MATH2069/2969: Discrete Mathematics and Graph Theory

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Time allowed: 2 hours, plus 10 minutes reading time

This booklet contains 22 pages.

This paper comprises 6 questions worth 15 marks each, for a total of 90 marks. Each question is divided into several parts; the marks assigned to each part are indicated at the end of the question.

Questions 1,3,5 are the same for MATH2069 and MATH2969. For questions 2,4,6, this paper contains both the normal-level MATH2069 question and the (completely different) advanced-level MATH2969 question. You may ONLY answer the questions for the unit you are enrolled in.

If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. A calculator is permitted.

SOLUTIONS
1. In this question, numerical answers need not be evaluated.

Suppose that a company has 3 case-workers and 100 clients. Every client has to be assigned to one of the three case-workers, who will manage their case. Since the clients all have different needs and the case-workers all have different skills, it matters which clients are assigned to which case-workers.

(a) How many different ways are there to assign the clients to their case-workers, allowing the possibility that a case-worker may have no clients? Explain your answer.

Solution: The answer is \(3^{100}\), because there are 3 possible choices of case-worker for each of the 100 clients, so \(3 \times 3 \times \cdots \times 3\) (100 factors) choices in total.

Marking Scheme. 2 marks for the correct answer, 1 for a brief explanation. The answer \( \binom{102}{100} \), with an explanation claiming that unordered selection was appropriate, got 2 out of 3. I gave 1 out of 3 for the answer \(100^3\), and 0 out of 3 for other incorrect answers such as \( \binom{100}{3} \).

(b) Explain why, no matter how they are assigned, there must be one case-worker who ends up managing at least 34 clients.

Solution: This is a special case of the Pigeonhole Principle (the number 34 = \(\left\lceil \frac{100}{3} \right\rceil\)). The proof is by contradiction: if every case-worker had \(\leq 33\) clients, the total number of clients would be \(\leq 3 \times 33 = 99\), contradicting the fact that there are 100 clients.

Marking Scheme. 3 marks for either a convincing appeal to the Pigeonhole Principle, explaining that 34 = \(\left\lceil \frac{100}{3} \right\rceil\), or the proof by contradiction.

(c) Now suppose there is a constraint that every case-worker must have at least one client. How many ways are there to assign the clients to the case-workers? Express your answer in terms of a suitable Stirling number.

Solution: We are effectively counting surjective functions from the set of clients to the set of case-workers, of which there are \(3!S(100, 3)\). (An alternative form of the answer, derived using the Inclusion/Exclusion Principle, is \(3^{100} - 3 \times 2^{100} + 3\), but the question specified the Stirling number form.)

Marking Scheme. 3 marks for the correct answer, with no explanation required, and no deduction for not using Stirling numbers. I gave 2 out of 3 for the most common incorrect answer \(S(100, 3)\), 1 out of 3 for \(3!S(97, 3)\), and 1 out of 3 for \(100 \times 99 \times 98 \times 3^{97}\). This last answer imagines assigning one client to each case-worker first and then assigning the rest arbitrarily; it overcounts, because it distinguishes the clients who were chosen first.

(d) If the constraint is that every case-worker must have at least two clients, how many ways are there to assign the clients? Explain your answer.

Solution: We need to subtract from \(3!S(100, 3)\) the number of assignations in which at least one of the case-workers gets exactly one client. Let \(A_1\) be the set of all assignations in which Case-worker 1 gets exactly one client (and all the others get at least one), and similarly define \(A_2, A_3\). Then \(|A_i| = 100 \times 2!S(99, 2)\) for all \(i\), \(|A_i \cap A_j| = 100 \times 99\) for all \(i \neq j\), and...
|A_1 \cap A_2 \cap A_3| = 0. So by the Inclusion/Exclusion Principle, the answer to the question is

\[3! S(100, 3) - 3 \times 100 \times 2! S(99, 2) + 3 \times 100 \times 99.\]

**Marking Scheme.** 3 marks, double-star difficulty. Among those who used the right method, the most common mistake (losing 1 mark) was to calculate |A_i| as 100 \times 2^{99}. I gave 2 out of 3 for a plausible explanation of the incorrect answers 100 \times 99 \times 98 \times 3! S(97, 3) and \(\binom{100}{2} \times \binom{98}{2} \times \binom{96}{2} \times 3^{94}\), both of which overcount for the same reason as the incorrect answer mentioned in the previous part.

(e) Suppose that 6 of the 100 clients are designated as special. If the constraint is that every case-worker must have two of these special clients, how many ways are there to assign all the clients to case-workers? Explain your answer.

**Solution:** There are \(\binom{6}{2,2,2}\) = 90 ways to assign the 6 special clients to the three case-workers and 3^{94} ways to assign the others, so the answer is 90 \times 3^{94}.

**Marking Scheme.** 3 marks, single-star difficulty. For incorrect answers, I gave 1 out of 3 if they at least had one of the factors \(\binom{6}{2,2,2}\) and 3^{94} correct.

\((3 + 3 + 3 + 3 + 3 = 15 \text{ marks})\)
2. This question is for MATH2069 students only.

Define a sequence by the initial conditions \( g_0 = 1 \), \( g_1 = 1 \), and the recurrence relation \( g_n = 5g_{n-1} - 6g_{n-2} \) for \( n \geq 2 \).

(a) Say whether this recurrence relation is homogeneous or non-homogeneous, and what its order is.

**Solution:** This is a second-order homogeneous linear recurrence relation.

**Marking Scheme.** 1 mark for each of “second-order” and “homogeneous”.

(b) Using any method, solve the recurrence relation (i.e. give a closed formula for \( g_n \)).

**Solution:** The characteristic polynomial is \( x^2 - 5x + 6 \) which has roots 2 and 3. So the general solution of \( g_n = 5g_{n-1} - 6g_{n-2} \) has the form

\[ g_n = C_12^n + C_23^n, \]

for some constants \( C_1, C_2 \).

