2008 MATH2069 Graph Theory Exam

Solutions/comments by Anthony Henderson

General note about conventions:

The lecturer in 2008 (Dr. Bill Palmer) allowed multiple edges & loops in the definition of a "graph". Thus what I call a "graph" is what he called a "simple graph". For most of the questions, the distinction does not matter.

He also included some topics which I have left out in my notes "Introduction to Graph Theory", such as planar graphs, dual graphs, Kruskal's algorithm (quite similar to Prim's algorithm), and adjacency matrices (mentioned in Tutorial 4).

He used some terminology which I didn't introduce:
"order" = "number of vertices"
"size" = "number of edges"
"labelled tree" = a tree with vertex set \{1, 2, \ldots, n\} for some \( n \)
"trail" = "walk in which no edge is repeated"
"Hamiltonian cycle" = "spanning cycle"
"contraction of a graph \( G \) along an edge \( e \)" = (multi)graph obtained by identifying ends of \( e \) & removing \( e \)"
Multiple Choice Section: Answers/comments

NOTE: there will be no multiple choice section in this year's exam.

Q1: (b) (6 vertices, so a spanning tree has 5 edges.)
Q2: (a) (graph is because \(2,3,1,3,2,3,3\) is graphic)
Q3: (c) (\(K_n\) has \(\binom{n}{2}\) edges)
Q4: (c) (\(K_{m,n}\) has \(mn\) edges)
Q5: (a) (\(K_n\) has \(n^{n-2}\) spanning trees)
Q6: (d) ("trail" = "walk with no repeated edges")
Q7: (c) (\(Q_k\) has \(k^{k-1}\) edges)
Q8: (d) (Eulerian circuit uses every edge exactly once)
Q9: (c) (Hamiltonian cycle contains every vertex)
Q10: (a) ("K" is a typo for "L")
Q11: (a) ("order" = "number of vertices")
Q12: (c) (irrelevant this year)
Q13: (c) (Ore's Theorem)
Q14: (d) (\(K_n\) has \(\frac{n!}{\Delta n}\) spanning cycles)
Q15: (b) (Cayley's Formula)
Q16: (b) (no. of times \(v\) appears in Prüfer = \(\deg(v) - 1\))
Q17: (a) (bipartite \(\iff\) \(\chi \leq 2\))
Q18: (e) (coefficient of \(t^{n-1}\) is = no. of edges)
Q19: (c) (degree of \(P_d(t) = \text{no. of vertices}\))
Q20: (d) (forests are bipartite)
Q21: (c) (irrelevant this year)
Q22: (c) (""")
Q23: (c) (sum of coefficients of \(P_d(t) = P_0(t) = 0\))
Q24: (c) (irrelevant this year)
1. (a) If we denote the vertices of $G$ and $H$ as shown:

Then the bijection $\{a, b, c, d, e, f, g, h\} \rightarrow \{p, q, r, s, t, u, v\}$

which sends $a \rightarrow q$, $b \rightarrow r$, $f \rightarrow s$, $g \rightarrow t$
$c \rightarrow p$, $d \rightarrow u$, $e \rightarrow v$

sends the edges of $G$ to the edges of $H$.
So $G$ and $H$ are isomorphic. [3 marks]

(b) (i) $\deg(v_i)$ is the number of edges which have $v_i$ as an end, so $\sum_{i=1}^{n} \deg(v_i)$ counts every edge exactly twice, once for each of its ends. Hence $\sum_{i=1}^{n} \deg(v_i) = 2|E|.$

[Comment: this is the Hand-shaking Lemma] [2 marks]

(ii) The previous part shows that the sum of all $\deg(v_i)$ is even. Hence the sum of just the odd degrees is even, which means there must be an even number of these odd degrees. [1 mark]

(c) The cycle $C_5$ is an example of such a graph:

It is clear that $\overline{C_5}$ is also a 5-cycle, i.e., it is isomorphic to $C_5.$ [3 marks]
1. (d) The following graph has degree sequence 
\( (1, 2, 2, 3, 3, 4, 5, 5) \) is:

```
  a   b   c   d   e   f
 /\   /\   /\   /\   /\   /\   /\
(   ) (   ) (   ) (   ) (   ) (   ) (   )
```

[3 marks]

[Comment: The sequence \((1, 1, 1, 2, 5, 5, 5)\) can be ruled out using the Havel-Hakimi Theorem, since the sequence \((0, 0, 1, 1, 4, 4)\) is clearly not graphic. The same operation applied to \(s\) gives:

\[
\begin{align*}
(1, 2, 2, 3, 3, 4, 5, 5) \rightarrow (1, 1, 1, 2, 3, 4) \\
(1, 1, 1, 2, 3, 4, 4) \rightarrow (0, 0, 1, 1, 2)
\end{align*}
\]

