THE UNIVERSITY OF
SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH2069/2969: Discrete Mathematics and Graph Theory

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Time allowed: 2 hours, plus 10 minutes reading time

This booklet contains 13 pages.

This paper comprises 6 questions worth 10 marks each, for a total of 60 marks. Each question is divided into several parts; the marks assigned to each part are indicated at the end of the question.

Questions 1,3,4,6 are the same for MATH2069 and MATH2969. For questions 2,5, this paper contains both the normal-level MATH2069 question and the (completely different) advanced-level MATH2969 question. You may ONLY answer the questions for the unit you are enrolled in.

If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. A calculator is permitted.

SOLUTIONS
1. Let $n$ be a positive integer. Denote by $X$ the set of all functions $f$ from the set $A = \{1, \ldots, n\}$ to the set $B = \{1, 2, 3\}$.

(a) State the cardinality of the set $X$.

**Solution:** $|X| = 3^n$ by a theorem from lectures.

(b) Denote by $X_1$ the set of functions $f : A \to B$ such that 1 does not belong to the range of $f$. Define the respective sets of functions $X_2$ and $X_3$ in the same way by replacing 1 by 2 or 3. Use the inclusion-exclusion formula to calculate the cardinality of the set $X_1 \cup X_2 \cup X_3$.

**Solution:**

$$|X_1 \cup X_2 \cup X_3| = |X_1| + |X_2| + |X_3|$$

$$- |X_1 \cap X_2| - |X_1 \cap X_3| - |X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3|$$

$$= 2^n + 2^n + 2^n - 1 - 1 - 1 + 0 = 3 \cdot 2^n - 3.$$

(c) Calculate the number of functions $f : A \to B$ satisfying the condition $f(1) \leq f(2) \leq \cdots \leq f(n)$.

**Solution:** The list of values $(f(1), \ldots, f(n))$ must have the form $(1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3)$. Let $k_i$ denote the number of entries equal to $i$ in such a list for $i = 1, 2, 3$. The number of lists of this form equals the number of triples $(k_1, k_2, k_3)$ such that $k_1 + k_2 + k_3 = n$. By a theorem from lectures, the number of such triples is $\binom{n+2}{2}$.

(d) Give the definition of the Stirling numbers $S(n, k)$.

**Solution:** The Stirling number $S(n, k)$ is the number of ways of writing an $n$-element set as a disjoint union of $k$ nonempty subsets.

(e) Explain why the number of surjective functions from a set of cardinality $n$ to a set of cardinality $k$ is found as $k! S(n, k)$.

**Solution:** Let $X$ and $Y$ be sets of cardinalities $n$ and $k$, respectively. Every function $f : X \to Y$ gives rise to a way of writing $X$ as a disjoint union, namely as the disjoint union of the preimages $f^{-1}(y)$ for $y \in Y$. Saying that $f$ is surjective is the same as saying that all these preimages are nonempty, so in that case we get a partition of $X$ into $k$ blocks. For a fixed partition of $X$ into $k$ blocks, there are $k!$ surjective functions $f : X \to Y$ which give rise to it, because once you have grouped together the elements of $X$ which get sent to the same element of $Y$ by $f$, you just need to decide which block gets sent to which element of $Y$. So the number of surjective functions is $k!$ times the number of partitions into $k$ blocks.

(f) Hence use your answers to parts (a) and (b) to write an explicit formula for $S(n, 3)$.

**Solution:** By (a) and (b), the number of surjective functions equals $|X| - |X_1 \cup X_2 \cup X_3| = 3^n - 3 \cdot 2^n + 3$. Hence,

$$S(n, 3) = \frac{1}{6} (3^n - 3 \cdot 2^n + 3).$$

SOLUTIONS
(1 + 2 + 2 + 1 + 3 + 1 = 10 marks)
2. This question is for MATH2069 students only.

(a) Use the characteristic polynomial to find the general solution of the recurrence relation \( b_n = b_{n-1} + 6b_{n-2}, \ n \geq 2. \)

**Solution:** The characteristic equation is \( x^2 - x - 6 = 0, \) its roots are \(-2\) and \(3\). Hence the general solution is \( b_n = A(-2)^n + B3^n, \) where \(A\) and \(B\) are arbitrary constants.