The given initial conditions result in the equations

\[ C_1 + C_2 = 1 \quad \text{and} \quad 2C_1 + 3C_2 = 1, \]

which have the unique solution \( C_1 = 2 \), \( C_2 = -1 \). So \( g_n = 2 \times 2^n - 3^n = 2^{n+1} - 3^n \).

**Marking Scheme.** 1 mark for finding the roots of the characteristic polynomial, 1 for the general solution \( g_n = C_12^n + C_23^n \), and 1 for determining the constants, with allowances for carrying on mistakes. Some students used the generating-function method, effectively answering this question and the next simultaneously.

(c) Hence or otherwise, find a closed formula for the generating function \( G(z) \) of the sequence \( g_0, g_1, g_2, \cdots \).

**Solution:** We have

\[ G(z) = \sum_{n=0}^{\infty} \left(2^{n+1} - 3^n\right) z^n = 2 \sum_{n=0}^{\infty} 2^n z^n - \sum_{n=0}^{\infty} 3^n z^n = \frac{2}{1 - 2z} - \frac{1}{1 - 3z}. \]

Alternatively, from the initial data we get

\[ G(z) = g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots \\
= 1 + z + (5g_1 - 6g_0)z^2 + (5g_2 - 6g_1)z^3 + \cdots \\
= 1 + z + 5(g_1 z^2 + g_2 z^3 + \cdots) - 6(g_0 z^2 + g_1 z^3 + \cdots) \\
= 1 + z + 5(zG(z) - z) - 6z^2G(z), \]

which rearranges to \( G(z) = \frac{1 - 4z}{1 - 5z + 6z^2} \).

**Marking Scheme.** 3 marks, with deductions for arithmetic mistakes. Those answers which deduced a correct formula for \( G(z) \) from the recurrence relation could still lose marks if they continued to find an incorrect partial-fractions decomposition (especially if they then deduced a formula for \( g_n \) conflicting with that in the previous question!).

**SOLUTIONS**
(d) Using generating functions or otherwise, find a closed formula for the sum
\[ f_n = g_0g_n + g_1g_{n-1} + \cdots + g_{n-1}g_1 + g_ng_0. \]

**Solution:** If we let \( F(z) \) be the generating function of the sequence \( f_0, f_1, f_2, \ldots \), then \( F(z) = G(z)^2 \) by the definition of multiplication of power series. Hence
\[
F(z) = \frac{4}{(1 - 2z)^2} + \frac{1}{(1 - 3z)^2} - \frac{4}{(1 - 2z)(1 - 3z)}
\]
\[
= \frac{4}{(1 - 2z)^2} + \frac{1}{(1 - 3z)^2} - \frac{12}{1 - 3z} + \frac{8}{1 - 2z},
\]
from which we can read off the answer
\[
f_n = 4(n + 1)2^n + (n + 1)3^n - 12 \times 3^n + 8 \times 2^n = 4(n + 3)2^n + (n - 11)3^n.
\]
Alternatively, using the above formula for \( g_n \) we find:
\[
f_n = \sum_{m=0}^{n} g_m g_{n-m}
\]
\[
= \sum_{m=0}^{n} (2^{m+1} - 3^m)(2^{n-m+1} - 3^{n-m})
\]
\[
= \sum_{m=0}^{n} 2^{n+2} + 3^n - 2^{m+1}3^{n-m} - 2^{n-m+1}3^m
\]
\[
= (n + 1)(2^{n+2} + 3^n) - 2 \sum_{m=0}^{n} 2^{m+1}3^{n-m}
\]
\[
= (n + 1)(2^{n+2} + 3^n) - 2(2 \times 3^{n+1} - 2^{n+2}),
\]
where the last equality uses the sum of a geometric progression.

**Marking Scheme.** 3 marks, double-star difficulty. 1 mark just for the idea of calculating \( G(z)^2 \).

(e) Using the generating function, or any other valid method, solve the recurrence relation
\[ a_n = 5a_{n-1} + 3 \times 2^n, \text{ for } n \geq 1, \]
with the initial condition \( a_0 = 3 \).

**Solution:** Let \( A(z) \) be the generating function of the sequence \( a_n \). Then
\[
A(z) = 3 + \sum_{n=1}^{\infty} a_n z^n
\]
\[
= 3 + \sum_{n=1}^{\infty} (5a_{n-1} + 3 \times 2^n) z^n
\]
\[
= 3 + 5 \sum_{n=1}^{\infty} a_{n-1} z^n + 3 \sum_{n=1}^{\infty} 2^n z^n
\]
\[
= 3 + 5zA(z) + \frac{3}{1 - 2z} - 3,
\]
which leads to

\[ A(z) = \frac{3}{(1 - 2z)(1 - 5z)} = \frac{-2}{1 - 2z} + \frac{5}{1 - 5z}. \]

So \( a_n = -2^{n+1} + 5^{n+1}. \)

Alternatively, we can use the particular-solution method for solving non-homogeneous recurrence relations. The general solution of the homogeneous relation \( b_n = 5b_{n-1} \) is \( b_n = C5^n \) for some constant \( C \). We try to find a particular solution of the form \( p_n = A2^n \); we must have \( A2^n = 5A2^{n-1} + 3 \cdot 2^n \) for all \( n \geq 1 \), which implies that \( A = 5A/2 + 3 \), leading to \( A = -2 \). Hence the general solution of the given recurrence relation is \( a_n = C5^n - 2^{n+1} \), and the initial condition \( a_0 = 3 \) leads to \( C = 5 \).

Alternatively, the solution can be found by unravelling the recurrence relation and summing the geometric progression which results.

**Marking Scheme.** 4 marks. Those who used generating functions got 2 marks for finding a closed formula for \( A(z) \), and 2 marks for extracting the coefficient of \( a_n \) from this formula (or whatever incorrect formula they had). Those who used the particular-solution method got 1 mark for solving \( b_n = 5b_{n-1} \), 2 marks for finding the particular solution and incorporating it correctly in the general solution for \( a_n \), and 1 mark for determining the constant.

\[ (2 + 3 + 3 + 3 + 4 = 15 \text{ marks}) \]

2. This question is for MATH2969 students only.
(a) Let $\lambda$ be any nonzero complex number. Prove that a sequence $a_0, a_1, a_2, \cdots$ satisfies the recurrence relation

$$a_n = 2\lambda a_{n-1} - \lambda^2 a_{n-2}, \text{ for } n \geq 2,$$

if and only if there are constants $C_1, C_2$ such that $a_n = (C_1 + C_2n)\lambda^n$ for all $n \geq 0$. (In other words, give the justification for the characteristic-polynomial method in this case.)

