THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH2069: Discrete Mathematics and Graph Theory

Paper 1: Discrete Mathematics

Lecturer: Anthony Henderson

Time allowed: 1$\frac{1}{2}$ hours

This booklet contains 12 pages.

This paper comprises 5 questions worth 14 marks each, for a total of 70 marks.

All questions should be attempted. If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. A calculator is permitted.

SOLUTIONS
1. In this question, your numerical answers need not be evaluated.
Suppose there are 80 students enrolled in a class, each of whom must be assigned to a tutorial. There are to be 3 tutorials, on Monday, Tuesday, and Wednesday.

(a) Explain why, no matter how they are assigned, there must be a tutorial containing at least 27 students.

Solution: This is a special case of the Pigeonhole Principle (the number 27 = \( \lceil \frac{80}{3} \rceil \)). The proof is by contradiction: if every tutorial had \( \leq 26 \) students, the total number of students would be \( \leq 3 \times 26 = 78 \), contradicting the fact that there are 80 students.

Marking scheme. 3 marks. I gave 1 mark for the fact that 27 = \( \lceil \frac{80}{3} \rceil \), 1 mark for mentioning the Pigeonhole Principle, and 1 mark for general coherence of the answer; alternatively, you could get full marks by explaining the proof by contradiction.

(b) How many different ways are there to assign the students, if there must be 35 students on Monday, 25 on Tuesday, and 20 on Wednesday?

Solution: This is counted by the multinomial coefficient \( \binom{80}{35, 25, 20} = \frac{80!}{35!25!20!} \), which can also be expressed as \( \binom{80}{35} \binom{45}{25} \).

Marking scheme. 2 marks. I gave 2 marks for the right answer, 1 mark for \( \binom{80}{35} + \binom{45}{25} \).

(c) How many different ways are there to assign the students, if the only restriction is that every tutorial has at least one student in it? Explain your answer.

Solution: The restriction means that we are counting the surjective functions from the set of students to the set of tutorials. By a general result proved in lectures, the number of such functions is \( 3! S(80, 3) = 6 S(80, 3) \), where \( S(80, 3) \) is the Stirling number. An alternative description of this answer is as follows. Let \( X \) be the set of all assignments of students to tutorials, \( A \) the subset where no students are assigned to Monday, \( B \) the subset where no students are assigned to Tuesday, and \( C \) the subset where no students are assigned to Wednesday. Then \( |X| = 3^{80} \) (the number of functions from a set of size 80 to a set of size 3), \( |A| = |B| = |C| = 2^{80} \), \( |A \cap B| = |A \cap C| = |B \cap C| = 1 \), and \( |A \cap B \cap C| = 0 \). By the Inclusion/Exclusion Principle,

\[
|A \cup B \cup C| = 2^{80} + 2^{80} + 2^{80} - 1 - 1 - 1 = 3 \times 2^{80} - 3.
\]

The question asks for \( |X \setminus (A \cup B \cup C)| \), which is therefore \( 3^{80} - 3 \times 2^{80} + 3 \).

(The fact that this equals \( 6 S(80, 3) \) is a special case of a formula for \( S(n, 3) \) found in lectures.)

Marking scheme. 3 marks, single-star difficulty. I gave 2 marks for the answer in either form, 1 mark for some explanation (even just mentioning “surjective functions” or “Inclusion/Exclusion”). I gave 1 out of 3 for the answer \( \binom{80}{77} \) (this treats the students as indistinguishable, which was obviously not the intention, or else the previous part would have been trivial). I also
gave 1 out of 3 for a common incorrect answer $80 \times 79 \times 78 \times 377$, obtained by first selecting one student to put in each tutorial and then allocating the rest arbitrarily (this overcounts, because it distinguishes one of the students in each tutorial). I gave 0 out of 3 for $80 \times 79 \times 78 \times \binom{79}{77}$ (mixing up the two in a fairly meaningless way). I also gave 0 out of 3 for any answer which tried to carry on the numbers 35, 25, 20 from the previous part, since it should have been clear that this would make the present part trivial.

