This paper comprises 6 questions worth 10 marks each, for a total of 60 marks. Each question is divided into several parts; the marks assigned to each part are indicated at the end of the question.

Questions 1, 3, 4, 6 are the same for MATH2069 and MATH2969. For questions 2, 5, this paper contains both the normal-level MATH2069 question and the (completely different) advanced-level MATH2969 question. You may ONLY answer the questions for the unit you are enrolled in.

If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. Approved calculators are permitted.
1. (a) In this part (1(a)) you are required to evaluate your answers: it is not sufficient to merely give a formula for the answer. You must justify your answers.

(i) Calculate the number of positive integers $n$ such that $n < 10000$ and each digit of $n$ is either 1, 2 or 3. (For example, 2332 and 13 are two such integers.)

**Solution:** The number of $k$-digit numbers with digits from $\{1, 2, 3\}$ is $3^k$, since there are three possible choices for each of the $k$-positions. Given that $0 < n < 10000$ the number of digits of $n$ must be 1, 2, 3 or 4. So the total number of such integers is $3 + 3^2 + 3^3 + 3^4 = 120$.

(ii) How many of the integers $n$ in Part (i) have the property that the digits of $n$ are distinct? (That is, it is not allowed for two of the digits of $n$ to be the same.)

**Solution:** Given that the digits are all different and there are only three different digits available, $n$ cannot have four digits. If $n$ has three digits then all of 1, 2 and 3 must be used, and the number of possibilities is $3! = 6$, the number of ways of assigning the three digits to the three positions. If $n$ has two digits then exactly one of 1, 2 or 3 is not used (three choices) and the remaining two can go in either order (two choices), giving six possibilities. Obviously there are three one-digit possibilities. So the total is $6 + 6 + 3 = 15$.

(iii) How many of the integers $n$ in Part (i) have the property that all of 1, 2 and 3 appear as digits of $n$?

**Solution:** If $n$ has exactly three digits then all of 1, 2 and 3 must be used, and as in Part (ii) this gives 6 possibilities. otherwise $n$ must have four digits and one of 1, 2 or 3 must be used twice, the others once each. There are three choices for the one that appears twice, and $\binom{4}{2} = 6$ possibilities for the two positions it occupies. Since there are two ways to assign the remaining two digits to the two remaining positions, the number of possibilities is $3 \times 6 \times 2 = 36$. So altogether there are $6 + 36 = 42$ numbers of the required form.

(b) (i) Let $n$ be a positive integer. Find the integer $k$ that satisfies the following equation:

$$\binom{2n}{n} = (2n + 2)\binom{2n}{n} - k\binom{2n + 1}{n + 1}.$$ 

**Solution:** Since

$$\binom{2n + 1}{n + 1} = \frac{(2n + 1)(2n)(n)}{(n + 1)!} = \frac{(2n + 1)(n + 1)!}{(n + 1)(n)!} = \frac{2n + 1}{n + 1}\binom{2n}{n}$$

it follows that

$$\binom{2n + 1}{n} = (n + 1)\binom{2n}{n + 1}.$$
and hence
\[(2n + 2)\binom{2n}{n} - (n + 1)\binom{2n + 1}{n + 1} = \binom{2n}{n}.\]

So the solution is \(k = n + 1\).

(ii) Using Part (i) or otherwise, show that \(\frac{1}{n + 1}\binom{2n}{n}\) is an integer.

**Solution:** Dividing both sides of the above equation by \(n + 1\) gives
\[
\frac{1}{n + 1}\binom{2n}{n} = 2\binom{2n}{n} - \binom{2n + 1}{n + 1}
\]

which is an integer since the binomial coefficients are integers.

(1 out of 2 for asserting without proof that \(\frac{1}{n + 1}\binom{2n}{n}\) is the Catalan number \(c_n\), which is an integer.)

\[2 + 2 + 2 + 2 + 2 = 10\text{ marks}\]
2. This question is for MATH2069 students only.

(a) Use the characteristic polynomial to find all sequences \(a_0, a_1, a_2, \ldots\) that satisfy the recurrence relation
\[ a_n = 8a_{n-1} - 15a_{n-2} \]
for all \(n \geq 2\).

