1. For vertices \( v \) and \( w \) in a connected weighted graph, \( d(v, w) \) denotes the minimum weight of a walk from \( v \) to \( w \). Explain why the ‘triangle inequality’

\[
d(u, v) + d(v, w) \geq d(u, w)
\]

holds for any vertices \( u, v, w \).

**Solution:** Suppose that \( u = u_0, u_1, \ldots, u_k = v \) is a walk from \( u \) to \( v \) which has the minimum weight \( d(u, v) \), and \( v = v_0, v_1, \ldots, v_\ell = w \) is a walk from \( v \) to \( w \) which has the minimum weight \( d(v, w) \). Then \( u_0, u_1, \ldots, u_k, v_1, \ldots, v_\ell \) is a walk from \( u \) to \( w \) which has weight \( d(u, v) + d(v, w) \). Hence \( d(u, v) + d(v, w) \) must be greater than or equal to the minimum weight of a walk from \( u \) to \( w \), which is \( d(u, w) \).

2. Use Dijkstra’s Algorithm to find all the minimal walks from \( A \) to \( Z \) in the following graph, and the weight \( d(A, Z) \) of such a minimal walk.

**Solution:** The algorithm starts by giving \( A \) the permanent label \([0]\), and then labelling each vertex adjacent to \( A \) with a temporary label equal to the weight of the intervening edge. We then find the smallest temporary label: in this case it is the label \((1)\) on the vertex \( B \). We make this permanent, and then create or adjust the temporary labels of all the vertices adjacent to that one (to avoid clutter, vertex names other than \( A \) and \( Z \) are omitted in this and subsequent pictures):

Now we find the next smallest temporary label (in this case \((3)\)), and repeat the process described above. The following diagram results after making the label \((3)\)
permanent, and labelling all adjacent vertices. (Note that the temporary label (6) has been reduced to (5).)

Repeating the process until all vertices have permanent labels, we obtain the following diagram:

The label on each vertex is now the minimum weight of a walk from $A$ to that vertex, and in particular $d(A, Z) = 14$.

To find all walks from $A$ to $Z$ of weight 14, we work backwards: we first consider what the last step of the walk could possibly be, then the step before that, and so on. In the present situation, the last step of the walk must be from the vertex $G$ labelled [12], because $10 + 5 > 14$, $8 + 8 > 14$, and $11 + 5 > 14$. Similarly, the step before that has to be from the vertex $F$ labelled [10]. Continuing in this way, we find two walks of weight 14 from $A$ to $Z$:

- $A, E, D, C, F, G, Z$
- $A, E, D, H, C, F, G, Z$

3. Find a solution to the Chinese Postman Problem in the following graph, where every edge has weight 1.

\[
\begin{array}{cccc}
  a & b & c & d \\
  e & f & g & h \\
  i & j & k & \ell \\
  m & n & p & q \\
\end{array}
\]

**Solution:** A solution to the Chinese Postman Problem is a walk which uses every edge, returns to its starting point, and has minimum weight subject to these conditions. This graph has exactly two vertices of odd degree, namely $b$ and $c$; so it contains an Eulerian trail from $b$ to $c$. Here is one such Eulerian trail:

- $b, c, d, h, g, k, \ell, q, p, k, j, n, m, i, j, f, e, a, b, f, g, c$.
By a result proved in lectures, if we follow this Eulerian trail with a minimal walk from $c$ to $b$ (which in this case clearly consists of a single step from $c$ to $b$), we will have a solution of the Chinese Postman Problem:

$$b, c, d, h, g, k, \ell, q, p, k, j, n, m, i, j, f, e, a, b, f, g, c, b.$$ 

4. Suppose that $G$ is a connected weighted graph and $u, v, w$ are distinct vertices. Show that any solution to the Travelling Salesman Problem for $G$ has weight at least $d(u, v) + d(v, w) + d(w, u)$.

**Solution:** A solution to the Travelling Salesman Problem is a walk which visits every vertex, returns to its starting point, and has minimum weight subject to these conditions. Shifting some steps of the walk from the beginning to the end if necessary, we can assume that the starting point is the vertex $u$. The walk visits both $v$ and $w$ at some point; without loss of generality, we can assume that it visits $v$ first. So the walk has the form $u, \cdots, v, \cdots, w, \cdots, u$, where the $\cdots$ indicate some other vertices (possibly including $u, v, w$ again). The weight of the first portion of the walk, namely $u, \cdots, v$, is at least $d(u, v)$; the weight of the second portion, namely $v, \cdots, w$, is at least $d(v, w)$; and the weight of the third portion, namely $w, \cdots, u$, is at least $d(w, u)$. Hence the total weight is at least $d(u, v) + d(v, w) + d(w, u)$, as claimed.