**Solution:** Suppose that the formula $a_n = (C_1 + C_2n)\lambda^n$ holds, for some $C_1$ and $C_2$. Then for any $n \geq 2$, we have:

$$2\lambda a_{n-1} - \lambda^2 a_{n-2} = 2\lambda\left[C_1 + C_2(n-1)\right]\lambda^{n-1} - \lambda^2\left[C_1 + C_2(n-2)\right]\lambda^{n-2}$$

$$= (2C_1 + 2C_2n - 2C_2 - C_1 - C_2n + 2C_2)\lambda^n$$

$$= (C_1 + C_2n)\lambda^n = a_n,$$

as claimed. Conversely, suppose that $a_0, a_1, a_2, \cdots$ satisfies the recurrence relation. Define a new sequence $b_0, b_1, b_2, \cdots$ by $b_n = (a_0 + \left(\frac{a_1}{\lambda} - a_0\right)n)\lambda^n$. By what we have just proved, $b_n$ satisfies the same recurrence relation as $a_n$. Moreover, $b_0 = a_0$ and $b_1 = (a_0 + \frac{a_1}{\lambda} - a_0)\lambda = a_1$. Hence (by induction, strictly speaking) $b_n = a_n$ for all $n$, and our claim is proved.

**Marking Scheme.** 2 marks for each implication in the “if and only if”. The more mechanical direction of checking that $(C_1 + C_2n)\lambda^n$ is a solution was usually fine, but students often failed to explain adequately why every solution was of this form (as the question made clear, appealing to the characteristic-polynomial result was begging the point).

(b) Using the generating function, or any other valid method, solve the recurrence relation

$$d_n = 4d_{n-1} - 4d_{n-2} + (n-1)2^n, \text{ for } n \geq 2,$$

with the initial conditions $d_0 = 0, d_1 = 0$.

**Solution:** Let $D(z)$ be the generating function. We have

$$D(z) = \sum_{n=2}^{\infty} d_n z^n$$

$$= \sum_{n=2}^{\infty} (4d_{n-1} - 4d_{n-2} + (n-1)2^n) z^n$$

$$= 4 \sum_{n=2}^{\infty} d_{n-1} z^n - 4 \sum_{n=2}^{\infty} d_{n-2} z^n + \sum_{n=2}^{\infty} (n-1)2^n z^n$$

$$= 4zD(z) - 4z^2D(z) + \frac{4z^2}{(1-2z)^2},$$

which rearranges to

$$D(z) = \frac{4z^2}{(1-2z)^4}.$$
Extracting the coefficient of $z^n$, we find that
\[ d_n = \left( \frac{n+1}{3} \right) 2^n. \]

This problem could also be solved using the particular-solution method, but you would have to realize (presumably by trial and error) that a particular solution incorporating $n^3 2^n$ and $n^2 2^n$ terms was needed.

**Marking Scheme.** 5 marks, double-star difficulty. 2 marks for the expression of $D(z)$ in terms of itself, 1 mark for deducing a closed formula for $D(z)$, 2 marks for extracting the coefficient of $z^n$.

(c) Recall that the Catalan generating function $C(z) = \sum_{n=0}^{\infty} c_n z^n$ satisfies the equation $C(z) = 1 + zC(z)^2$. Deduce from this that
\[ 2z^2C(z)C'(z) = zC'(z) - C(z) + 1. \]

**Solution:** Differentiating both sides of $C(z) = 1 + zC(z)^2$ (and using the Product Rule on the right-hand side), we obtain $C'(z) = C(z)^2 + 2zC(z)C'(z)$. Hence
\[ 2z^2C(z)C'(z) = zC'(z) - zC(z)^2 = zC'(z) - (C(z) - 1), \]
proving the claim.

**Marking Scheme.** 3 marks, 2 for differentiating the original equation and 1 for the manipulations needed to reach the desired result.

(d) Hence or otherwise prove that for all $n \geq 0$,
\[ \sum_{m=1}^{n} mc_m c_{n-m} = \frac{n}{2} c_{n+1}. \]

Explain carefully what general facts you are using.

**Solution:** The coefficient of $z^m$ in $C'(z)$ is by definition $(m+1)c_{m+1}$. From the formula for the product of two power series, the coefficient of $z^{n-1}$ in $C(z)C'(z) = C'(z)C(z)$ is
\[ \sum_{m=0}^{n-1} (m+1)c_{m+1}c_{n-1-m} = \sum_{m=1}^{n} mc_m c_{n-m}. \]
(When $n = 0$ this sum is to be interpreted as zero, which is indeed the coefficient of $z^{-1}$ in $C(z)C'(z)$.) So taking the coefficient of $z^{n+1}$ on both sides of the equation proved in the previous part, we obtain
\[ 2 \sum_{m=1}^{n} mc_m c_{n-m} = (n+1)c_{n+1} - c_{n+1} = nc_{n+1}, \]
which implies the desired equation.

**Solutions**
An alternative way to prove this is the following:

\[
2 \sum_{m=1}^{n} mc_{m} c_{n-m} = 2 \sum_{m=0}^{n} mc_{m} c_{n-m} = \sum_{m=0}^{n} mc_{m} c_{n-m} + \sum_{m=0}^{n} (n-m)c_{m} c_{n-m} = n \sum_{m=0}^{n} c_{m} c_{n-m} = nc_{n+1},
\]

where the last step is the Catalan recurrence relation.