This last is clearly graphic: the graph with vertices \(\{a, b, c, d, e, f\}\) on the top line of the above picture has this degree sequence.
Then you can add a vertex \(g\), adjacent to the appropriate existing vertices, to get a graph with degree sequence \((1, 1, 1, 2, 3, 4)\); and then another vertex \(h\), to get the graph above. The question should not have said "the graph which realizes it", since there are several different isomorphism classes with this degree sequence.]
2. (a) [REPLACING Kruskal’s algorithm with Prim’s]
Prim’s algorithm constructs a minimal spanning tree by starting with a single vertex and successively adding new vertices along with single edges which join them to the existing vertices of the tree. At each stage, the new edge added is chosen to have minimal weight among all edges joining an existing vertex with one not yet in the tree.

Applying this algorithm to the graph L, starting with the vertex A and making arbitrary choices in case of equal weights, we get the following spanning tree:

![Graph L](image)

The weight of this minimal spanning tree is 15. [5 marks]

[Comment: Kruskal’s algorithm builds up the spanning tree by successively adding edges of minimal weight, but those edges don’t have to connect with the edges already chosen (so the result may not be a connected subgraph at some stages). The only rule is that an edge must not be added if it would form a cycle with already existing edges. In the case of the graph L, the spanning tree shown above could equally well be produced by Kruskal’s algorithm.]
2. (b) (For clarity, the names of the vertices are not indicated in the following picture.)

So, the minimal weight of a path from A to J is 9.

The step before the end must be from H to J,

since A-J is ruled out by the fact that 8+3>9.

Continuing to work backwards in this way,

we see that there are two minimal walks from A to J:

A, B, E, F, H, J and A, D, E, F, H, J.

[5 marks]

[Comment: the question asks about "least weight paths", which amounts to the same thing as "minimal walks", because a minimal walk must be a walk along a path.]
3. (a) [NB: the correction referred to in the boldface heading was read out in the exam itself, and has been incorporated into this version of the paper]

A tree is a connected graph with no cycles.

A leaf of a tree is a vertex of degree 1.

A spanning tree of a graph is a subgraph which contains all the vertices and is a tree.

[3 marks]

(b) [This question asks for a proof from lectures. You can find the proof in the notes: Theorem 3.5]

[3 marks]

(c) Since $T$ is a tree, the number of edges is one less than the number of vertices, i.e.

$$r + 3 + 2 + s - 1 = r + s + 4.$$

By the Hand-shaking Lemma,

$$5r + 3 + 2 + 2 + s = 2(r + s + 4).$$

So

$$3r + 16 = s + 8,$$

i.e. $s = 3r + 8$.

[3 marks]
3. (d) There are 7 vertices, 1, 2, 3, 4, 5, 6, 7, of which 5 and 6 are leaves. The smallest leaf, 5, must be adjacent to 1, the first entry in the sequence. So, T = 5 has vertices 1, 2, 3, 4, 6, 7, and the Praüfer sequence is (3, 3, 4, 7), so 1 and 6 are leaves.

Continuing in this way, we find that T is:

```
     5
 1      2
      3
        4
          7
```

[3 marks]

[Comment: since no vertex appears more than once in the Praüfer sequence, the tree has no vertices of degree \( \geq 3 \) and has to be a path.]

(e) The picture and Laplacian matrix of \( K_{2,3} \) are:

```
\[ M = \begin{pmatrix}
 3 & 0 & -1 & 1 & -1 \\
 0 & 3 & -1 & 1 & -1 \\
 -1 & -1 & 2 & 0 & 0 \\
 -1 & 0 & 2 & 0 & 0 \\
 -1 & 1 & 0 & 0 & 2
\end{pmatrix} \]
```

The \((1,1)\)-cofactor of \( M \) is

\[
\det \begin{pmatrix}
 3 & -1 & -1 & -1 \\
 -1 & 2 & 0 & 0 \\
 -1 & 0 & 2 & 0 \\
 -1 & 1 & 0 & 0 \\
\end{pmatrix} = \det \begin{pmatrix}
 \frac{3}{2} & 0 & 0 & 0 \\
 -1 & 2 & 0 & 0 \\
 -1 & 0 & 2 & 0 \\
 -1 & 0 & 0 & 2
\end{pmatrix} = \frac{3}{2} \times 2 \times 2 \times 2 = 12.
\]

So, \( K_{2,3} \) has 12 spanning trees.