**Solution:** The characteristic polynomial is
\[ x^2 - 8x + 15 = (x-3)(x-5), \]
and so the general solution is
\[ a_n = H3^n + K5^n \]
for all \(n\), where \(H\) and \(K\) are arbitrary constants.

(b) Let \(C\) be any constant. Find the solution to the recurrence relation in Part (a) satisfying the initial conditions \(a_0 = C\) and \(a_1 = 7C\).

**Solution:** Since \(a_n = H3^n + K5^n\), the initial conditions give
\[ C = a_0 = H3^0 + K5^0 = H + K \]
and
\[ 7C = a_1 = H3^1 + K5^1 = 3H + 5K. \]
This gives \(7C - 3C = 2K\), so that \(K = 2C\), and hence \(H = -C\). So \(a_n = 2C5^n - 3C^n\).

(c) Find a closed formula for the generating function \(\sum_{n=0}^{\infty} a_n z^n\) when the sequence \(a_0, a_1, a_2, \ldots\) satisfies the conditions of Part (b).

**Solution:** Let \(F(z) = \sum_{n=0}^{\infty} a_n z^n\), where \(a_n = 2C5^n - 3C^n\) as above. Then
\[ F(z) = 2C \sum_{n=0}^{\infty} 5^n z^n - C \sum_{n=0}^{\infty} 3^n z^n = \frac{2C}{1-5z} - \frac{C}{1-3z}. \]

(d) Let \(a_n\) be as in Parts (b) and (c), and define
\[ b_n = a_0 + a_1 + \cdots + a_n \]
for all integers \(n \geq 0\).

Using generating functions or otherwise, show that \(b_n\) satisfies the same recurrence relation as \(a_n\).

**Solution:** Writing \(G(z) = \sum_{n=0}^{\infty} b_n z^n\), we have that
\[ \frac{F(z)}{1-z} = \left( \sum_{n=0}^{\infty} z^n \right) \left( \sum_{n=0}^{\infty} a_n z^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k \right) z^n = G(z), \]
so that
\[ G(z) = \frac{C}{1-z} \left( \frac{2}{1-5z} - \frac{1}{1-3z} \right) \]
\[ = \frac{C}{1-z} \left( \frac{2(1-3z) - (1-5z)}{(1-5z)(1-3z)} \right) \]
\[ = \frac{C}{(1-5z)(1-3z)} \frac{5C/2 - 3C/2}{1-5z - 1-3z} = \frac{C}{1-5z - 1-3z}. \]
giving \( b_n = (5C/2)5^n - (3C/2)3^n \). Since this is of the form \( H3^n + K5^n \) it follows from Part (a) that \( b_n \) satisfies the same recurrence as \( a_n \).

Alternatively, observe that if \( n \geq 2 \) then

\[
\begin{align*}
 b_n - 8b_{n-1} + 15b_{n-2} &= \sum_{k=0}^{n} a_k - 8 \sum_{k=0}^{n-1} a_k + 15 \sum_{k=0}^{n-2} a_k \\
&= \sum_{k=0}^{n} a_k - 8 \sum_{k=1}^{n} a_{k-1} + 15 \sum_{k=2}^{n} a_{k-2} \\
&= a_0 + a_1 - 8a_0 + \sum_{k=2}^{n} (a_k - 8a_{k-1} + 15a_{k-2}) \\
&= a_1 - 7a_0 \\
&= 0
\end{align*}
\]

since \( a_0 = C \) and \( a_1 = 7C \).

\[
(3 + 2 + 3 + 2 = 10 \text{ marks})
\]

2. This question is for MATH2969 students only.

Suppose that the sequences \( a_0, a_1, a_2, \ldots \) and \( b_0, b_1, b_2, \ldots \) are related by the rules that

\[
\begin{align*}
a_n &= \sqrt{5}b_{n-1} + 2b_{n-2} \\
b_n &= \sqrt{5}a_{n-1} - 2a_{n-2}
\end{align*}
\]

for all \( n \geq 2 \). (1)

(a) Use induction to prove that there is a unique solution for any specified values of \( a_0, a_1, b_0 \) and \( b_1 \).

**Solution:** We use induction on \( n \) to show that for each nonnegative integer \( n \) there exist unique \( a_n \) and \( b_n \) satisfying the recurrence (if \( n \geq 2 \)) or having the specified value (if \( n = 0 \) or \( n = 1 \)). It is immediate that this is true if \( n = 0 \) or if \( n = 1 \); this observation starts the induction.