**Marking Scheme.** 3 marks, single-star difficulty. 1 mark just for displaying knowledge of relevant power series facts, such as the formulas for derivative and product.

\[(4 + 5 + 3 + 3 = 15 \text{ marks})\]
3. Let \( f(n) \) and \( g(n) \) be two functions of a nonnegative integer variable such that
\[ \lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \infty. \]
Recall that \( f(n) \asymp g(n) \) means that there are some positive constants \( C, D, N \) such that
\[ C \leq \frac{f(n)}{g(n)} \leq D, \quad \text{for all } n \geq N. \]

(a) Is it true that \( \binom{n}{2} \asymp n^2 \)? Explain your answer.

**Solution:** Yes, because
\[ \lim_{n \to \infty} \frac{n(n-1)}{n^2} = \lim_{n \to \infty} \frac{1 - \frac{1}{n}}{2} = \frac{1}{2}, \]
from which it follows that for all sufficiently large \( n \), the quotient \( \binom{n}{2} / n^2 \) lies between, say, \( \frac{1}{4} \) and \( \frac{3}{4} \).

**Marking Scheme.** 3 marks for observing that the limit is \( \frac{1}{2} \), and that this implies the result. Alternatively, 3 marks could be obtained by explaining that, like any polynomial function of \( n \), \( \binom{n}{2} = \frac{1}{2}n^2 - \frac{1}{2}n \) has the same growth rate as its leading term \( \frac{1}{2}n^2 \). Marks were deducted for vagueness which impaired the reasoning, e.g. suggesting that the mere fact that \( \binom{n}{2} / n^2 \) had a finite limit was enough (the point is that the limit is positive). Any answer claiming the statement was false got 0 out of 3, as did saying “yes” with no explanation.

(b) Give the rest of this definition: “We say that \( f(n) \) grows at a slower rate than \( g(n) \), the notation for which is \( f(n) \prec g(n) \), if . . . “.

**Solution:** \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0. \)

**Marking Scheme.** 2 marks for this or for the equivalent \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty. \)

0 out of 2 for saying “. . . \( g(n) \) grows at a faster rate than \( f(n) \)”!

(c) Prove by induction that \( 3^n < n! \) for all \( n \geq 7. \)

**Solution:** The base case is true because \( 3^7 = 2187 < 5040 = 7! \). We can now assume that \( n \geq 8 \) and that \( 3^{n-1} < (n-1)! \). Then
\[ 3^n = 3 \times 3^{n-1} < 3 \times (n-1)! < n \times (n-1)! = n!, \]
completing the induction step.

**Marking Scheme.** 1 mark for the base case (which had to be shown with the explicit numbers rather than merely asserted), and 2 marks for the induction step. Marks were deducted if the logic was obtrusively backwards (that is, if the argument appeared to deduce \( 3^{n-1} < (n-1)! \) from \( 3^n < n! \)).

(d) Hence or otherwise prove that \( 2^n \prec n! \).

**Solution:** By the previous part, for all \( n \geq 7 \) we have \( 0 < \frac{2^n}{n!} < \frac{2^7}{7!} = \left(\frac{2}{3}\right)^7 \).

Since \( \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0 \), we have \( \lim_{n \to \infty} \frac{2^n}{n!} = 0 \) by the Squeeze Law. There
are many alternative proofs: for instance, one can prove by induction that $2^n < n!$ for all $n \geq 4$ (this was done in lectures), and it then follows that $\frac{2^n}{n!} = \frac{2 \times 2^{n-1}}{n(n-1)!} < \frac{2}{n}$ for all $n \geq 5$, so again $\lim_{n \to \infty} \frac{2^n}{n!} = 0$ by the Squeeze Law.

**Marking Scheme.** 3 marks, single-star difficulty (although this result was proved in lectures). I gave 1 out of 3 for a vague appeal to the fact that when $n$ increases by 1, $2^n$ only doubles whereas $n!$ is multiplied by $n + 1$. Note that the first proof amounts to saying that $2^n \prec 3^n$ and $3^n < n!$ (for sufficiently large $n$), from which it follows that $2^n \prec n!$. Arguments appealing to this quasi-transitivity principle (which happened not to have been stated in lectures) rather than deducing it from the Squeeze Law got 3 or 2 marks depending on their convincingness. By contrast, it was definitely not sufficient to say that $2^n < 3^n < n!$ for sufficiently large $n$, because $f(n) < g(n)$ does not imply $f(n) \prec g(n)$.

(e) Define a sequence $a_n$ by the initial condition $a_0 = 0$ and the recurrence relation

$$a_n = a_{n-1} + \lceil \sqrt{n} \rceil + \frac{n^3}{3}, \text{ for } n \geq 1.$$ 

Prove that $a_n \asymp n^2$.

**Solution:** We use a ‘squeezing’ argument, bounding $a_n$ above and below by sequences which grow at the same rate as $n^2$. Define a sequence $\ell_n$ by the initial condition $\ell_0 = 0$ and the recurrence relation $\ell_n = \ell_{n-1} + \frac{2}{3}$, for $n \geq 1$. Since this differs from the recurrence in the question by the omission of the nonnegative term $\lceil \sqrt{n} \rceil$, we have $a_n \geq \ell_n$ for all $n \geq 0$. Unravelling gives

$$\ell_n = \frac{0 + 1 + 2 + \cdots + n}{3} = \frac{n(n+1)}{6},$$

which grows at the same rate as $n^2$. Now define a sequence $u_n$ by the initial condition $u_0 = 0$ and the recurrence relation $u_n = u_{n-1} + 2n$, for $n \geq 1$. Since this is obtained from the recurrence in the question by increasing both $\lceil \sqrt{n} \rceil$ and $\frac{2}{3}$ to $n$, we have $a_n \leq u_n$ for all $n \geq 0$. Unravelling gives

$$u_n = 0 + 2 + 2 \times 2 + \cdots + 2n = n(n+1),$$

which also grows at the same rate as $n^2$. It follows that $a_n$ grows at the same rate as $n^2$. Other proofs are possible.

**Marking Scheme.** 4 marks, single-star difficulty, with 1 or 2 marks deducted if only one of the bounds was devised correctly. In incomplete answers, I gave 1 out of 4 for correctly unravelling the recurrence relation to obtain the non-closed formula $a_n = \sum_{m=1}^{\sqrt{n}} \lceil \sqrt{m} \rceil + \frac{4}{3}$.

$$(3 + 2 + 3 + 3 + 4 = 15 \text{ marks})$$
4. This question is for MATH2069 students only.