Separately from the tutorials, the students have to complete a group-work assignment, so they need to be split into 20 groups of size 4. (There is no ordering on the groups.) Suppose that 20 of the students are designated advanced-level and the other 60 are designated normal-level.

(d) In how many different ways can the splitting into groups be done, if there are no restrictions on how many advanced-level students each group contains?

**Solution:** If the groups were ordered, this would be counted by the multinomial coefficient $\binom{80}{4,4,\ldots,4} = \frac{80!}{(4!)^{20}}$. By the Overcounting Principle, we get the answer by dividing this by $20!$, so the answer is $\frac{80!}{(4!)^{20}20!}$.

**Marking scheme.** 3 marks, borderline single-star difficulty. I gave full marks for the correct answer; 2 marks for the answer $\frac{80!}{(4!)^{20}}$; 1 mark for the answers $\binom{80}{4}$ and $S(80,20)$, zero for anything else.

(e) In how many different ways can the splitting into groups be done, if each group must contain one advanced-level student and three normal-level students? Explain your answer.

**Solution:** Imagine numbering the advanced-level students $S_1, S_2, \ldots, S_{20}$. Then the groups are determined by specifying which three normal-level students go with $S_1$, which three go with $S_2$, and so on. So this is counted by the multinomial coefficient $\binom{60}{3,3,\ldots,3} = \frac{60!}{(3!)^{20}}$.

**Marking scheme.** 3 marks, single-star difficulty. 2 marks for the answer and 1 mark for the explanation. I only took off 1 mark for being out by a factor of $20!$.

$$3 + 2 + 3 + 3 + 3 = 14 \text{ marks}$$

2. Recall that the Fibonacci sequence is defined by the initial conditions $F_0 = 0, F_1 = 1$, and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

**SOLUTIONS**
(a) Prove by induction that

\[ F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}, \quad \text{for all } n \geq 0. \]

(If \( n = 0 \), the left-hand side has 0 terms, so should be interpreted as 0.)

**Solution:** When \( n = 0 \), both sides are zero. Assume that \( n \geq 1 \) and that the statement is known for \( n - 1 \): that is, \( F_1 + F_3 + F_5 + \cdots + F_{2n-3} = F_{2n-2} \). Then we have

\[ F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n-2} + F_{2n-1} = F_{2n}, \]

where the second equation uses the Fibonacci recurrence relation. This completes the inductive step and proves the result.

**Marking scheme.** 4 marks. I gave 1 mark for the base case (proving \( n = 1 \) instead, or proving both \( n = 0 \) and \( n = 1 \), were also OK), 1 mark for a clear explicit or implicit statement of the inductive hypothesis, and 2 marks for the rest. No marks were deducted for the usual faults of logic arising from leaving out explanatory words.

(b) Give the rest of this definition: “The sequence \( a_n \) satisfies a second-order homogeneous linear recurrence relation if . . . ”.

**Solution:** \( a_n = ra_{n-1} + sa_{n-2} \) for all \( n \geq 2 \), where \( r, s \) are fixed numbers and \( s \neq 0 \).

**Marking scheme.** 2 marks. It was not necessary to include the phrases defining the notation (and of course other letters could be used in place of \( r \) and \( s \)). I deducted 1 mark for giving instead the definition of second-order non-homogeneous, or for general-order homogeneous. I gave 1 out of 2 to the answer “it has characteristic polynomial \( x^2 - rx - s \)”, which is true but not really the definition.

(c) Suppose that \( a_n \) and \( b_n \) are two sequences which satisfy the same recurrence relation as the Fibonacci sequence (but with possibly different initial conditions). Explain why any sequence which is a linear combination of \( a_n \) and \( b_n \) also satisfies this recurrence relation.