Now suppose that \( n \geq 2 \) and assume inductively that for each positive integer \( j < n \) there exist exactly one \( a_j \) and one \( b_j \) satisfying the conditions. In particular, \( a_{n-1}, b_{n-1}, a_{n-2} \) and \( b_{n-2} \) exist and are uniquely determined.

To complete the induction we just need to prove the existence and uniqueness of \( a_n \) and \( b_n \) satisfying

\[
\begin{align*}
a_n &= \sqrt{5}b_{n-1} + 2b_{n-2} \\
b_n &= \sqrt{5}a_{n-1} - 2a_{n-2},
\end{align*}
\]

and this is obvious since these equations explicitly define \( a_n \) and \( b_n \) in terms of uniquely determined quantities.

**SOLUTIONS**
(b) Find a 4th-order homogeneous linear recurrence relation satisfied by the sequence $a_0, a_1, a_2, \ldots$, and hence find all numbers $\lambda$ such that
\[
\begin{align*}
a_n &= \lambda^n \\
b_n &= \sqrt{5}\lambda^{n-1} - 2\lambda^{n-2}
\end{align*}
\]
gives a solution to (1).

**Solution:** If $n \geq 4$ then $n - 1 \geq n - 2 \geq 2$, and so by Equation (1)
\[
a_n = \sqrt{5}b_{n-1} + 2b_{n-2} = \sqrt{5}(\sqrt{5}a_{n-2} - 2a_{n-3}) + 2(\sqrt{5}a_{n-3} - 2a_{n-4})
= 5a_{n-2} - 4a_{n-4}.
\]
This is the desired 4th order homogeneous linear recurrence satisfied by the sequence $a_0, a_1, a_2, \ldots$. The characteristic polynomial of this recurrence is
\[
x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2).
\]
Hence $a_n = \lambda^n$ satisfies the 4th order recurrence (2) if and only if $\lambda$ is $\pm 1$ or $\pm 2$. In particular, since (1) implies that (2) holds for all $n \geq 4$, these four values of $\lambda$ are the only ones for which
\[
\begin{align*}
a_n &= \lambda^n \\
b_n &= \sqrt{5}\lambda^{n-1} - 2\lambda^{n-2}
\end{align*}
\]
can possibly give a solution to (1).

Conversely, let $\lambda$ be one of $\pm 1$ or $\pm 2$, and for all nonnegative integers $n$ use the equations (3) to define $a_n$ and $b_n$. Then for all $n \geq 2$,
\[
\sqrt{5}b_{n-1} + 2b_{n-2} = \sqrt{5}(\sqrt{5}\lambda^{n-2} - 2\lambda^{n-3}) + 2(\sqrt{5}\lambda^{n-3} - 2\lambda^{n-4})
= 5\lambda^{n-2} - 4\lambda^{n-4}
= \lambda^{n-4}(5\lambda^2 - 4)
= \lambda^n
= a_n
\]
since $5\lambda^2 - 4 = \lambda^4$. Furthermore,
\[
b_n = \sqrt{5}\lambda^{n-1} - 2\lambda^{n-2} = \sqrt{5}a_{n-1} - 2a_{n-2}.
\]
Hence (1) does indeed hold four these four values of $\lambda$.

(c) Show that if $a_1 = 0$ and $b_1 = \sqrt{5}a_0$ then $a_n = 0$ for all odd $n$, and find all such solutions $a_n$, $b_n$ of (1).

**Solution:** From Part (b) we know that the sequence $a_0, a_1, a_2, \ldots$ satisfies the recurrence $a_n = 5a_{n-2} - 4a_{n-4}$ for all $n \geq 4$, and hence
\[
a_n = A1^n + B(-1)^n + C2^n + D(-2)^n
\]
for all $n \geq 0$.
for some constants $A$, $B$, $C$ and $D$ determined by $a_0$, $a_1$, $a_2$ and $a_3$. The condition $a_1 = 0$ gives

$$A - B + 2C - 2D = 0. \quad (4)$$

Since $a_3 = \sqrt{5} b_2 + 2b_1 = \sqrt{5} (\sqrt{5} a_1 - 2a_0) + 2b_1$ the condition $b_1 = \sqrt{5} a_0$ implies that $a_3 = 5a_1$. Combined with $a_1 = 0$ this gives $a_3 = 0$, and so

$$A - B + 8C - 8D = 0. \quad (5)$$

Subtracting (4) from (5) gives $C = D$, and substituting this back in (4) gives $A = B$. So

$$a_n = A1^n + A(-1)^n + C2^n + D(-2)^n$$

and when $n$ is odd this gives $a_n = A - A + C2^n - C2^n = 0$, as required.

