graph of order \( n \), size \( e \) and with \( f \) faces.

(b) [3 Marks] Let \( G \), of order \( n \), be a connected 3-regular plane graph in which every vertex lies on one face of length 4, one face of length 6 and one face of length 8. Determine the number of faces of \( G \).

(c) [3 Marks] Show that the order of a self-complementary simple graph is either \( 4k \) or \( 4k + 1 \), where \( k \) is a positive integer.

(d) [3 Marks] Show that \( K_{n,n} \) is Hamiltonian if and only if \( n \geq 2 \). Hence show that \( K_{n,n} \) has \( \frac{n!(n-1)!}{2} \) Hamiltonian cycles.

End of Extended Answer Section
Family Name: .................................................................

Other Names: .................................................................

Seat Number: ..............

Indicate your answer to each question by filling in the appropriate oval.

Answers →

| Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 | Q9 | Q10 | Q11 | Q12 | Q13 | Q14 | Q15 | Q16 | Q17 | Q18 | Q19 | Q20 | Q21 | Q22 | Q23 | Q24 |
|----|----|----|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| a  | b  | c  | d  | e  | a  | b  | c  | d  | e   | a  | b  | c  | d  | e   | a  | b  | c  | d  | e   | a  | b  | c  | d  | e   | a  | b  | c  | d  | e   | a  | b  | c  | d  | e   |

Write your SID here →

Code your SID into the columns below each digit, by filling in the appropriate oval.

SID into ...

Faculties of Arts, Economics, Mathematics & Graph Theory
MATH2069 Discrete
Mathematics & Graph Theory
Paper 2 Graph Theory

This is the first and last page of this answer sheet
Correct Responses to MC Component of
MATH2069 Discrete Mathematics & Graph Theory Paper 2 Graph Theory
8040: Semester 1 2007

Q1 $\rightarrow$ b
Q2 $\rightarrow$ a
Q3 $\rightarrow$ c
Q4 $\rightarrow$ c
Q5 $\rightarrow$ a
Q6 $\rightarrow$ d
Q7 $\rightarrow$ c
Q8 $\rightarrow$ d
Q9 $\rightarrow$ c
Q10 $\rightarrow$ a
Q11 $\rightarrow$ e
Q12 $\rightarrow$ c
Q13 $\rightarrow$ c
Q14 $\rightarrow$ d
Q15 $\rightarrow$ b
Q16 $\rightarrow$ b
Q17 $\rightarrow$ a
Q18 $\rightarrow$ e
Q19 $\rightarrow$ c
Q20 $\rightarrow$ d
Q21 $\rightarrow$ c
Q22 $\rightarrow$ c
Q23 $\rightarrow$ d
Q24 $\rightarrow$ d