This paper comprises 5 questions worth 14 marks each, for a total of 70 marks.

All questions should be attempted. If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. A calculator is permitted.
1. In this question, your numerical answers need not be evaluated.
Suppose there are 80 students enrolled in a class, each of whom must be assigned to
a tutorial. There are to be 3 tutorials, on Monday, Tuesday, and Wednesday.
(a) Explain why, no matter how they are assigned, there must be a tutorial
containing at least 27 students.
(b) How many different ways are there to assign the students, if there must be
35 students on Monday, 25 on Tuesday, and 20 on Wednesday?
(c) How many different ways are there to assign the students, if the only re-
striction is that every tutorial has at least one student in it? Explain your
answer.

Separately from the tutorials, the students have to complete a group-work assign-
ment, so they need to be split into 20 groups of size 4. (There is no ordering on
the groups.) Suppose that 20 of the students are designated advanced-level and
the other 60 are designated normal-level.
(d) In how many different ways can the splitting into groups be done, if there are
no restrictions on how many advanced-level students each group contains?
(e) In how many different ways can the splitting into groups be done, if each
group must contain one advanced-level student and three normal-level stu-
dents? Explain your answer.

(3 + 2 + 3 + 3 + 3 = 14 marks)

2. Recall that the Fibonacci sequence is defined by the initial conditions $F_0 = 0,$
$F_1 = 1,$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2.$
(a) Prove by induction that
$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}, \quad \text{for all } n \geq 0.$
(If $n = 0,$ the left-hand side has 0 terms, so should be interpreted as 0.)
(b) Give the rest of this definition: “The sequence $a_n$ satisfies a second-order
homogeneous linear recurrence relation if \ldots.”
(c) Suppose that $a_n$ and $b_n$ are two sequences which satisfy the same recurrence
relation as the Fibonacci sequence (but with possibly different initial condi-
tions). Explain why any sequence which is a linear combination of $a_n$ and
$b_n$ also satisfies this recurrence relation.
(d) State the formula for the general solution of $a_n = a_{n-1} + a_{n-2}$ ($n \geq 2$).
(e) From your answer to the previous part, derive a closed formula for $F_n.$

(4 + 2 + 3 + 2 + 3 = 14 marks)

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3. (a) Give the rest of this definition: “The generating function $A(z)$ of a sequence $a_0, a_1, a_2, \cdots$ is . . .”.

(b) Assuming that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, prove that $\sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}$. (Explain carefully what general facts you are using.)

(c) Using the generating function, or any other valid method, solve the recurrence relation

$$a_n = 2a_{n-1} + 3^n, \text{ for } n \geq 1,$$

with the initial condition $a_0 = 1$.

(d) Using any method, solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 2, \text{ for } n \geq 2,$$

with the initial conditions $a_0 = 8, a_1 = 20$.

(2 + 3 + 4 + 5 = 14 marks)

4. (a) Give the rest of this definition: “The Stirling number $S(n, k)$ is the number of . . .”.

(b) Prove that $S(n, n-1) = \binom{n}{2}$ for all $n \geq 2$.

(c) Is it always true that $S(n, k) \leq \frac{k^n}{k!}$? Explain your answer.

(d) Using the facts that $S(4, 1) = 1$, $S(4, 2) = 7$, $S(4, 3) = 6$, and $S(4, 4) = 1$, write $n^4$ as a linear combination of binomial coefficients $\binom{n}{k}$.

(e) Hence or otherwise find a formula for $\sum_{n=0}^{\infty} n^4 z^n$ of the form $\frac{P(z)}{(1-z)^5}$ where $P(z)$ is a polynomial in $z$.

(2 + 3 + 3 + 3 = 14 marks)

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5. Let \( f(n) \) and \( g(n) \) be two functions of a nonnegative integer variable such that \( \lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \infty \). Recall that:

\[
f(n) \prec g(n) \quad \text{means that} \quad \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.\]

(a) Prove that \( n \prec 2^n \), using L’Hopital’s Rule or otherwise.

(b) Is it true that \( \binom{n}{2} \prec n^2 \)? Explain your answer.

(c) Give the rest of this definition: “We say that \( f(n) \) and \( g(n) \) grow at the same rate, the notation for which is \( f(n) \asymp g(n) \), if . . .”.

(d) Define a sequence \( a_n \) by the initial condition \( a_0 = 0 \) and the recurrence relation

\[
a_n = a_{n-1} + n + \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{3} \right\rceil, \quad \text{for } n \geq 1.\]

Prove that \( a_n \asymp n^2 \).

\[ (3 + 3 + 2 + 6 = 14 \text{ marks}) \]