**Solution:** The assumptions are that

\[ a_n = a_{n-1} + a_{n-2}, \quad \text{for } n \geq 2, \quad \text{and} \]
\[ b_n = b_{n-1} + b_{n-2}, \quad \text{for } n \geq 2. \]

For any constants \( C_1 \) and \( C_2 \), taking \( C_1 \) times the first equation plus \( C_2 \) times the second equation shows that

\[ C_1a_n + C_2b_n = (C_1a_{n-1} + C_2b_{n-1}) + (C_1a_{n-2} + C_2b_{n-2}), \quad \text{for } n \geq 2. \]

That is, the sequence \( C_1a_n + C_2b_n \) satisfies the Fibonacci recurrence relation.

**Marking scheme.** 3 marks. (This repeats an argument made in lectures.) I gave 1 mark for displaying knowledge that the assumption meant \( a_n = a_{n-1} + a_{n-2} \) and \( b_n = b_{n-1} + b_{n-2} \), and 2 marks for the argument. Proving it
only for a particular linear combination, e.g. $a_n + b_n$ or $a_n - b_n$, got 1 out of 2. Some students used the formula for the general solution of the recurrence relation: I deducted a mark for this, because it relies on what we’re proving here.

(d) State the formula for the general solution of $a_n = a_{n-1} + a_{n-2} \ (n \geq 2)$.

**Solution:** $a_n = C_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + C_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n$ for constants $C_1, C_2$.

**Marking scheme.** 2 marks. No marks were deducted for failing to clarify that $C_1$ and $C_2$ were independent of $n$ (and of course using other letters was fine). I gave 1 out of 2 for the answer $C_1 \lambda_1^n + C_2 \lambda_2^n$ with $\lambda_1$ and $\lambda_2$ not evaluated. I also gave 1 out of 2 if the formula (with roots evaluated or un-evaluated) was present in the student’s answer to the following part. (Parts (d) and (e) go together, and some students explicitly combined their answers, which was fine. The justification for not giving a full 2 out of 2 if the formula was present in the answer to (e) but not in the answer to (d) is that such students had not demonstrated that they understood one of the key ideas, namely that the only difference between sequences satisfying the same recurrence relation is in the constants $C_1, C_2$.)

(e) From your answer to the previous part, derive a closed formula for $F_n$.

**Solution:** Write $F_n = C_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + C_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n$. The $n = 0$ case of this statement says that $0 = C_1 + C_2$, and the $n = 1$ case says that $1 = C_1 \left(\frac{1 + \sqrt{5}}{2}\right) + C_2 \left(\frac{1 - \sqrt{5}}{2}\right)$. So we have $C_2 = -C_1$, and $1 = C_1 \left(\frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2}\right)$, showing that $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$. So

$$F_n = \frac{1}{\sqrt{5}} \left( \left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n \right).$$

**Marking scheme.** 3 marks. I gave 1 mark for using the initial conditions to get equations for the constants, 1 mark for solving them, and 1 mark for putting together the final formula. Allowance was made for carrying forward an incorrect answer to the previous part, but the final mark was still deducted if the formula did not satisfy the initial conditions.

$(4 + 2 + 3 + 2 + 3 = 14$ marks)
3. (a) Give the rest of this definition: “The generating function $A(z)$ of a sequence $a_0, a_1, a_2, \cdots$ is . . .”.

**Solution:** $a_0 + a_1 z + a_2 z^2 + \cdots$, which can also be written $\sum_{n=0}^{\infty} a_n z^n$.

**Marking scheme.** 2 marks, for either form. I gave 1 out of 2 for something which was close enough to being correct that the problem could simply have been a slip of the pen.

(b) Assuming that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, prove that $\sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}$. (Explain carefully what general facts you are using.)

