1. Consider a $k$-th order homogeneous linear recurrence relation

$$a_n = r_1 a_{n-1} + \cdots + r_k a_{n-k}, \quad n \geq k; \quad (1)$$

such that the roots of its characteristic polynomial are all distinct.

(a) Prove that for any positive integer $m$ there exists a homogeneous linear recurrence relation of order not exceeding $(k + m - 1)$ such that for every sequence $a_n$ satisfying (1), the sequence of powers $a_n^m$ satisfies that relation.

**Solution:** Let $\lambda_1, \ldots, \lambda_k$ be the roots of the characteristic polynomial. Since they are all distinct, the general solution of (1) has the form

$$a_n = C_1 \lambda_1^n + \cdots + C_k \lambda_k^n,$$

where $C_1, \ldots, C_k$ are arbitrary constants. Using the multinomial theorem we get

$$a_n^m = (C_1 \lambda_1^n + \cdots + C_k \lambda_k^n)^m = \sum_{m_1 + \cdots + m_k = m} \binom{m}{m_1, \ldots, m_k} C_1^{m_1} \cdots C_k^{m_k} (\lambda_1^{m_1} \cdots \lambda_k^{m_k})^n.$$

This shows that $a_n^m$ is a linear combination of the sequences $(\lambda_1^{m_1} \cdots \lambda_k^{m_k})^n$. The number of powers $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$ is the number of solutions of the equation $m_1 + \cdots + m_k = m$ which equals $\binom{k + m - 1}{m}$. This is the maximum possible number of distinct values of the powers (some of the powers may coincide).

Hence there exists a polynomial $p(x)$ of degree not exceeding $\binom{k + m - 1}{m}$ whose roots are the distinct numbers among the powers $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$. Every sequence $(\lambda_1^{m_1} \cdots \lambda_k^{m_k})^n$ is a solution of the homogeneous linear recurrence relation with the characteristic polynomials $p(x)$. Therefore, by a theorem from lectures, the sequence $a_n^m$, which is a linear combination of these sequences, is also a solution. The order of the homogeneous linear recurrence relation coincides with the degree of $p(x)$ and so does not exceed $\binom{k + m - 1}{m}$.

(b) Write down a homogeneous linear recurrence relation of order 3 satisfied by the sequence of squares of the Lucas numbers $L_n^2$.

**Solution:** The roots of the characteristic polynomial $x^2 - x - 1$ are $\lambda_{1,2} = (1 \pm \sqrt{5})/2$. As in the solution of the previous part with $m = 2$, consider the polynomial $p(x) = (x - \lambda_1^2)(x - \lambda_2^2)(x - \lambda_1 \lambda_2)$ with the roots

$$\lambda_1^2 = \frac{3 + \sqrt{5}}{2}, \quad \lambda_2^2 = \frac{3 - \sqrt{5}}{2}, \quad \lambda_1 \lambda_2 = -1.$$
We have \( p(x) = (x+1)(x^2-3x+1) = x^3 - 2x^2 - 2x + 1 \) and so the recurrence relation is \( a_n = 2a_{n-2} + 2a_{n-2} - a_{n-3} \) with \( n \geq 3 \).

(5 marks)

2. Define formal power series by the formulas

\[
\begin{align*}
\sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \\
\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \\
\tan z &= \frac{\sin z}{\cos z}, \\
\arctan z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}.
\end{align*}
\]

Prove the identities
(a) \( \arctan(\tan z) = z \);

**Solution:** Observe that the derivative of \( \arctan z \) we have

\[
(\arctan z)' = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{2n+1} z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = \frac{1}{1+z^2}.
\]

Furthermore, as pointed out in lectures and easily checked,

\[
(\sin z)' = \cos z, \quad (\cos z)' = -\sin z.
\]

Hence, by the quotient rule,

\[
(\tan z)' = \frac{(\sin z)\cos z - (\cos z)'\sin z}{(\cos z)^2} = \frac{(\cos z)^2 + (\sin z)^2}{(\cos z)^2} = 1 + (\tan z)^2.
\]

Now, using the chain rule we get

\[
(\arctan(\tan z))' = \frac{1}{1+(\tan z)^2} (\tan z)' = \frac{1}{1+(\tan z)^2} (1 + (\tan z)^2) = 1.
\]

By a theorem from lectures, we can conclude that \( \arctan(\tan z) = c + z \), where \( c \) is a constant. To find the value of \( c \), write

\[
\tan z = a_0 + a_1 z + \ldots.
\]

We have \( \sin z = \tan z \cos z \) so that

\[
z - \frac{z^3}{6} + \ldots = (a_0 + a_1 z + \ldots)(1 - \frac{z^2}{2} + \ldots)
\]

which implies that \( a_0 = 0 \) and \( a_1 = 1 \). Since

\[
\arctan z = z - \frac{z^3}{3} + \ldots \quad \text{and} \quad \tan z = z + \ldots
\]

we find that the constant term \( c \) of the series \( \arctan(\tan z) \) is zero, thus proving that \( \arctan(\tan z) = z \).
(b) \( \tan(\arctan z) = z. \)

**Solution:** Set \( A(z) = \tan(\arctan z). \) Using the chain rule we get

\[
A'(z) = \left( 1 + (\tan(\arctan z))^2 \right) \frac{1}{1 + z^2} = \frac{1 + A(z)^2}{1 + z^2}.
\]

Thus, the formal power series \( A(z) \) satisfies the differential equation

\[
(1 + z^2) A'(z) = 1 + A(z)^2.
\]

We will show that given a value of \( a_0 \), the differential equation has a unique solution. The equation reads

\[
(1 + z^2) \sum_{n=0}^{\infty} (n + 1) a_{n+1} z^n = 1 + \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} a_m a_{n-m} \right) z^n.
\]

Comparing the coefficients of \( z^n \) on both sides, we get the relations

\[
(n + 1) a_{n+1} = \sum_{m=0}^{n} a_m a_{n-m} - (n - 1) a_{n-1}, \quad n \geq 1
\]

and \( a_1 = 1 + a_0^2. \) Thus we get a recurrence relation which determines all elements \( a_n \) uniquely from the given value of \( a_0. \) So \( A(z) \) is uniquely determined by the value of \( a_0. \) Since

\[
\arctan z = z - \frac{z^3}{3} + \ldots \quad \text{and} \quad \tan z = z + \ldots
\]

we find that the constant term \( a_0 \) of the series \( \tan(\arctan z) \) is zero. On the other hand, the formal power series \( A(z) = z \) is clearly a solution of the differential equation and its constant term is also zero. By the uniqueness of the solution, we may conclude that \( \tan(\arctan z) = z. \)

(5 marks)