(b) Consider the nonhomogeneous recurrence relation \( a_n = a_{n-1} + 6a_{n-2} + 2^n, \ \ n \geq 2, \) and suppose that \(a_n\) and \(p_n\) are two solutions of this relation. Show that the sequence \(a_n - p_n\) is a solution of the homogeneous recurrence relation in part (a).

**Solution:** For \( n \geq 2 \) we have

\[
a_n = a_{n-1} + 6a_{n-2} + 2^n \quad \text{and} \quad p_n = p_{n-1} + 6p_{n-2} + 2^n.
\]

Subtracting, we obtain

\[
a_n - p_n = a_{n-1} - p_{n-1} + 6(a_{n-2} - p_{n-2})
\]

so that \(b_n = a_n - p_n\) satisfies the relation in part (a).

(c) Find a particular solution of the recurrence relation in part (b) in the form \( p_n = C2^n. \)

**Solution:** Substituting into the relation we get

\[
C2^n = C2^{n-1} + 6C2^{n-2} + 2^n.
\]

Dividing by \(2^n\) we find \(C = -1.\)

(d) Hence find the solution of the recurrence relation in part (b) with the initial conditions \(a_0 = a_1 = 12.\)

**Solution:** The general solution is \( a_n = A(-2)^n + B3^n - 2^n. \) The initial conditions give \(A + B = 13\) and \(-2A + 3B = 14.\) Solving the system, we find \(A = 5\) and \(B = 8.\) Thus, \( a_n = 5(-2)^n + 8 \cdot 3^n - 2^n.\)

(e) Write a closed form of the generating function for the sequence \(a_n\) you found in part (d).

**Solution:** The generating function is

\[
A(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (5(-2)^n + 8 \cdot 3^n - 2^n) z^n = \frac{5}{1+2z} + \frac{8}{1-3z} - \frac{1}{1-2z}.
\]

\[
(2 + 2 + 2 + 2 + 2 = 10 \text{ marks})
\]

2. This question is for MATH2969 students only.

**SOLUTIONS**
(a) Consider the nonhomogeneous recurrence relation

\[ a_n = -2n a_{n-1} + \frac{3^n}{n!}, \quad n \geq 1. \]

Prove that the general solution of this relation can be written in the form

\[ a_n = b_n + p_n, \]

where \( b_n \) is the general solution of the corresponding homogeneous relation and \( p_n \) is a particular solution of the nonhomogeneous relation.

**Solution:** Fix a particular solution \( p_n \) of the nonhomogeneous relation. Then for any solution \( b_n \) of the homogeneous recurrence relation

\[ b_n = -2 \frac{n}{n} b_{n-1}, \quad n \geq 1, \]

the sequence \( a_n = b_n + p_n \) solves the nonhomogeneous relation which follows by substitution. Furthermore, any solution \( a_n \) of the nonhomogeneous relation can be written in this form, since \( a_n - p_n \) is a solution of the homogeneous relation.

(b) Find the solution of the recurrence relation in part (a) satisfying the initial condition \( a_0 = 1 \).

**Solution:** First find the general solution of the recurrence relation

\[ b_n = -2 \frac{n}{n} b_{n-1}, \quad n \geq 1. \]

Unraveling, we find that \( b_n = \frac{(-2)^n}{n!} b_0 \). Now find a particular solution \( p_n \) of the recurrence relation in part (a). We will try to find \( p_n \) in the form \( p_n = A \frac{3^n}{n!} \). Substituting, we get

\[ A \frac{3^n}{n!} = -A \frac{2 \cdot 3^{n-1}}{n(n-1)!} + \frac{3^n}{n!} \]

which gives \( A = 3/5 \). Thus, \( p_n = \frac{3^{n+1}}{5 n!} \). Hence, the general solution is

\[ a_n = \frac{3^{n+1}}{5 n!} + B \frac{(-2)^n}{n!}. \]

The initial condition gives \( B = 2/5 \) and so

\[ a_n = \frac{3^{n+1} - (-2)^{n+1}}{5 n!}. \]