(a) Suppose that $G$ is a graph with 7 vertices which is regular of degree 4. How many edges does $G$ have, and how many edges does its complement $\overline{G}$ have?

**Solution:** By the Hand-shaking Lemma, the number of edges in $G$ is $\frac{7 \times 4}{2} = 14$. Hence the number of edges in $\overline{G}$ is $\left(\frac{7}{2}\right) - 14 = 7$.

**Marking Scheme.** 2 marks for the number of edges of $G$, 1 mark for that of $\overline{G}$. The most common incorrect answer, failing to divide anything by 2 (so $G$ had 28 edges and $\overline{G}$ had 14), got 1 out of 3.

(b) Is $(1, 2, 3, 4, 4)$ the degree sequence of a graph? If your answer is yes, draw a picture of such a graph; if your answer is no, explain why.

**Solution:** If $(1, 2, 3, 4, 4)$ were the degree sequence of a graph $G$, then $G$ would have two vertices, say $u$ and $v$, of degree 4, and a third vertex, say $w$, of degree 1. But then $u$ and $v$ both have to be adjacent to all the other vertices, which means in particular that $u$ and $v$ are both adjacent to $w$, contradicting the fact that $\deg(w) = 1$. So the answer is no. An alternative way to phrase this argument uses the idea of the Havel–Hakimi Theorem (though we don’t actually need the content of the theorem in this case): if we deleted the vertex $v$ of degree 4, then the remaining graph would have degree sequence $(0, 1, 2, 3)$, which is impossible because there cannot be a vertex of degree 3 if there are only two other vertices with nonzero degree.

**Marking Scheme.** 3 marks for either of these arguments, deducting a mark for not explaining why $(0, 1, 2, 3)$ is not graphic.

(c) Either write down an Eulerian circuit in the following graph, or explain why none exists.

![Graph](image)

**Solution:** An example of an Eulerian circuit in this graph is:

$e, a, b, f, e, g, h, i, c, f, i, j, f, d, c, h, e$.

**Marking Scheme.** 3 marks, deducting 1 for errors in the circuit which were more than just mistaking or omitting one symbol. Answers which deduced that there was no Eulerian circuit from an incorrect count of a vertex degree got 1 out of 3.

(d) Prove that if $G$ is a Hamiltonian graph, and $v$ is a vertex of $G$, then $G - v$ is connected.

**Solution:** Since $G$ is Hamiltonian, it contains a spanning cycle $C$. The graph $C - v$ is a path, so every vertex of $C - v$ is linked to every other vertex in $C - v$. Since the vertices of $G - v$ are the same as those of $C - v$, and all
edges in $C - v$ are still edges in $G - v$, this implies that $G - v$ is connected.

**Marking Scheme.** 3 marks for this proof from lectures, deducting 1 for flaws such as carelessly failing to distinguish between $G$ and $C$. Incorrect answers got 1 out of 3 if they displayed knowledge that a graph is Hamiltonian if and only if it contains a spanning cycle. The most common incorrect argument appealed to the fact that Hamiltonian graphs contain no bridges: however, a graph can have no bridges and still contain cut-vertices. The circular reasoning of using the more general form of this result (also proved in lectures), that if $S$ is any subset of vertices then $G - S$ has at most $|S|$ connected components, got 0 out of 3; it is implicit in the request to prove the $|S| = 1$ case that it is not allowable to use the general case.

(e) Show that the following graph is not Hamiltonian.

```
 a--------b
 |        |
 |        |
 e        f
 |        |
 |        |
 d--------c
```

**Solution:** Suppose for a contradiction that there was a spanning cycle in this graph. This cycle must use exactly two of the edges ending at each vertex. Hence it must include both edges ending at every vertex of degree 2, namely $g, h, i, j$. Since it includes both $\{f, h\}$ and $\{f, i\}$, it cannot include $\{a, f\}$, so it must use the other two edges ending at $a$. Hence the spanning 10-cycle includes the 5-cycle with vertices $a, b, g, j, e$, which is a contradiction.

**Marking Scheme.** 3 marks, single-star difficulty. Less rigorous arguments got between 0 and 2 depending on their convincingness: simply re-stating the definition (“the graph is not Hamiltonian because there is no walk which visits every vertex exactly once and returns to its starting point”) got 0, and brute-force lists of the cycles in the graph needed to be complete to be allowable. Appealing to a non-existent converse of Ore’s Theorem got 0 out of 3 (if the degree condition in Ore’s Theorem is not satisfied, it tells us nothing about whether the graph is Hamiltonian or not). Reluctantly, I gave full marks for the argument that this graph is a spanning subgraph of the Petersen graph, which was shown in lectures to be non-Hamiltonian (the objection to this argument is that it is harder to prove non-Hamiltonianess for the Petersen graph than for the graph in question, so it verges on the sort of circularity mentioned in the previous part).

$(3 + 3 + 3 + 3 + 3 = 15 \text{ marks})$

4. This question is for MATH2969 students only.

(a) Is $(2, 2, 2, 3, 3, 4)$ the degree sequence of a graph? If your answer is yes, draw a picture of such a graph; if your answer is no, explain why.
Solution: The answer is yes: the following picture is of such a graph.

![Graph Image]

Marking Scheme. 3 marks simply for drawing such a picture. 2 out of 3 for a convincing proof using the Havel–Hakimi idea (the Havel–Hakimi Theorem itself is not logically involved) without an accompanying picture.

(b) Prove that if $G$ is a graph with $n$ vertices, $k$ edges, and $s$ connected components, then $k \leq \binom{n - s + 1}{2}$.