**Solution:** Differentiating $\sum_{n=0}^{\infty} z^n$, we get $\sum_{n=0}^{\infty} (n+1) z^n$ by the definition of derivative. Differentiating $\frac{1}{1-z}$, we get $\frac{1}{(1-z)^2}$ using the Quotient Rule (which is valid for derivatives of power series). So the desired equation follows by differentiating both sides of the assumed equation. Alternatively, it follows by squaring both sides of the original equation: using the multiplication rule for power series, we see that

$$\left(\sum_{n=0}^{\infty} z^n\right)^2 = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} 1\right) z^n = \sum_{n=0}^{\infty} (n+1) z^n.$$

**Marking scheme.** 3 marks, of single-star difficulty. (Both methods of derivation were explained in lectures.) I gave 2 out of 3 for a correct proof which failed to “explain carefully” what general facts were being used (it was enough to mention either the definition of derivative for power series or the validity of the quotient rule for them; or, in the other method, the multiplication rule). I gave 1 out of 3 for a proof which assumed that $\sum_{n=0}^{\infty} n z^n = \frac{z}{(1-z)^2}$, since this was more or less the same statement as what was being proved.

(c) Using the generating function, or any other valid method, solve the recurrence relation

$$a_n = 2a_{n-1} + 3^n, \text{ for } n \geq 1,$$

with the initial condition $a_0 = 1.$
Solution: Let \( A(z) \) be the generating function of the sequence \( a_n \). Then

\[
A(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \\
= 1 + \sum_{n=1}^{\infty} (2a_{n-1} + 3^n) z^n \\
= 1 + 2 \sum_{n=1}^{\infty} a_{n-1} z^n + \sum_{n=1}^{\infty} 3^n z^n \\
= 1 + 2zA(z) + \frac{1}{1-3z} - 1,
\]

which leads to

\[
A(z) = \frac{1}{(1-2z)(1-3z)} = \frac{-2}{1-2z} + \frac{3}{1-3z}.
\]

So \( a_n = -2^{n+1} + 3^{n+1} \). This answer can also be obtained by unravelling the recurrence relation and using the formula for the sum of a geometric progression; alternatively, one can note that the sequence \( p_n = 3^{n+1} \) is a solution of the recurrence relation, so the general solution is \( a_n = C2^n + 3^{n+1} \), and the initial condition gives \( C = -2 \).

Marking scheme. 4 marks. For those who used generating functions, I gave 3 marks for a correct closed formula for \( A(z) \) (with 1 or 2 marks deducted for errors in the calculation), and 1 mark for a correct extraction of the coefficient of \( z^n \) from whatever formula for \( A(z) \) was obtained, provided that it satisfied the initial condition \( a_0 = 1 \). No students used the unravelling method. For those who used the particular-solution method, I gave 2 marks for finding a particular solution and 2 marks for deducing the formula for \( a_n \) (students could still get the latter 2 marks using an incorrect particular solution, if they ensured that \( a_0 = 1 \)). Students who tried more than one approach got the maximum of their marks for the different approaches.

(d) Using any method, solve the recurrence relation

\[
a_n = 5a_{n-1} - 6a_{n-2} + 2, \text{ for } n \geq 2,
\]

with the initial conditions \( a_0 = 8, \ a_1 = 20 \).

Solution: Since this is a non-homogeneous linear recurrence relation, we first look for a particular solution of the form \( p_n = C \) (a constant sequence). We must have \( C = 5C - 6C + 2 \), which is satisfied when \( C = 1 \). Since the general solution of the homogeneous recurrence relation \( b_n = 5b_{n-1} - 6b_{n-2} \) is \( b_n = C_12^n + C_23^n \) for some constants \( C_1, C_2 \), the general solution of our non-homogeneous recurrence relation is

\[
a_n = C_12^n + C_23^n + 1, \text{ for some constants } C_1, C_2.
\]