(c) Using generating functions or otherwise, for each \( n \geq 0 \) calculate the sum

\[ \sum_{k=0}^{n} \frac{2^k a_{n-k}}{k!}, \]

where the sequence \( a_n \) is found in part (b).
**Solution:** We have

\[ A(z) = \frac{3}{5} \exp(3z) + \frac{2}{5} \exp(-2z) \]

and so the generating function of the sequence

\[ c_n = \sum_{k=0}^{n} \frac{2^k a_{n-k}}{k!} \]

is found by \( C(z) = A(z) \exp(2z) = \frac{3}{5} \exp(5z) + \frac{2}{5} \). (We can rely on the formula \( \exp(cz) \exp(dz) = \exp((c+d)z) \) proved in the assignment). Hence, \( c_0 = 1 \) and \( c_n = \frac{3 \cdot 5^{n-1}}{n!} \) for \( n \geq 1 \).

(d) Define the generating function \( \log(1 + z) \) by

\[ \log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n. \]

Use the substitution \( z \mapsto \exp(z) - 1 \) to define the series \( \log(\exp(z)) \). Prove that \( \log(\exp(z)) = z \).

**Solution:** Set \( A(z) = \log(\exp(z)) \) and note that the constant term \( a_0 \) of \( A(z) \) is zero. Calculating the derivative, we find that \( (\log(1 + z))' = \frac{1}{1 + z} \) (this is also known from tutorials). Hence, using the chain rule, we find

\[ A'(z) = \frac{1}{\exp(z)} (\exp(z))' = 1. \]

The unique solution of this equation is \( A(z) = z \), thus proving the identity.

(2 + 3 + 3 + 2 = 10 marks)
3. Recall that $F_n$ is the Fibonacci sequence $0, 1, 1, 2, \ldots$ with the generating function

$$F(z) = \frac{z}{1 - z - z^2}$$

and consider the sequence $a_n$ defined by the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + F_n$$

for $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$.

(a) Show that the generating function $A(z)$ of the sequence $a_n$ is given by

$$A(z) = \frac{z}{1 - z - z^2}.$$  

**Solution:** We have

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2} + F_n) z^n = z + zA(z) + z^2A(z) + F(z) - z.$$  

Hence,

$$A(z) = \frac{F(z)}{1 - z - z^2} = \frac{z}{(1 - z - z^2)^2}.$$  

(b) Hence prove the identity $a_n = F_0 F_{n+1} + F_1 F_n + \cdots + F_{n+1} F_0$ for $n \geq 0$.

**Solution:** We have

$$zA(z) = F(z)^2 = \sum_{n=0}^{\infty} (F_0 F_n + F_1 F_{n-1} + \cdots + F_n F_0) z^n.$$  

Equating the coefficients of $z^n$ for $n \geq 1$ we find that

$$a_{n-1} = F_0 F_n + F_1 F_{n-1} + \cdots + F_n F_0$$

which is the desired identity.

(c) Find constants $E, F, G, H$ such that

$$A(z) = \frac{E}{1 - x_1 z} + \frac{F}{(1 - x_1 z)^2} + \frac{G}{1 - x_2 z} + \frac{H}{(1 - x_2 z)^2},$$

where $x_1 = \frac{1 + \sqrt{5}}{2}$ and $x_2 = \frac{1 - \sqrt{5}}{2}$.

**Solution:** Multiplying both sides of the above relation by $(1 - z - z^2)^2,$ we get

$$z = E(1 - x_1 z)(1 - x_2 z)^2 + F(1 - x_2 z)^2 + G(1 - x_2 z)(1 - x_1 z)^2 + H(1 - x_1 z)^2.$$  

Both sides are polynomials in $z$ so that taking $z = x_1^{-1}$ we find

$$F = \frac{x_1^{-1}}{(1 - x_2 x_1^{-1})^2} = \frac{x_1}{(x_1 - x_2)^2} = \frac{x_1}{5} = \frac{1 + \sqrt{5}}{10},$$

while taking $z = x_2^{-1}$ we get

$$H = \frac{x_2^{-1}}{(1 - x_1 x_2^{-1})^2} = \frac{x_2}{(x_1 - x_2)^2} = \frac{x_2}{5} = \frac{1 - \sqrt{5}}{10}.$$
Furthermore, taking the derivative over \( z \) and then setting \( z = x_1^{-1} \) we find

\[
1 = E(-x_1)(1-x_2x_1^{-1})^2 + 2F(1-x_2x_1^{-1})(-x_2)
\]

which gives

\[
E = -\frac{x_1}{5\sqrt{5}} = -\frac{1 + \sqrt{5}}{10\sqrt{5}}.
\]