[3 marks]

[Comment: alternatively, to obtain a spanning tree, we must delete two edges not meeting at one of 3, 4, 5. So, the answer is \( (\binom{3}{2}) = 3 = 12 \).]
4. (a) "... if and only if every vertex has even degree." [2 marks]

(b) This is true. In fact, the number 19 is irrelevant; every bipartite Eulerian graph has an even number of edges. If the vertices are coloured white & black so that every edge joins a white vertex with a black vertex, an Eulerian circuit (starting with a white vertex, say) must alternate while = black - white - black - ... = black - white, so it must use an even number of edges. But by definition it uses every edge in the graph exactly once, so the graph has an even number of edges. [3 marks]

(c) H: 

![Graph Image]

is obviously connected.

The degrees of the vertices are:

\[ \text{deg}(a) = 4, \text{deg}(b) = 2, \text{deg}(c) = 2, \text{deg}(d) = 6, \]
\[ \text{deg}(e) = 2, \text{deg}(f) = 4, \text{deg}(g) = 4. \]

Since these degrees are all even, H is Eulerian. If there were a spanning cycle C in H, it would have to use 2 of the edges at each vertex. So it would have to use both edges ending at b, both edges ending at c, & both edges ending at e. But this means it would use \{b,d\}, \{c,d\}, & \{d,e\}, more than the possible 2 edges at d. So no spanning cycle exists, & H is not Hamiltonian. [3 marks]
6. (a) The chromatic number \( \chi(G) \) is the smallest nonnegative integer \( t \) such that \( G \) has a \textit{vertex colouring with} \( t \) colours.

The edge chromatic number \( \chi'(G) \) is the smallest nonnegative integer \( t \) such that \( G \) has an edge colouring with \( t \) colours.

[2 marks]

(b) In the complete graph \( K_p \), every vertex is adjacent to every other vertex. So a vertex colouring must use \( p \) colours, one for each vertex.

Hence \( \chi(K_p) = p \).

[3 marks]

Comment: this seems too easy, and the hypothesis that \( p \) is odd is irrelevant, so I believe that \( \chi(K_p) \) was a typo for the edge chromatic number \( \chi'(K_p) \).

In that case the question is asking for a proof which you can find in the notes: Theorem 4.30 (first two paragraphs).

(c) This question asks for a proof which you can find in the notes: Theorem 4.9.

[3 marks]
6. (d) $G$ has a vertex colouring with four colours $R, W, B, Y$ as follows:

It has no vertex colouring with three colours, because if it was possible to give a vertex colouring with only $R, W, B$, then we can assume the vertices of the leftmost 3-cycle are coloured:

and one can then deduce, from left to right, lower that the other vertices must be coloured:

but now the remaining vertex cannot be coloured $R, W, or B$. So $\chi(G) = 4$.

[2 marks]

[Comment: this is a case where the Welsh-Powell Algorithm does give a vertex colouring with the minimal number of colours. However, since there is no general rule that the Algorithm always does this, it is necessary to give a special argument for why 4 is the minimum.]
6. (d) continued.

A has an edge colouring with four colours R, W, B, Y as follows:

Since A has vertices of degree 4 (i.e., places where 4 edges meet), it cannot have an edge colouring with fewer than 4 colours. So \( \chi'(A) = 4 \).

[2 marks]

[Comment: the above edge colouring was found by adopting the heuristic approach of attempting to colour "parallel" edges the same colour.]

(e) Let \( G \) be a graph. The two "reduction formulae" are that:

- if \( v \) and \( w \) are non-adjacent vertices of \( G \), then \( P_G(t) = P_{G + \{v, w\}}(t) + P_{G_{[v, w]}}(t) \);

- if \( v \) and \( w \) are adjacent vertices of \( G \), then \( P_G(t) = P_{G - \{v, w\}}(t) - P_{G_{[v, w]}}(t) \).

[2 marks]
6 (c) continued. Let \( G \) be the graph, with vertices denoted as follows:

\[ a \quad b \]
\[ c \quad d \]
\[ e \]

Since the two vertices adjacent to \( b \) are adjacent to each other, we have:
\[ P_G(t) = P_{G-b}(t) (t-2). \]

In \( G-b \), the two vertices adjacent to \( c \) are adjacent to each other, so
\[ P_{G-b}(t) = P_{(G-b)-c}(t) (t-2). \]

Since \( (G-b)-c \) is isomorphic to \( K_3 \),
\[ P_{(G-b)-c}(t) = t(t-1)(t-2). \]

Hence
\[ P_G(t) = t(t-1)(t-2)^3. \]

[3 marks]

Comment: you could also apply the above formulae, either using the second repeatedly to reduce to graphs with fewer edges until you reach trees, or using the first repeatedly to reduce to graphs with fewer non-edges until you reach complete graphs. But in the case of the given \( G \), the argument given here is quicker.]