The \( n = 0,1 \) initial conditions force \( C_1 + C_2 = 7 \) and \( 2C_1 + 3C_2 = 19 \), which implies that \( C_1 = 2, \ C_2 = 5 \). So the solution to our problem is
$a_n = 2^{n+1} + 5 \times 3^n + 1$. Alternatively, let $A(z)$ be the generating function of the sequence $a_n$. Then

$$A(z) = 8 + 20z + \sum_{n=2}^{\infty} a_n z^n$$

$$= 8 + 20z + \sum_{n=2}^{\infty} (5a_{n-1} - 6a_{n-2} + 2) z^n$$

$$= 8 + 20z + 5 \sum_{n=2}^{\infty} a_{n-1} z^n - 6 \sum_{n=2}^{\infty} a_{n-2} z^n + 2 \sum_{n=2}^{\infty} z^n$$

$$= 8 + 20z + 5(zA(z) - 8z) - 6z^2A(z) + \frac{2}{1-z} - 2 - 2z$$

$$= 5zA(z) - 6z^2A(z) + \frac{8 - 28z + 22z^2}{1-z},$$

which leads to

$$A(z) = \frac{8 - 28z + 22z^2}{(1-z)(1-2z)(1-3z)} = \frac{C_1}{1-z} + \frac{C_2}{1-2z} + \frac{C_3}{1-3z},$$

for some constants $C_1$, $C_2$, and $C_3$. Clearing denominators gives

$$8 - 28z + 22z^2 = C_1(1-2z)(1-3z) + C_2(1-z)(1-3z) + C_3(1-z)(1-2z).$$

Setting $z = 1, \frac{1}{2}, \frac{1}{3}$ respectively shows that $C_1 = 1$, $C_2 = 2$, $C_3 = 5$. So $a_n = 2^{n+1} + 5 \times 3^n + 1$ as before.

**Marking scheme.** 5 marks. For those who used the particular-solution method, I gave 1 mark for finding that $p_n = 1$ was a solution, 2 marks for deducing the general solution, and 2 marks for determining the constants (with appropriate allowances for carrying through mistakes). Students who made the common error of determining the constants for the homogeneous part, getting $4 \times 2^n + 4 \times 3^n + 1$, got 3 out of 5. For those who used generating functions, I gave 3 marks for a correct closed formula for $A(z)$, and 2 marks for deducing the formula for $a_n$.

$$(2 + 3 + 4 + 5 = 14 \text{ marks})$$

4. (a) Give the rest of this definition: “The Stirling number $S(n,k)$ is the number of . . . ”.

**Solution:** partitions of the set $\{1,2,\ldots,n\}$ with $k$ blocks.

**Marking scheme.** 2 marks for this or anything which could be interpreted as equivalent. I gave 1 mark for something close, e.g. the number of surjective functions from an $n$-element set to a $k$-element set or equivalent, or something which sounded more like the number of partitions of the number $n$ with $k$ parts.

**SOLUTIONS**
(b) Prove that $S(n, n - 1) = \binom{n}{2}$ for all $n \geq 2$.

**Solution:** If a partition of $\{1, 2, \ldots, n\}$ has $n - 1$ blocks, then two elements must be together in one block, and all the other elements are alone in their block. So the only thing to choose is which two elements to put together; this can be done in $\binom{n}{2}$ ways. Alternatively, one can proceed by induction, using the base case $S(2, 1) = 1$ and the recurrence relation for the Stirling numbers, a special case of which is that

$$S(n, n - 1) = S(n - 1, n - 2) + (n - 1)S(n - 1, n - 1) = S(n - 1, n - 2) + n - 1.$$

**Marking scheme.** 3 marks. (This repeats an argument made in lectures.) I gave 1 mark for something useful for the induction approach, such as a correct statement of the recurrence relation in general or specifically for $S(n, n - 1)$, or a set-up with proof of the base case.

(c) Is it always true that $S(n, k) \leq \frac{k^n}{k!}$? Explain your answer.

**Solution:** Yes, because $k!S(n, k)$ is the number of surjective functions from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, k\}$; hence it cannot exceed $k^n$, which is the total number of functions from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, k\}$.