Solution: Let $G_1, \ldots, G_s$ be the connected components of $G$, and suppose that $G_i$ has $n_i$ vertices and $k_i$ edges, so that $n = n_1 + \cdots + n_s$ and $k = k_1 + \cdots + k_s$. We have $k_i \leq \binom{n_i}{2}$ for all $i$, so

$$k \leq \binom{n_1}{2} + \cdots + \binom{n_s}{2} = \frac{1}{2}(n_1(n_1 - 1) + \cdots + n_s(n_s - 1)) \leq \frac{1}{2}(n - s + 1)(n_1 - 1 + \cdots + n_s - 1) \quad \text{(since each } n_i \leq n - s + 1)$$

$$= \frac{1}{2}(n - s + 1)(n - s) = \binom{n - s + 1}{2},$$

proving the required upper bound.

Marking Scheme. 4 marks for this or an equivalent proof (the result was proved in lectures). For incomplete answers, I gave 1 mark just for knowing the $s = 1$ case, and 2 out of 4 for getting the inequality $k \leq \binom{n_1}{2} + \cdots + \binom{n_s}{2}$.

(c) Determine (with proof) the maximum number of edges a connected graph with 5 vertices can have without being Hamiltonian.

Solution: The following graph with 5 vertices and 7 edges is clearly not Hamiltonian, because $\deg(a) = 1$:

![Graph Image]

To show that 7 is the maximum, it suffices to show that any graph $G$ with 5 vertices and 8 edges is Hamiltonian. Note that $\overline{G}$ has $\binom{5}{2} - 8 = 2$ edges, which either have an end in common or do not; hence there are only two possible isomorphism classes for $\overline{G}$, and thus only two possible isomorphism classes
for $G$. The following pictures represent the possible isomorphism classes for $G$, and in each case there is visibly a spanning cycle:

So the answer is indeed 7.

**Marking Scheme.** 4 marks, double-star difficulty. I gave 2 marks for finding an example with 7 edges, and 1 further mark for displaying knowledge of potentially useful facts such as Ore’s Theorem (which could be used instead of the classification to show that graphs with 5 vertices and 8 edges are Hamiltonian). Note that it is not enough to say that one particular graph with 7 edges cannot add a new edge without becoming Hamiltonian, because there are other graphs with 7 edges which conceivably could.

(d) Determine (with proof) the minimum weight of a spanning cycle in the following weighted graph $H$.

**Solution:** There are actually only four spanning cycles in the graph $H$, so the simplest solution is to list them and compare their weights. An argument using more general facts is as follows. If $C$ is any spanning cycle of $H$, then $C - d$ is a spanning path of $H - d$. Now we can build a minimal spanning tree of $H - d$ using Prim’s Algorithm, starting with the vertex $a$, say: the first edge to be added is $\{a, b\}$ of weight 2, and the next two edges would both have weight 3. So $C - d$ must have weight at least 8. Since the edges of $H$ ending at $d$ have weights 4, 5, 5, the two edges of $C$ ending at $d$ have a combined weight of at least $4 + 5 = 9$. So $C$ must have weight at least $8 + 9 = 17$. Since the cycle with successive vertices $a, b, c, e, d$ has weight 17, this is the minimum weight of a spanning cycle.

**Marking Scheme.** 4 marks, single-star difficulty (because hardly any students realized how few spanning cycles there were). I gave 1 out of 4 for finding a spanning cycle of weight 17, a further 1 mark for displaying knowledge of Prim’s Algorithm, and a further 1 mark for displaying knowledge of a general lower bound on the minimum weight of a spanning cycle.

$(3 + 4 + 4 + 4 = 15$ marks)
5. (a) Complete the following definition: “A tree is . . . ”

**Solution:** a connected graph containing no cycles.

**Marking Scheme.** 2 marks, deducting 1 for the most common mistake of omitting “connected”.

(b) Prove that if $T$ is a tree with $n$ vertices, then $T$ has $n - 1$ edges. (You may assume that any tree with at least two vertices has a leaf.)

**Solution:** We prove this by induction. The $n = 1$ base case is obvious. Assume that $n \geq 2$ and that the result is known for trees with fewer than $n$ vertices. Let $v$ be a leaf of $T$; then $T - v$ is a tree with $n - 1$ vertices, so by the induction hypothesis it has $n - 2$ edges. Since $T$ has one more edge than $T - v$ (namely the unique edge ending at $v$), $T$ has $n - 1$ edges as required.

**Marking Scheme.** 3 marks. I deducted a mark if it was assumed without justification that a tree with $n$ vertices must arise by attaching a single edge to a tree with $n - 1$ vertices (i.e. if no reference was made to the existence of leaves). Some vaguer answers got 1 out of 3 if they had the right idea, 0 out of 3 if they only applied to paths.

(c) Draw the tree $T$ with vertex set \{1, 2, 3, 4, 5, 6\} whose Prüfer sequence is (3, 6, 1, 6). Show your working.

**Solution:** The leaves are the vertices which do not appear in the Prüfer sequence – in this case, 2, 4, and 5. The smallest leaf 2 must be adjacent to 3, and the Prüfer sequence of $T - 2$ must be (6, 1, 6); its smallest leaf 3 must be adjacent to 6, and the Prüfer sequence of $(T - 2) - 3$ must be (1, 6); its smallest leaf 4 must be adjacent to 1, and the Prüfer sequence of $((T - 2) - 3) - 4$ must be (6). Hence we can successively determine $((T - 2) - 3) - 4, (T - 2) - 3$, and $(T - 2)$, finding eventually that $T$ itself is the following tree:

```
4 1 6 3 2
\downarrow
5
```

**Marking Scheme.** 3 marks, deducting 1 if no sensible working was evident. I gave 1 out of 3 for incorrect answers which showed an understanding of the idea.

(d) Determine how many trees with vertex set \{1, 2, 3, 4, 5, 6\} have the property that the vertex 6 has degree 3. Explain what general facts you are using.

**Solution:** In the Prüfer sequence of a tree, the number of occurrences of each vertex is one less than its degree. So under the Prüfer bijection between trees with vertex set \{1, 2, 3, 4, 5, 6\} and sequences $(p_1, p_2, p_3, p_4)$ where each $p_i \in \{1, 2, 3, 4, 5, 6\}$, the trees in which the vertex 6 has degree 3 correspond to the sequences in which 6 occurs exactly twice. We can enumerate these by choosing the positions of the two 6’s in $\binom{4}{2} = 6$ ways, and then choosing the other two entries in $5^2$ ways. So the answer is $\binom{4}{2} \times 5^2 = 150$.