Similarly, taking the derivative over \( z \) and then setting \( z = x_2^{-1} \) we find

\[
1 = G(-x_2)(1-x_1x_2^{-1})^2 + 2H(1-x_1x_2^{-1})(-x_1)
\]

which gives

\[
G = \frac{x_2}{5\sqrt{5}} = \frac{1 - \sqrt{5}}{10\sqrt{5}}.
\]

(d) Determine the growth rate of the sequence \( a_n \).

**Solution:** Use the partial fraction decomposition found in the previous part,

\[
\frac{z}{(1-z-z^2)^2} = \frac{E}{1-x_1z} + \frac{F}{(1-x_1z)^2} + \frac{G}{1-x_2z} + \frac{H}{(1-x_2z)^2}
\]

with \( x_1 = \frac{1 + \sqrt{5}}{2} \) and \( x_2 = \frac{1 - \sqrt{5}}{2} \). Then

\[
a_n = Ex_1^n + F(n+1)x_1^n + Gx_2^n + H(n+1)x_2^n.
\]

Observe that \( x_1 > 1 \), while \( |x_2| < 1 \). Hence, since \( F \) is a positive constant we can conclude that \( a_n \asymp nx_1^n \).

\[
(2 + 2 + 3 + 3 = 10 \text{ marks})
\]

4. (a) Use mathematical induction to prove that any connected graph with \( n \) vertices has at least \( n - 1 \) edges.

**Solution:** The \( n = 1 \) base case is obvious. Assume that \( n \geq 2 \) and that we know the result for graphs with fewer than \( n \) vertices. Suppose that \( G \) contains a bridge \( e = \{v, w\} \). Since \( G \) is connected, every vertex in \( G-e \) must be linked to either \( v \) or \( w \); that is, \( G-e \) has two connected components, the component \( G_1 \) containing \( v \) and the component \( G_2 \) containing \( w \). Suppose that \( G_i \) has \( n_i \) vertices and \( k_i \) edges, for \( i = 1, 2 \). Then \( n_1 + n_2 = n \), so \( n_1, n_2 < n \); thus \( G_1 \) and \( G_2 \) are graphs to which the induction hypothesis applies, and we conclude that \( k_1 \geq n_1 - 1 \) and \( k_2 \geq n_2 - 1 \). Hence

\[
k = k_1 + k_2 + 1 \geq (n_1 - 1) + (n_2 - 1) + 1 = n_1 + n_2 - 1 = n - 1.
\]
On the other hand, if $G$ does not contain a bridge, then if we delete any edge it remains connected; if we continue to delete edges, we must eventually reach a spanning subgraph which does contain a bridge. Since the number of edges in this subgraph is at least $n - 1$, the number of edges in $G$ is even more. So in either case, $k \geq n - 1$ and the inductive step is complete.

(b) For the following graph $H$ write its degree sequence.

![Graph H](image)

**Solution:** The degree sequence of the given graph is $(1, 2, 3, 3, 3, 4)$.

(c) State the Havel–Hakimi Theorem.

**Solution:** Let $(d_1, d_2, \ldots, d_n)$ be a weakly increasing sequence of nonnegative integers. If $(d_1, d_2, \ldots, d_n)$ is graphic, then there is a graph $G$ with vertices $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and the vertices adjacent to $v_n$ are $v_{n-1}, v_{n-2}, \ldots, v_{n-d_n}$.

(d) Draw two non-isomorphic graphs $G$ and $G'$ whose degree sequences coincide with the one you found in part (b) so that each of $G$ and $G'$ satisfies the conditions provided by the Havel–Hakimi Theorem. Explain why the graphs $H$, $G$ are $G'$ are pairwise non-isomorphic.