**Marking scheme.** 3 marks, single-star difficulty. There were no marks for saying yes without explanation, or with only a few numerical examples as evidence. I gave 2 out of 3 for the argument that $\frac{k^n}{k!}$ counts the partitions of a set of size $n$ into $k$ possibly empty parts; this is not quite right (and indeed $\frac{k^n}{k!}$ is not usually an integer), because the empty parts ruin the validity of dividing by $k!$ to un-order the parts.

(d) Using the facts that $S(4, 1) = 1$, $S(4, 2) = 7$, $S(4, 3) = 6$, and $S(4, 4) = 1$, write $n^4$ as a linear combination of binomial coefficients $\binom{n}{k}$.

**Solution:** Using the $a = 4$ case of the general formula

$$n^a = \sum_{k=1}^{a} k!S(a, k)\binom{n}{k},$$

we see that $n^4 = \binom{n}{1} + 14\binom{n}{2} + 36\binom{n}{3} + 24\binom{n}{4}$.

**Marking scheme.** 3 marks, single-star difficulty (in that the formula involved was rated single-star). I gave 1 out of 3 for slightly misremembered versions, as long as they made sense.

(e) Hence or otherwise find a formula for $\sum_{n=0}^{\infty} n^4 z^n$ of the form $\frac{P(z)}{(1 - z)^5}$ where $P(z)$ is a polynomial in $z$.

SOLUTIONS
Solution: From the previous part, we have

\[
\sum_{n=0}^{\infty} n^4 z^n = \sum_{n=0}^{\infty} \binom{n}{1} z^n + 14 \sum_{n=0}^{\infty} \binom{n}{2} z^n + 36 \sum_{n=0}^{\infty} \binom{n}{3} z^n + 24 \sum_{n=0}^{\infty} \binom{n}{4} z^n
\]

\[
= \frac{z}{(1-z)^2} + \frac{14z^2}{(1-z)^3} + \frac{36z^3}{(1-z)^4} + \frac{24z^4}{(1-z)^5}
\]

\[
= \frac{z(1-z)^2 + 14z^2(1-z)^2 + 36z^3(1-z) + 24z^4}{(1-z)^5}
\]

\[
= \frac{z + 11z^2 + 11z^3 + z^4}{(1-z)^5}
\]

Marking scheme. 3 marks, single-star difficulty. Stopping at the penultimate line, as soon as the form \( \frac{P(z)}{(1-z)^5} \) appears, was fine. I deducted marks for mistakes in the calculation, but allowed full carry-over of an incorrect answer to the previous part. There were no marks just for stating the key formula \( \sum_{n=0}^{\infty} \binom{n}{k} z^n = \frac{z^k}{(1-z)^{k+r}} \).

\[ (2 + 3 + 3 + 3 = 14 \text{ marks}) \]
5. Let \( f(n) \) and \( g(n) \) be two functions of a nonnegative integer variable such that \( \lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \infty \). Recall that:

\[ f(n) \prec g(n) \text{ means that } \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0. \]

(a) Prove that \( n \prec 2^n \), using L’Hopital’s Rule or otherwise.

**Solution:** L’Hopital’s Rule tells us that

\[ \lim_{x \to \infty} \frac{x^2}{2^x} = \lim_{x \to \infty} \frac{1}{(\ln 2)^2} = 0, \]

from which it follows also that \( \lim_{n \to \infty} \frac{n^2}{2^n} = 0 \).

**Marking scheme.** 3 marks. (This repeats an argument made in lectures.) Merely stating a more general result (e.g. that fixed-power growth is slower than exponential growth) got no marks. I gave 2 marks for the use of L’Hopital’s Rule to prove that \( \lim_{x \to \infty} \frac{x^2}{2^x} = 0 \), and 1 mark (usually lost) for the realization that the limit through integer values is a different statement following from this. Incorrect derivatives of \( 2^x \) lost 1 or 2 marks depending on their heinousness; the most common (with \( n \) instead of \( x \)) was \( n^2 n^2 - 1^n \).