(d) Find all solutions $a_n$, $b_n$ of (1) satisfying $0 \leq a_n \leq 2$ for all $n$.

**Solution:** Every solution $a_n$, $b_n$ of (1) must satisfy

$$a_n = A + B(-1)^n + C2^n + D(-2)^n \quad \text{for all } n \geq 0$$

for some constants $A$, $B$, $C$ and $D$, and in particular this means that for all $n \geq 0$,

$$2a_{2n} - a_{2n+1} = 2(A + B + 2^n C + 2^n D) - (A - B + 2^{n+1} C - 2^{n+1} D)$$

$$= A + 3B + 2^{n+2}D.$$

If $D \neq 0$ then the absolute value of this expression tends to infinity as $n \to \infty$, whereas $|2a_{2n} - a_{2n+1}| \leq 4$ for all $n$ if $0 \leq a_n \leq 2$ for all $n$. So this assumption forces $D = 0$. Similarly, since

$$2a_{2n} + a_{2n+1} = 2(A + B + 2^n C + 2^n D) + (A - B + 2^{n+1} C - 2^{n+1} D)$$

$$= 3A + B + 2^{n+2}C$$

it follows that if $0 \leq a_n \leq 2$ for all $n$ then $C = 0$.

So if $0 \leq a_n \leq 2$ for all $n$ it follows that $a_n = A + (-1)^n B$ for all $n$, where $A + B$ and $A - B$ both lie in the interval $[0, 2]$. We then obtain

$$b_n = \sqrt{5}a_{n-1} - 2a_{n-2} = \sqrt{5}(A - B(-1)^n) - 2(A + (-1)^n B)$$

$$= (\sqrt{5} - 2)A - (\sqrt{5} + 2)(-1)^n B.$$

Conversely, it is readily checked that

$$a_n = A + (-1)^n B$$

$$b_n = (\sqrt{5} - 2)A - (\sqrt{5} + 2)(-1)^n B \quad \text{for all } n \geq 0$$

gives a solution to (1). And $0 \leq a_n \leq 2$ holds if and only if $0 \leq A + B \leq 2$ and $0 \leq A - B \leq 2$ both hold. These conditions are equivalent to $0 \leq A \leq 2$ and $|B| \leq \min(A, 2 - A)$.

$(2 + 3 + 3 + 2 = 10$ marks)
3. (a) (i) Define \( a_n = \frac{101^n}{n!} \), for all integers \( n \geq 0 \). Determine all values of \( n \) for which \( a_{n+1} < a_n \) and all values of \( n \) for which \( a_{n+1} > a_n \), and hence find the maximum value of \( a_n \). [Formula, not numerical value.]

**Solution:** Let \( n \) be a nonnegative integer. Multiplying through by \( \frac{(n+1)!}{101^n} \) shows that \( \frac{101^{n+1}}{(n+1)!} < \frac{101^n}{n!} \) if and only if \( 101 < n+1 \), or (equivalently) \( n < 100 \). Exactly similar reasoning shows that \( \frac{101^{n+1}}{(n+1)!} > \frac{101^n}{n!} \) if and only if \( n > 100 \). So

\[
a_0 < a_1 < \cdots < a_{99} < a_{100} = a_{101} > a_{102} > a_{103} > \cdots,
\]

and hence the maximum value of \( a_n \) is \( a_{100} = \frac{101^{100}}{101!} \).

(ii) Find a positive number \( K \) such that

\[
\frac{100^n}{n!} \leq K \left( \frac{100}{101} \right)^n
\]

for all nonnegative integers \( n \), and hence show that \( 100^n < n! \).

