MATH2969: Discrete Mathematics and Graph Theory (Advanced)

Paper 1: Discrete Mathematics

Lecturer: Anthony Henderson

Time allowed: 1½ hours

This booklet contains 4 pages.

This paper comprises 5 questions worth 14 marks each, for a total of 70 marks.

All questions should be attempted. If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. A calculator is permitted.
1. In this question, your numerical answers need not be evaluated.
Suppose there are 80 students enrolled in a class, each of whom must be assigned to a tutorial. There are to be 3 tutorials, on Monday, Tuesday, and Wednesday.

(a) Explain why, no matter how they are assigned, there must be a tutorial containing at least 27 students.

(b) How many different ways are there to assign the students, if there must be 35 students on Monday, 25 on Tuesday, and 20 on Wednesday?

(c) How many different ways are there to assign the students, if the only restriction is that every tutorial has at least one student in it? Explain your answer.

Separately from the tutorials, the students have to complete a group-work assignment, so they need to be split into 20 groups of size 4. (There is no ordering on the groups.) Suppose that 20 of the students are designated advanced-level and the other 60 are designated normal-level.

(d) In how many different ways can the splitting into groups be done, if the members of each group must be either all advanced-level or all normal-level?

(e) In how many different ways can the splitting into groups be done, if each group must contain one advanced-level student and three normal-level students? Explain your answer.

(3 + 2 + 3 + 3 + 3 = 14 marks)

2. The first four parts of this question relate to a \( k \)-th-order homogeneous linear recurrence relation

\[ a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}, \quad \text{for } n \geq k, \]

where \( r_1, \cdots, r_k \) are some fixed numbers, and \( r_k \neq 0 \).

(a) Give the definition of the characteristic polynomial of this recurrence relation.

(b) If \( \lambda \) is a root of the characteristic polynomial, prove that \( a_n = \lambda^n \) is a solution of the recurrence relation.

(c) If \( \lambda \) is a repeated root of the characteristic polynomial, prove that \( a_n = n \lambda^n \) is a solution of the recurrence relation.

(d) If \( a_n \) and \( b_n \) are two solutions of the recurrence relation, prove that any sequence which is a linear combination of \( a_n \) and \( b_n \) is also a solution.

(e) Using the previous parts, or otherwise, prove that a sequence \( g_n \) satisfies the recurrence relation

\[ g_n = 4g_{n-1} - 4g_{n-2}, \quad \text{for } n \geq 2, \]

if and only if \( g_n = (C_1 + C_2 n)2^n \) for some constants \( C_1, C_2 \).

(2 + 2 + 3 + 3 + 4 = 14 marks)
3. (a) Give the rest of this definition: "The generating function \( A(z) \) of a sequence \( a_0, a_1, a_2, \ldots \) is . . . ."

(b) Prove by induction on \( k \) that
\[
\sum_{n=0}^{\infty} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}}, \quad \text{for all } k \geq 0.
\]

(c) Find a formula for the generating function \( A(z) \) of the sequence defined by the initial conditions \( a_0 = a_1 = 1 \) and the recurrence relation
\[
a_n = \frac{1}{n}(a_{n-1} + 2a_{n-2}), \quad \text{for } n \geq 2.
\]

(d) Using the generating function, or any other valid method, solve the recurrence relation
\[
a_n = 6a_{n-1} - 9a_{n-2} + 3^n, \quad \text{for } n \geq 2,
\]
with the initial conditions \( a_0 = 0, a_1 = 0 \).

\((2 + 3 + 4 + 5 = 14 \text{ marks})\)

4. (a) Give the rest of this definition: "The Stirling number \( S(n,k) \) is the number of . . . ."

(b) Prove that \( S(n,2) = 2^{n-1} - 1 \) for all \( n \geq 1 \). (Explain carefully what general facts you are using.)

(c) Is it always true that \( S(n,k) \leq \frac{k^n}{k!} \)? Explain your answer.

(d) Using the facts that \( S(4,1) = 1, S(4,2) = 7, S(4,3) = 6, \) and \( S(4,4) = 1, \)
write \( n^4 \) as a linear combination of binomial coefficients \( \binom{n}{k} \).

(e) Hence or otherwise find a formula for \( \sum_{n=0}^{\infty} n^4 z^n \) of the form \( \frac{P(z)}{(1-z)^5} \) where \( P(z) \) is a polynomial in \( z \).

\((2 + 3 + 3 + 3 + 3 = 14 \text{ marks})\)

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5. Let $f(n)$ and $g(n)$ be two functions of a nonnegative integer variable such that $\lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \infty$. Recall that:

$$f(n) \prec g(n) \text{ means that } \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$ 

(a) Prove that $n \prec 2^n$, using L’Hopital’s Rule or otherwise.

(b) Is it true that $F_n \prec 2^n$, where $F_n$ means the Fibonacci number? Explain your answer.

(c) Give the rest of this definition: “We say that $f(n)$ and $g(n)$ grow at the same rate, the notation for which is $f(n) \asymp g(n)$, if . . .”.

(d) Define a sequence $a_n$ by the initial condition $a_0 = 0$ and the recurrence relation

$$a_n = a_{n-1} + n + \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{3} \right\rfloor,$$

for $n \geq 1$.

Prove that $a_n \asymp n^2$.

(3 + 3 + 2 + 6 = 14 marks)

This is the end of the examination paper