Alternatively, one can choose the neighbours of vertex 6 in $\binom{5}{3} = 10$ ways, and it is not hard to see that there are then 15 possible ways to attach the
other two vertices to these neighbours (6 ways in which they get attached to different neighbours, and 9 in which they get attached to the same neighbour). So the answer is \(10 \times 15 = 150\).

**Marking Scheme.** 4 marks, single-star difficulty. I gave 1 out of 4 for making a first step (such as choosing the neighbours of vertex 6), and 2 or 3 out of 4 for fuller answers which missed some cases. There were no marks just for knowing Cayley’s Formula.

(e) Using any method, find the number of spanning trees of the following graph.

\[\begin{array}{ccc}
1 & 5 & 2 \\
4 & & 3 \\
\end{array}\]

**Solution:** Since there are five vertices, any spanning tree has to use four of the six edges of the graph; so two of the six edges must be deleted. If the edge \(\{1, 4\}\) is included in the spanning tree, then one of the two edges \(\{1, 5\}\) and \(\{4, 5\}\) must be deleted to prevent there being a 3-cycle; similarly, one of the other three edges must be deleted to prevent there being a 4-cycle. So there are \(2 \times 3 = 6\) spanning trees which include the edge \(\{1, 4\}\). If the edge \(\{1, 4\}\) is one of the ones deleted, then what remains is a 5-cycle, and we can delete any of its 5 edges. So there are 5 spanning trees which do not include \(\{1, 4\}\), and \(6 + 5 = 11\) spanning trees in total.

Alternatively, we can use the Matrix–Tree Theorem. The Laplacian matrix of this graph is

\[
M = \begin{pmatrix}
3 & -1 & 0 & -1 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
-1 & 0 & -1 & 3 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{pmatrix},
\]

and the number of spanning trees equals any cofactor of \(M\). The (1,1)-cofactor is:

\[
\det\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 3 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix} = 2\det\begin{pmatrix}
2 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 2
\end{pmatrix} + \det\begin{pmatrix}
-1 & -1 & 0 \\
0 & 3 & -1 \\
0 & -1 & 2
\end{pmatrix}
\]

\[
= 2(2 \times 5 - (-1) \times (-2)) + ((-1) \times 5) = 11.
\]

**Marking Scheme.** 3 marks, deducting 1 for a minor arithmetical mistake. Just forming a cofactor correctly was worth 1 out of 3. I also gave full marks to a complete listing of the eleven spanning trees, on the grounds that the graph was small enough for it to be considered obvious that there were no more; however, those who attempted a listing but fell short of eleven got 0 out of 3, having demonstrated the unreliableness of their own method.
(2 + 3 + 3 + 4 + 3 = 15 marks)
6. This question is for MATH2069 students only.

Recall that a vertex colouring of a graph $G$ is a way of assigning a colour to each vertex in such a way that no two adjacent vertices have the same colour; and for any nonnegative integer $t$, the chromatic polynomial $P_G(t)$ is the number of vertex colourings of $G$ using a fixed set of $t$ possible colours.

(a) State (without proof) the formula for the chromatic polynomial of a tree with $n$ vertices.

**Solution:** If $T$ is a tree with $n$ vertices, $P_T(t) = t(t-1)^{n-1}$.

**Marking Scheme.** 2 marks, deducting 1 if the exponent of the $t-1$ factor was incorrect. Giving the formula for the chromatic polynomial of a complete graph, or giving the chromatic number of a tree, got 0 out of 2.

(b) Let $G$ be a graph and $\{v, w\}$ an edge of $G$. Prove that

$$P_G(t) = P_{G-\{v, w\}}(t) - P_{G[\{v, w\}])(t),$$

where $G[\{v, w\}]$ is the graph obtained from $G$ by fusing the vertices $v$ and $w$.

**Solution:** Fix a colour set with $t$ colours. The vertex colourings of $G-\{v, w\}$ can be split into two kinds: those where $v$ and $w$ have the same colour, and those where they have different colours. The vertex colourings where $v$ and $w$ have the same colour are in bijection with the vertex colourings of $G[\{v, w\}]$, while the vertex colourings where $v$ and $w$ have different colours are in bijection with the vertex colourings of $G$ (in which the edge $\{v, w\}$ forces the colours of $v$ and $w$ to be different). Hence $P_{G-\{v, w\}}(t) = P_{G[\{v, w\}]}(t) + P_G(t)$, which rearranges to the desired equation.

**Marking Scheme.** 3 marks for this proof from lectures. Deducing the result from the other reduction formula (which is essentially just the trivial last step of the argument) got 1 out of 3.

(c) Using any method, find $P_G(t)$ where $G$ is the following graph.

![Graph Image]

**Solution:** Since the vertices $b, c, e, f$ are all adjacent to each other, they must be given different colours in any vertex colouring. So the number of ways to colour these vertices with a fixed set of $t$ colours is $t(t-1)(t-2)(t-3)$.

Then $a$ can be given any colour different from those of $b$ and $f$, and $d$ can be given any colour different from those of $c$ and $e$, so (assuming that $t \geq 2$) there are $t - 2$ choices for the colour of $a$ and for the colour of $d$, and $P_G(t) = t(t-1)(t-2)^3(t-3)$ (note that this gives the correct answer, i.e. zero, in the case that $t < 2$ also).

**Marking Scheme.** 3 marks for the right answer, with no explanation.
required; incorrect answers got 1 or 2 out of 3 depending on how much of the idea was understood.

(d) Complete the following definition: “If $G$ is a graph, the edge chromatic number $\chi'(G)$ is . . .”

Solution: the smallest nonnegative integer $t$ such that $G$ has an edge colouring with $t$ colours (i.e. the edges can be coloured with $t$ colours in such a way that no two edges with the same colour have an end in common).