**Solution:** By Part (i) above, \( \frac{101^n}{n!} \leq \frac{101^{100}}{100!} \) for all nonnegative integers \( n \), and hence

\[
\frac{100^n}{n!} = \left( \frac{101^n}{n!} \right) \left( \frac{100}{101} \right)^n \leq \left( \frac{101^{100}}{100!} \right) \left( \frac{100}{101} \right)^n.
\]

So the positive number \( K = \frac{101^{100}}{100!} \) has the required property. Taking limits as \( n \to \infty \) we deduce that

\[
0 \leq \lim_{n \to \infty} \frac{100^n}{n!} \leq K \lim_{n \to \infty} \left( \frac{100}{101} \right)^n = 0,
\]

and consequently \( \lim_{n \to \infty} \left( \frac{100^n}{n!} \right) = 0 \). By the definition this means that \( 100^n < n! \).

(b) Let \( a_n \) be the \( n \)-th digit in the decimal expansion of \( \pi = 3.14159 \ldots \) (so that \( a_0 = 3 \), \( a_1 = 1 \), \( a_2 = 4 \), and so on). Define a sequence \( b_0, b_1, b_2, \ldots \) by \( b_0 = 0 \) and \( b_n = (1 + a_n)n + b_{n-1} \) for all \( n \geq 1 \).

(i) Find sequences \( \ell_0, \ell_1, \ell_2, \ldots \) and \( u_0, u_1, u_2, \ldots \) having the same growth rate and satisfying \( \ell_n \leq b_n \leq u_n \) for all nonnegative integers \( n \), using induction to prove that these inequalities are satisfied.

**Solution:** Define \( \ell_n \) and \( u_n \) by \( \ell_0 = u_0 = 0 \) and

\[
\ell_n = n + \ell_{n-1}, \quad u_n = 10n + u_{n-1}
\]

for all positive integers \( n \). Then \( \ell_0 = b_0 = u_0 \), and so \( \ell_n \leq b_n \leq u_n \) holds when \( n = 0 \). Proceeding inductively, let \( n \) be an integer greater than zero and assume that

\[
\ell_{n-1} \leq b_{n-1} \leq u_{n-1}.
\]

\[ (6) \]
Since $a_n$ is a decimal digit we have that

$$1 \leq 1 + a_n \leq 10. \quad (7)$$

Multiplying (7) through by $n$ and adding to (6) gives

$$n + \ell_{n-1} \leq (1 + a_n)n + b_{n-1} \leq 10n + u_{n-1},$$

so that $\ell_n \leq b_n \leq u_n$, as required.

(ii) Show that $b_n \asymp n^2$.

**Solution:** Unravelling the inductive definitions of $\ell_n$ and $u_n$ gives

$$\ell_n = 0 + 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

and

$$u_n = 0 + 10 + 20 + \cdots + 10n = 5n(n+1).$$

It follows that for all $n > 0$,

$$\frac{1}{2} + \frac{1}{2n} = \frac{\ell_n}{n^2} \leq \frac{b_n}{n^2} \leq \frac{u_n}{n^2} = 5 + \frac{1}{n},$$

and since $\frac{1}{n} \leq 1$ we deduce that

$$\frac{1}{2} \leq \frac{b_n}{n^2} \leq 6.$$

By the definition of $\sim$ this means that $b_n \asymp n^2$, as claimed.

$(2 + 3 + 3 + 2 = 10$ marks$)$

4. (a) Complete the definition: “Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomorphic if . . . ”

**Solution:** Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomorphic if there is a bijection $f : V \to V'$ such that $f(v), f(w)$ are adjacent in $G'$ if and only if $v, w$ are adjacent in $G$.

(b) The three pictures represent graphs $G$, $G'$ and $G''$:

![Graphs](image_url)
(i) Determine which two of the pictures represent isomorphic graphs. Copy both of them into your answer booklet and label the vertices so that they become two pictures of the same graph.

**Solution:** The graph $G$ is isomorphic to $G''$:

![Graphs G and G''](image)

(ii) Explain why the third graph is not isomorphic to those two.

**Solution:** The graph $G'$ is not isomorphic to $G$ or $G''$ because $G$ (and $G''$) contains a vertex of degree 4 while $G'$ does not.

(c) Give the definition of a connected graph.

**Solution:** We say that $G$ is connected if every vertex is linked to every other vertex.

(d) Suppose that a graph $G$ has $n$ vertices and $s$ connected components.

(i) Show that the number of vertices in each connected component does not exceed $n - s + 1$.