**Solution:**

![Graph G](image)

![Graph G'](image)

The graph $H$ is not isomorphic to $G$ or $G'$ because the only vertex of $H$ of degree 4 is adjacent to vertices of degrees 1, 3, 3, 3, while in $G$ and $G'$ such a vertex is adjacent to vertices of degrees 2, 3, 3, 3. The graph $G$ is not isomorphic to $G'$ because the only vertex of $G$ of degree 1 is adjacent to a vertex of degree 2, while in $G'$ such a vertex is adjacent to a vertex of degree 3.

$(3 + 1 + 2 + 4 = 10 \text{ marks})$
5. This question is for MATH2069 students only.

(a) State the theorem which provides necessary and sufficient conditions for a connected graph to be Eulerian.

**Solution:** A connected graph $G$ is Eulerian if and only if the degree $deg(v)$ is even for every vertex $v$ of $G$.

(b) Either write down an Eulerian circuit in the following graph, or explain why none exists.

```
 a----b----c----d----e
 |   |   |   |   |
 f----g----h----i----j
```

**Solution:** All vertices have even degrees so that an Eulerian circuit exists. For instance, $a, g, c, j, i, h, g, f, b, h, e, d, c, b, a$ is such a circuit.

(c) Complete the definition: “A connected graph with at least 3 vertices is Hamiltonian if . . .”

**Solution:** it has a walk which visits every vertex exactly once and then returns to its starting point.

(d) Prove that if a graph $G$ is Hamiltonian, then for any vertex $v$ of $G$ the graph $G - v$ is connected.

**Solution:** Since $G$ is Hamiltonian, it contains a spanning cycle $C$. For any vertex $v$ of $G$, $v$ is also a vertex of $C$, and the graph $C - v$ is a path and hence connected. But then $G - v$, which has all the edges of $C - v$ and more, is also connected.

(e) Determine whether or not the following graph is Hamiltonian. Give your reasons.

```
 a----b----c----d
 |   |   |   |
 e----g----h----i
     |   |   |
     f----j
```

**Solution:** The graph is not Hamiltonian since $G - f$ is not connected.

(2 + 2 + 1 + 3 + 2 = 10 marks)

5. This question is for MATH2969 students only.

(a) State the theorem which provides necessary and sufficient conditions for a connected graph to have an Eulerian trail.

**Solution:** A connected graph $G$ has an Eulerian trail if and only if it has exactly two vertices of odd degree.
(b) Give a proof of the sufficiency of these conditions. (You may use the theorem providing the conditions for the existence of an Eulerian circuit without proving it.)

**Solution:** Suppose that \( G \) is connected and has exactly two vertices of odd degree, say \( v \) and \( w \). If \( v \) and \( w \) are not adjacent, then we can add the edge \( \{v, w\} \) to form the graph \( G + \{v, w\} \); since this increases the degrees of \( v \) and \( w \) by 1, all vertices in \( G + \{v, w\} \) have even degree. By the Euler Theorem, there is an Eulerian circuit in \( G + \{v, w\} \). Since this circuit includes the edge \( \{v, w\} \) at some point, we may as well suppose that it starts \( v, w, \ldots, v \); we then obtain an Eulerian trail \( w, \ldots, v \) in \( G \) by deleting \( \{v, w\} \) from the circuit. If \( v \) and \( w \) are adjacent in \( G \), instead we add a new vertex \( x \) as well as the edges \( \{v, x\} \) and \( \{w, x\} \). Since all the degrees in the new graph are even, it has an Eulerian circuit. Since this circuit visits \( x \) only once, we may as well suppose that it starts \( v, x, w, \ldots, v \); again, deleting the superfluous edges gives an Eulerian trail in \( G \).

(c) Either write down an Eulerian trail in the following graph, or explain why none exists.

```
 a  b  c  d  e
 f  g  h  i  j
```

**Solution:** An Eulerian trail is \( e, d, c, b, a, g, c, j, i, h, g, f, b, h, e, i \).

(d) State Ore’s Theorem.

**Solution:** Let \( G \) be a connected graph with \( n \) vertices where \( n \geq 3 \). If every non-adjacent pair of vertices \( \{v, w\} \) satisfies \( \text{deg}(v) + \text{deg}(w) \geq n \), then \( G \) is Hamiltonian.