5. Draw pictures of all the isomorphism classes of trees with 5 vertices.

   **Solution:** The easiest way to classify these trees is to think about the maximal degree of a vertex, remembering that the tree has 4 edges. If there is a vertex of degree 4, then all the edges end at that vertex, so the tree is isomorphic to $K_1, 4$. If there is a vertex of degree 3, then there is one edge which does not end at that vertex. If there is no vertex of degree $\geq 2$, then the tree must be a path. So there are just three isomorphism classes of trees with 5 vertices:

   *6. Let $T$ be a tree with $p \geq 2$ leaves and $q$ vertices of degree $\geq 3$. Prove that $q \leq p - 2$.

   **Solution:** Let $n$ be the total number of vertices of $T$. Since $T$ has at least two vertices and is connected, it has no vertices of degree 0. So $T$ has $p$ vertices of degree 1, $n - p - q$ vertices of degree 2, and $q$ vertices of degree $\geq 3$. Thus the sum of all the degrees is at least $p + 2(n - p - q) + 3q = 2n - p + q$. On the other hand, we know that $T$ has $n - 1$ edges, so the sum of all the degrees is $2n - 2$ by the Hand-shaking Lemma. We conclude that $2n - p + q \leq 2n - 2$, which rearranges to give the desired inequality $q \leq p - 2$.

7. An alcohol molecule has formula $C_kH_{2k+2}O$ where $k$ is a positive integer. Picture the molecule as a connected graph where the atoms are the vertices; $C$ atoms have degree 4, $H$ atoms have degree 1, and the $O$ atom has degree 2.

   (a) Show that no matter what $k$ is, the graph is always a tree.
**Solution:** Since the graph has $k$ vertices of degree 4, $2k + 2$ vertices of degree 1, and one vertex of degree 2, the Hand-shaking Lemma tells us that the number of edges is $\frac{1}{2}(4k + 2k + 2 + 2) = 3k + 2$, which is one less than the number of vertices. But a connected graph in which the number of edges is one less than the number of vertices must be a tree.

(b) Draw the graph for the methanol molecule, in which $k = 1$.

**Solution:** In this case there is one $C$ atom, one $O$ atom, and four $H$ atoms. Since the $H$ atoms are leaves of the tree, if we delete them all we still have a tree; thus the $C$ atom must be adjacent to the $O$ atom. The other three edges ending at the $C$ atom must join it to three of the $H$ atoms, and the remaining $H$ atom must be joined to the $O$ atom:

```
     H
  H----C----O----H
     H
```

8. Find the Prüfer sequences of the following trees.

(a)  

```
   6 1 5 2
  4
  3
```

**Solution:** Recall the procedure for finding the Prüfer sequence: we delete the smallest leaf (in this case 2), and record in the sequence the vertex it was adjacent to (in this case 4). After 2 has been deleted, the smallest leaf remaining is 3, which is also adjacent to 4; after 3 has been deleted, the smallest leaf remaining is 5, which is adjacent to 4; after 5 has been deleted, the smallest leaf remaining is 4, which is adjacent to 1. At this point we stop forming the sequence, because there are only two vertices left. The answer is $(4, 4, 4, 1)$.

(b)  

```
   3 2 4 1
  5
  6
```

**Solution:** The Prüfer sequence is $(4, 2, 2, 4)$.

(c)  

```
   1
  2 3 4 5
  6 7 8 9 10 11 12 13
```

**Solution:** The Prüfer sequence is $(2, 2, 1, 3, 3, 1, 4, 4, 1, 5, 5)$.

9. Draw the trees with the following Prüfer sequences, where the vertex set is always $\{1, 2, \ldots, n\}$ for some $n$. 


(a) \((1, 2, 3, 4, 5)\)

**Solution:** Since there are 5 terms in the sequence, there must be 7 vertices, so \(n = 7\). The leaves of the tree are the vertices which do not occur in the sequence, i.e. 6 and 7. So the smallest leaf is 6, and the first term of the sequence tells us that 6 must be adjacent to 1. After we delete 6, we have a tree with vertex set \(\{1, 2, 3, 4, 5, 7\}\) and Prüfer sequence \((2, 3, 4, 5)\); here the smallest leaf is 1, which must be adjacent to 2. After we delete 1, we have a tree with vertex set \(\{2, 3, 4, 5, 7\}\) and Prüfer sequence \((3, 4, 5)\); here the smallest leaf is 2, which must be adjacent to 3. After we delete 2, the smallest leaf is 3, adjacent to 4; after we delete 3, the smallest leaf is 4, adjacent to 5; after we delete 4, only the vertices 5 and 7 remain, so they must be adjacent. Hence the tree is:

```
6 1 2 3 4 5 7
```

(b) \((3, 3, 3, 3, 3)\)

**Solution:** Again \(n = 7\). Reasoning as in (a), we find that the tree is:

```
7
/
|
6
/
|
5
/
|
3
/
|
1
/
|
2
/
|
4
```

(c) \((2, 8, 6, 3, 1, 2)\)

**Solution:** Here \(n = 8\). Reasoning as in (a), we find that the tree is:

```
7 6 3 1 2 8 5 4
```

10. For any positive integers \(m < n\), let \(T(n, m)\) denote the number of trees with vertex set \(\{1, 2, \cdots, n\}\) in which \(\deg(n) = m\).