**Solution:** Each connected component contains at least one vertex. Hence any given connected component can contain at most $n - s + 1$ vertices.

(ii) Hence derive that the number of edges in $G$ does not exceed $\binom{n - s + 1}{2}$.

**Solution:** Let $n_1, \ldots, n_s$ be the numbers of vertices in the connected components so that $n_1 + \cdots + n_s = n$. The number of edges in the component with $n_i$ vertices does not exceed $\binom{n_i}{2}$. Hence, the total number of edges does not exceed

$$\binom{n_1}{2} + \cdots + \binom{n_s}{2}$$

$$= \frac{1}{2}(n_1(n_1 - 1) + \cdots + n_s(n_s - 1))$$

$$\leq \frac{1}{2}(n - s + 1)((n_1 - 1) + \cdots + (n_s - 1)) \quad \text{(since } n_i \leq n - s + 1 \text{ for all } i)$$

$$= \frac{1}{2}(n - s + 1)(n - s)$$

$$= \binom{n - s + 1}{2};$$

as required.

(e) Find the number of the isomorphism classes of connected graphs with 5 vertices and 5 edges. Draw a representative from each class and explain why your list is complete.
**Solution:** Since the number of edges exceeds 4, such graphs are not trees. If one edge belonging to a cycle is deleted then we get a tree. We know (e.g. from tutorials) that the number of isomorphism classes of trees with 5 vertices is 3:

Hence each connected graph in question can be obtained from one of these trees by adding one edge. Adding one edge in all possible ways to the trees and leaving one representative from each isomorphism class we get the full list of 5 isomorphism classes:

\[
\begin{align*}
\text{[Diagrams of trees]} \\
(1 + 3 + 1 + 2 + 3 = 10 \text{ marks})
\end{align*}
\]
5. This question is for MATH2069 students only.

(a) Give the definition of an Eulerian circuit in a connected graph.

**Solution:** An Eulerian circuit in a connected graph $G$ is a walk in $G$ which uses every edge exactly once and returns to its starting point.

(b) State the theorem from lectures which implies that the following graph $G$ is not Eulerian.

![Graph](image)

**Solution:** A connected graph $G$ is Eulerian if and only if the degree $\deg(v)$ is even for every vertex $v$ of $G$.

(c) Draw an Eulerian graph $H$ by adding the minimum number of edges to $G$ and find an Eulerian circuit in $H$.

**Solution:** The graph $G$ contains four vertices of odd degree: $a, b, h, i$. Hence the minimum number of edges to be added to $G$ to get an Eulerian graph $H$ cannot be less than 2. It is possible to add either the edges $\{a, h\}$, $\{b, i\}$ or $\{a, i\}$, $\{b, h\}$. So we can take $H$ in the form

![Graph](image)

A possible Eulerian circuit is $a, b, c, d, f, j, i, e, h, g, e, f, i, b, e, a, f, c, h, a$.

(d) Give the definition of a Hamiltonian graph.

**Solution:** A graph is Hamiltonian if it has a walk which visits every vertex exactly once and then returns to its starting point.

(e) Explain why the complete bipartite graphs $K_{p,p}$ with $p \geq 2$ are Hamiltonian.

**Solution:** Let the vertices of the graph be $1, 2, \ldots, p$ and $p + 1, \ldots, 2p$ so that the edges of $K_{p,p}$ are of the form $\{i, p + j\}$ with $i, j \in \{1, \ldots, p\}$. The required walk is provided by $1, 2, p + 1, \ldots, p, 2p, 1$.

(f) Prove that any graph which is obtained from $K_{5,3}$ by adding one edge is not Hamiltonian.

**Solution:** Suppose that the vertices of the graph $K_{5,3}$ are $1, 2, 3, 4, 5$ and $a, b, c$ so that the edges are $\{v, w\}$ with $v \in \{1, 2, 3, 4, 5\}$ and $w \in \{a, b, c\}$. Suppose first that we added one of the edges $\{1, 2\}$, $\{1, 3\}, \ldots, \{4, 5\}$ and that we have a spanning cycle in the new graph. Such a cycle can be written in the form

$$a, [\ldots], b, [\ldots], c, [\ldots], a,$$
where the symbols \([-\] denote some nonempty sequences of vertices 1, 2, 3, 4, 5 and each of the vertices occurs exactly once. Hence, there must be either a sequence \([-\] containing three vertices or two sequences containing two vertices each. However, in each of these cases, there are at least two edges of the form \(\{i, j\}\) with \(i, j \in \{1, 2, 3, 4, 5\}\), a contradiction.