(b) Is it true that \( \binom{n}{2} \prec n^2 \)? Explain your answer.

**Solution:** No, because \( \binom{n}{2} = \frac{1}{2}n^2 - \frac{1}{2}n \), so

\[ \lim_{n \to \infty} \frac{\binom{n}{2}}{n^2} = \lim_{n \to \infty} \frac{1}{2} - \frac{1}{2n} = \frac{1}{2} \neq 0. \]

Alternatively, \( \binom{n}{2} \asymp n^2 \) because the growth rate of a polynomial is simply that of its leading term.

**Marking scheme.** 3 marks. Nothing for simply answering “no” (or, obviously, “yes”) without explanation; also nothing for answering “no” with an alleged explanation that \( \binom{n}{2} \asymp n^2 \).

(c) Give the rest of this definition: “We say that \( f(n) \) and \( g(n) \) grow at the same rate, the notation for which is \( f(n) \sim g(n) \), if . . .”.

**Solution:** there are some positive constants \( C, D, N \) such that

\[ Cg(n) \leq f(n) \leq Dg(n), \text{ for all } n \geq N. \]

**Marking scheme.** 2 marks. The part about \( n \) being sufficiently large was not needed, but I deducted 1 mark for omitting the positivity of \( C \) (if \( C \) is allowed to be zero or negative, then the definition would include cases where \( f(n) \prec g(n) \)). I gave 1 out of 2 for a limit statement that would imply \( f(n) \asymp g(n) \), such as \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = L \) for \( L > 0 \) (or specifically \( L = 1 \), which is properly the definition of \( f(n) \sim g(n) \)). I gave 0 marks for “\( \lim_{n \to \infty} \frac{f(n)}{g(n)} \) is a constant” (whatever that means, it doesn’t exclude zero).
(d) Define a sequence \( a_n \) by the initial condition \( a_0 = 0 \) and the recurrence relation
\[
a_n = a_{n-1} + n + \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{3} \right\rfloor, \quad \text{for } n \geq 1.
\]
Prove that \( a_n \sim n^2 \).

**Solution:** We use a ‘squeezing’ argument, bounding \( a_n \) above and below by sequences which grow at the same rate as \( n^2 \). Define a sequence \( \ell_n \) by the initial condition \( \ell_0 = 0 \) and the recurrence relation \( \ell_n = \ell_{n-1} + n \), for \( n \geq 1 \). Since this differs from the recurrence in the question by the omission of the nonnegative terms \( \left\lceil \frac{n}{2} \right\rceil \) and \( \left\lfloor \frac{n}{3} \right\rfloor \), we have \( a_n \geq \ell_n \) for all \( n \geq 0 \). Obviously
\[
\ell_n = 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2},
\]
which grows at the same rate as \( n^2 \). Now define a sequence \( u_n \) by the initial condition \( u_0 = 0 \) and the recurrence relation \( u_n = u_{n-1} + 3n \), for \( n \geq 1 \). Since this is obtained from the recurrence in the question by increasing both \( \left\lceil \frac{n}{2} \right\rceil \) and \( \left\lfloor \frac{n}{3} \right\rfloor \) to \( n \), we have \( a_n \leq u_n \) for all \( n \geq 0 \). Obviously
\[
u_n = 0 + 3 + 3 \times 2 + \cdots + 3n = \frac{3n(n+1)}{2},
\]
which also grows at the same rate as \( n^2 \). It follows that \( a_n \) grows at the same rate as \( n^2 \). Other proofs are possible.

**Marking scheme.** 6 marks, double-star difficulty. I gave 1 mark for the idea of using a squeezing argument, and a further 3 marks for those who managed one bound but not the other, with deductions for mistakes.
\[
(3 + 3 + 2 + 6 = 14 \text{ marks})
\]