(a) Using the bijection between trees and Prüfer sequences, show that

\[
T(n, m) = \binom{n-2}{m-1}(n-1)^{n-m-1}.
\]

**Solution:** In the Prüfer sequence of a tree with vertex set \(\{1, 2, \cdots, n\}\), the number of times \(n\) occurs is one less than the degree of the vertex \(n\) in the tree. So \(T(n, m)\) equals the number of sequences of length \(n - 2\) in which every term belongs to \(\{1, 2, \cdots, n\}\), and \(n\) occurs \(m - 1\) times. We can count such sequences by first choosing the positions in which \(n\) occurs in \(\binom{n-2}{m-1}\) ways, and then choosing the other \(n - m - 1\) terms of the sequence in \((n - 1)^{n-m-1}\) ways. This gives the result.
(b) Hence show that for all $2 \leq m \leq n - 1$,
\[(n - m)T(n, m - 1) = (n - 1)(m - 1)T(n, m).\]

**Solution:** From the previous part, we have
\[
(n - m)T(n, m - 1) = (n - m)\frac{(n - 2)!}{(m - 2)!(n - m)!}(n - 1)^{n-m}
\]
\[
= \frac{(n - 2)!}{(m - 2)!(n - m - 1)!}(n - 1)^{n-m}
\]
\[
= (m - 1)\frac{(n - 2)!}{(m - 1)!(n - m - 1)!}(n - 1)^{n-m} = (n - 1)(m - 1)T(n, m).
\]

**(c)** Give a direct combinatorial proof of part (b), without using Prüfer sequences or the result of part (a). Hence give an alternative proof of part (a) and of Cayley’s Formula.

**Solution:** We aim to prove the desired equation by interpreting the two sides as the sizes of sets $X$ and $Y$, and defining a bijection between $X$ and $Y$. The left-hand side $(n - m)T(n, m - 1)$ is the size of the set $X$ of pairs $(u, T)$ where $T$ is a tree with vertex set $\{1, 2, \ldots, n\}$ in which $\deg(u) = m - 1$ and $u$ is a vertex which is neither equal to $n$ nor adjacent to $n$. The right-hand side $(n - 1)(m - 1)T(n, m)$ is the size of the set $Y$ of triples $(u, v, T')$ where $T'$ is a tree with vertex set $\{1, 2, \ldots, n\}$ in which $\deg(v) = m$, $v$ is a vertex not equal to $n$, and $u$ is a vertex which is adjacent to $n$ but does not lie in the unique path between $v$ and $n$. We define a function $f : X \to Y$ as follows: given $(u, T) \in X$, define $v$ to be the vertex adjacent to $u$ on the path between $u$ and $n$, and let $T'$ be the tree obtained from $T$ by deleting the edge $\{u, v\}$ and adding the edge $\{u, n\}$; then $f(u, T) = (u, v, T')$. The inverse function $g : Y \to X$ is defined as follows: given $(u, v, T') \in Y$, let $T$ be the tree obtained from $T'$ by deleting the edge $\{u, n\}$ and adding the edge $\{u, v\}$; then $g(u, v, T') = (u, T)$. It is easy to check that these maps are well defined and inverse to each other, so $|X| = |Y|$ and (b) is proved. It follows that for any positive integers $m < n$,
\[
T(n, m) = \frac{(n - 1)m}{n - m - 1}T(n, m + 1)
\]
\[
= \frac{(n - 1)m}{n - m - 1} \frac{(n - 1)(m + 1)}{n - m - 2} T(n, m + 2)
\]
\[
= \frac{(n - 1)^{n-m-1}m(m+1)}{(n-m-1)!} T(n, n-1)
\]
\[
= \frac{(n - 1)^{n-m-1}(n - 2)!}{(m - 1)!(n - m - 1)!} = (n - 1)^{n-m-1} \binom{n - 2}{m - 1},
\]
because clearly $T(n, n - 1) = 1$. So we have re-proved part (a). Finally, we can deduce Cayley’s Formula: for $n \geq 2$, the number of trees with vertex set $\{1, 2, \ldots, n\}$ is
\[
\sum_{m=1}^{n-1} (n - 1)^{n-m-1} \binom{n - 2}{m - 1} = (1 + (n - 1))^{n-2} = n^{n-2}.
\]