Similarly, if we add one of the edges \(\{a, b\}\), \(\{a, c\}\), or \(\{b, c\}\) to the graph \(K_{5,3}\) then renumbering the vertices if necessary, we may assume that a spanning cycle has the form

\[
1, [-], 2, [-], 3, [-], 4, [-], 5, [-], 1,
\]

where the symbols \([-\] indicate some sequences of vertices \(a, b, c\) and each of the vertices occurs exactly once. This is clearly impossible.

\((1 + 2 + 2 + 1 + 2 + 2 = 10 \text{ marks})\)

5. This question is for MATH2969 students only.

(a) Prove that if \(G\) is a connected graph such that all vertices have even degrees, then \(G\) contains a cycle.

\[\text{Solution:}\] Pick any vertex \(v_0\) of \(G\). Since \(G\) is connected, there must be some other vertex \(v_1\) which is adjacent to \(v_0\); since \(\deg(v_1)\) can’t be 1, there must be some vertex \(v_2 \neq v_0\) which is adjacent to \(v_1\); since \(\deg(v_2)\) can’t be 1, there must be some vertex \(v_3 \neq v_1\) which is adjacent to \(v_2\); and we can continue in this way indefinitely. Because there are only finitely many vertices, there must be some \(k < \ell\) such that \(v_k = v_{\ell}\), and we can assume that \(v_k, v_{k+1}, \ldots, v_{\ell-1}\) are all distinct. The subgraph \(C\) which consists of these distinct vertices and the edges between them (including the edge \(\{v_{\ell-1}, v_k\}\)) is a cycle.

(b) List all isomorphism classes of Eulerian graphs with 5 vertices. Explain why your list is complete.

\[\text{Solution:}\] By a theorem from lectures, a connected graph is Eulerian if and only if all vertices have even degrees. Therefore, possible degree sequences of Eulerian graphs are \((2, 2, 2, 2, 2), (2, 2, 2, 2, 4), (2, 2, 2, 4, 4), (2, 2, 4, 4, 4), (2, 4, 4, 4, 4),\) and \((4, 4, 4, 4, 4)\). Up to an isomorphism, the only graph with the degree sequence \((2, 2, 2, 2, 2)\) is the cycle graph \(C_5\). The complement to a graph with the sequence \((2, 2, 2, 2, 4)\) has the degree sequence \((0, 2, 2, 2, 2)\) and hence it is isomorphic to the graph with two connected components: a vertex and the cycle graph \(C_4\). This shows that the only graph with the degree sequence \((2, 2, 2, 2, 4)\) is obtained from the complete graph \(K_5\) by deleting a 4-cycle. Similarly, the only graph with the degree sequence \((2, 2, 2, 4, 4)\) is obtained from the complete graph \(K_5\) by deleting a 3-cycle. The sequences \((2, 2, 4, 4, 4)\) and \((2, 4, 4, 4, 4)\) are not graphic, since otherwise the complements to the graphs would have the respective degree sequences...
(0, 0, 2, 2) and (0, 0, 0, 2) which is impossible. The only graph with the sequence (4, 4, 4, 4, 4) is $K_5$. Thus, the complete list is

![Graphs](image)

(c) Suppose that $G$ is a connected graph with 7 vertices such that every vertex has degree $\geq 3$. Show that at least one vertex has degree $\geq 4$.

**Solution:** Suppose that all vertices have degree 3. Then the sum of the degrees of the vertices is $3 \cdot 7 = 21$ which is an odd number. This contradicts the Hand-shaking Lemma.

(d) (i) Show that the graph in part (c) does not have to be Hamiltonian.

**Solution:** The complete bipartite graph $K_{3,4}$ satisfies the conditions of part (c), but this graph is not Hamiltonian.

(ii) Prove that the graph $G$ in part (c) can be made into a Hamiltonian graph by adding at most three edges.