(e) Determine and explain whether or not the following graph is Hamiltonian.

```
```

**Solution:** The graph is not Hamiltonian. It is a bipartite graph: colour the vertices in the middle row white and remaining vertices black. Any spanning cycle would have to alternate between white and black vertices. But there are 6 white vertices and 7 black vertices, so this is impossible.

\[ 2 + 3 + 2 + 1 + 2 = 10 \text{ marks} \]
6. (a) Find a walk which solves the Travelling Salesman Problem for the following weighted graph. Justify your answer.

\[
\begin{array}{c}
\text{a} \\
\text{b} & 6 \\
\text{c} \\
\text{d} \\
\text{e} & 2 \\
\text{f} \\
\end{array}
\]

\[a, b, f, c, d, c, f, e, b, a\]

**Solution:** The walk \(a, b, f, c, d, c, f, e, b, a\) of weight 17 is a solution. The distance from \(a\) to \(d\) is 8, the distance from \(e\) to \(d\) is 5 and the distance from \(a\) to \(e\) is 4. Hence the weight of a solution of the Travelling Salesman Problem cannot be less than 17.

(b) Draw the tree with 6 vertices whose Prüfer sequence is \((2, 5, 5, 6)\).

**Solution:**

(c) Suppose that \(v\) and \(w\) are adjacent vertices of a graph \(G\). State and prove the theorem providing an expression for the chromatic polynomial \(P_G(t)\) in terms of the chromatic polynomials associated with the graphs \(G - \{v, w\}\) and \(G[v, w]\), where \(G[v, w]\) is obtained from \(G\) by fusing of \(v\) and \(w\).

**Solution:** Let \(G\) be a graph and \(t\) a nonnegative integer. If \(v\) and \(w\) are adjacent vertices of \(G\), then \(P_G(t) = P_{G - \{v, w\}}(t) - P_{G[v, w]}(t)\).

If \(v\) and \(w\) are not adjacent, then in a vertex colouring of \(G\) there is no constraint on whether the colours of \(v\) and \(w\) are different or not. Thus the set of all vertex colourings of \(G\) (with a fixed colour set of size \(t\)) is the disjoint union of the set of vertex colourings of \(G\) in which the colours of \(v\) and \(w\) are different, and the set of vertex colourings of \(G\) in which the colours of \(v\) and \(w\) are the same. The vertex colourings of \(G\) in which the colours of \(v\) and \(w\) are different are obviously in bijection with the vertex colourings of \(G + \{v, w\}\), because the extra edge \(\{v, w\}\) imposes exactly the condition that the colours of \(v\) and \(w\) are different. The vertex colourings of \(G\) in which the colours of \(v\) and \(w\) are the same are obviously in bijection with the vertex colourings of \(G[v, w]\), because if their colours have to be the same, the two vertices may as well be fused into one. This shows that in this case \(P_G(t) = P_{G + \{v, w\}}(t) + P_{G[v, w]}(t)\).

Now if \(v\) and \(w\) are adjacent in \(G\), then applying this to the graph \(G - \{v, w\}\), we get \(P_{G - \{v, w\}}(t) = P_G(t) + P_{G[v, w]}(t)\) because it makes no difference to the fused graph \(G[v, w]\) whether \(v\) and \(w\) are adjacent in \(G\) or not. Rearranging this equation gives the result.
(d) Find the chromatic polynomial for the graph

\begin{center}
\includegraphics[width=0.1\textwidth]{graph.png}
\end{center}

**Solution:** Use another theorem from lectures: If $v$ is a vertex of $G$ such that the vertices adjacent to $v$ are all adjacent to each other, then $P_G(t) = P_{G-v}(t)(t - \deg(v))$. Applying this twice, first to the top vertex then to the vertex adjacent to it, we get $P_G(t) = (t - 2)^2 P_{C_4}(t)$. The chromatic polynomial $P_{C_4}(t)$ was found in lectures (it can be calculated with the use of the theorem in part (c)), $P_{C_4}(t) = t(t - 1)(t^2 - 3t + 3)$. Thus, $P_G(t) = t(t - 1)(t - 2)^2(t^2 - 3t + 3)$.

\[ (3 + 2 + 3 + 2 = 10 \text{ marks}) \]