Marking Scheme. 2 marks, allowing less precise versions. Completing the statement of Vizing’s Theorem, instead of defining $\chi'(G)$, got 0 out of 2.

(e) Draw a picture of a graph $G$ in which every vertex has degree $\leq 5$, such that $\chi'(G) = 6$; and explain why $\chi'(G) = 6$ for your graph $G$. You may use any results from lectures that you require.

Solution: An example of such a $G$ is the following graph:

By Vizing’s Theorem, since the maximum degree of a vertex is 5, the edge chromatic number is at most $5+1 = 6$, so to prove that it is equal to 6 we need only show that there is no edge colouring of $G$ with 5 colours. Suppose for a contradiction that there is. Restricting attention to the complete subgraph $K$ with vertices 1, 2, 3, 4, 5, we see that at each of these vertices, all but one of the colours must be used for the edges of $K$ ending there; and the omitted colour must be different at each of these vertices. In the full graph $G$ each of these vertices is at the end of another edge not contained in $K$, so the omitted colour must in fact be used for this other edge. Hence the colours on the edges $\{1, 6\}$, $\{2, 7\}$, $\{3, 7\}$, $\{4, 6\}$, and $\{5, 6\}$ must all be different. This creates the desired contradiction, because the edge $\{6, 7\}$ cannot have any of these 5 colours.

Marking Scheme. 5 marks, double-star difficulty. The most common incorrect example offered was $K_6$, to which I gave 1 out of 5; justice then required giving 2 out of 5 for the observation that $K_6$ is not an example, since $\chi'(K_6) = 5$ as shown in lectures. (This implies that any example must have at least 7 vertices; the above example was obtained by splitting the vertex 6 of $K_6$ into two vertices.)

$(2 + 3 + 3 + 2 + 5 = 15$ marks)
6. This question is for MATH2969 students only.

(a) Complete the following definition: “If $G$ is a graph, the chromatic number $\chi(G)$ is . . .”

**Solution:** the smallest nonnegative integer $t$ such that $G$ has a vertex colouring with $t$ colours (i.e. the vertices can be coloured with $t$ colours in such a way that no two adjacent vertices have the same colour).

**Marking Scheme.** 2 marks, allowing less precise phrasing.

(b) Suppose that $G$ has $n \geq 1$ vertices and is regular of degree $d$. Prove that $\chi(G) \geq \frac{n}{n-d}$.

**Solution:** Suppose for a contradiction that $\chi(G) < \frac{n}{n-d}$; equivalently, $n - d < \frac{n}{\chi(G)}$ (we know that $n - d$ must be a positive integer). Consider a vertex colouring of $G$ with $\chi(G)$ colours, which exists by definition. By the Pigeonhole Principle, there must be some colour – say red – which is used for at least $\left\lceil \frac{n}{\chi(G)} \right\rceil$ vertices, and we are assuming that this number is at least $n - d + 1$. But if $v$ is any red vertex, the $d$ vertices adjacent to $v$ cannot be red, so there are at least $d$ non-red vertices. This gives the contradictory conclusion that the total number of vertices is at least $(n - d + 1) + d = n + 1$.

**Marking Scheme.** 4 marks, double-star difficulty. I gave 1 out of 4 just for the idea of proof by contradiction.

(c) Which connected graphs $G$ have the property that their chromatic number $\chi(G)$ is strictly greater than their edge chromatic number $\chi'(G)$? Explain your answer; you may use any of the results given in lectures.

**Solution:** We know that $\chi'(G) \geq \Delta(G)$ where $\Delta(G)$ is the maximum degree of a vertex of $G$. On the other hand, Brooks’ Theorem says that $\chi(G) \leq \Delta(G)$ unless $G$ is a complete graph or an odd cycle. So the inequality $\chi(G) > \chi'(G)$ is only possible when $G$ is complete or an odd cycle. But for $n \geq 3$ odd, $\chi(C_n) = \chi'(C_n) = 3$ and $\chi(K_n) = \chi'(K_n) = n$, so those cases are also ruled out. Since $\chi(K_n) = n$ is indeed greater than $\chi'(K_n) = n - 1$ for all $n \geq 2$ even and for $n = 1$, the answer is exactly the complete graphs where the number of vertices is either even or 1.

**Marking Scheme.** 4 marks, single-star difficulty. I gave 2 out of 4 for the example of the even complete graphs, and 2 out of 4 for the proof that such a graph $G$ must have $\chi(G) = \Delta(G) + 1$ and $\chi'(G) = \Delta(G)$. No marks were deducted for forgetting the case of the graph with one vertex.

(d) Let $t$ be a positive integer. Determine the smallest positive integer $n$ such that there exists a graph $G$ with $n$ vertices satisfying $\chi'(G) = t$. You may use any of the results given in lectures.

**Solution:** Recall from lectures that for $n \geq 2$, $\chi'(K_n)$ is $n$ if $n$ is odd and $n - 1$ if $n$ is even.

Now if $t = 1$, then $n = 2$, because $\chi'(K_2) = 1$, and it is clear that $\chi'(G) = 0$ for any graph $G$ with fewer than 2 vertices. If $t \geq 3$ is odd, then $n = t$, and
because $\chi'(K_t) = t$, and any graph $G$ with fewer than $t$ vertices is a subgraph of $K_{t-1}$, whence we have $\chi'(G) \leq \chi'(K_{t-1}) = t - 2$. If $t \geq 2$ is even, then $n = t + 1$, because $\chi'(K_{1,t}) = t$, and any graph $G$ with fewer than $t + 1$ vertices is a subgraph of $K_t$, whence we have $\chi'(G) \leq \chi'(K_t) = t - 1$.

**Marking Scheme.** 5 marks, double-star difficulty. I gave 1 mark for the $t = 1$ case, 2 marks for the $t \geq 3$ odd case (of which 1 was for the example of $K_t$ and 1 was for the reason why $t$ is the minimum), and 2 marks for the $t \geq 2$ even case (of which 1 was for the example of $K_{1,t}$ and 1 was for the reason why $t + 1$ is the minimum).

$(2 + 4 + 4 + 5 = 15$ marks)