**Solution:** There are at most three pairs of vertices of degree 3. Hence we can add at most three pairs of edges to get a new graph satisfying the condition of the Ore Theorem. Therefore, the new graph is Hamiltonian.

$$2 + 3 + 2 + 3 = 10 \text{ marks}$$
6. (a) Complete the formulation of the Chinese Postman Problem: “Given a connected weighted graph, find a walk which . . .”.

**Solution:** Given a connected weighted graph, find a walk which uses every edge, returns to its starting point, and has minimum weight subject to these two conditions.

(b) Find a walk which solves the Chinese Postman Problem for the following weighted graph. Justify your answer.

![Graph](image)

**Solution:** There are exactly two vertices of odd degree, $b$ and $f$. Hence, the graph has an Eulerian trail. By a theorem from lectures, an Eulerian trail from $b$ to $f$, followed by a minimal walk from $f$ to $b$, is a solution of the Chinese Postman Problem. An Eulerian trail from $b$ to $f$ is given by $b, a, e, b, c, e, f, c, d, f$ and a minimal walk from $f$ to $b$ is $f, e, c, b$. The solution of the Chinese Postman Problem is $b, a, e, b, c, e, f, c, d, f, e, c, b$.

(c) Consider all trees with 8 vertices $\{1, 2, 3, 4, 5, 6, 7, 8\}$, where the vertex 5 has degree 3, and the vertex 8 has degree 5.

(i) Prove that each of these trees has 6 leaves.

**Solution:** The number of edges in each tree is 7. By the Handshaking Lemma, the sum of the degrees of all vertices is 14. Hence, the sum of the degrees of the six remaining vertices is $14 - 3 - 5 = 6$. Therefore, each of the 6 vertices has degree 1, so it is a leaf.

(ii) Calculate the number of the trees.

**Solution:** By a theorem from lectures, the number of times a vertex $v$ of a tree occurs in the Prüfer sequence equals $\deg(v) - 1$. Hence the Prüfer sequences of the trees in question contain 4 entries equal to 8 and 2 entries equal to 5. The number of such sequences is $\binom{6}{2} = 15$.

(d) Consider the polynomial $Q(t) = t^4 - 3t^3 + at^2 + bt$, where $a$ and $b$ are integers. Find all values of $a$ and $b$ such that $Q(t)$ coincides with the chromatic polynomial $P_G(t)$ of a certain graph $G$.

**Solution:** By a theorem from lectures, if $Q(t) = P_G(t)$ then $G$ has 4 vertices and 3 edges. If $G$ is a connected graph then $G$ is a tree so that $P_G(t) = t(t-1)^3 = t^4 - 3t^3 + 3t^2 - t$. If $G$ is not connected then it must have two connected components; one of them is isomorphic to $K_3$ and the other is one-vertex graph $K_1$. In this case,

$$P_G(t) = P_{K_3}(t) P_{K_1}(t) = t \cdot t(t-1)(t-2) = t^4 - 3t^3 + 2t^2.$$  

Thus, the possible values of the coefficients are $a = 3$, $b = -1$ and $a = 2$, $b = 0$.  

**SOLUTIONS** turn to page 16
(e) For any $n \geq 2$ consider the graph $M_n$ with $\binom{n}{2}$ vertices $v_{ij}$ labelled by all pairs $\{i, j\}$ such that $1 \leq i < j \leq n$. Two vertices $v_{ij}$ and $v_{kl}$ are connected by an edge if and only if the pairs $\{i, j\}$ and $\{k, l\}$ have a common element. Find the chromatic number $\chi(M_n)$.

Solution: Observe that the chromatic number $\chi(M_n)$ coincides with the edge chromatic number $\chi'(K_n)$ of the complete graph $K_n$. Indeed, given a proper edge colouring of $K_n$, for any edge $\{i, j\}$ of $K_n$ use the same colour for the vertex $v_{ij}$ of $M_n$. This provides a bijection between the vertex colourings of $M_n$ and edge colourings of $K_n$. By a theorem from lectures, for any $n \geq 2$,

$$\chi(M_n) = \chi'(K_n) = \begin{cases} n & \text{if } n \text{ is odd}, \\ n - 1 & \text{if } n \text{ is even.} \end{cases}$$

(In fact, the graph $M_n$ is isomorphic to the dual graph $\hat{K}_n$ thus yielding the relation $\chi(M_n) = \chi'(K_n)$.)

$(1 + 2 + 2 + 2 + 3 = 10$ marks)